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DEMIMATROIDS ON GRAPHS

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Abstract

Demimatroids are a combinatorial structure arising as a generalization of linear codes, simplicial complexes and matroids. We show that several combinatorially defined invariants on graphs may be viewed as demimatroids, say matching number, edge chromatic number, edge list chromatic number, degree of a graph, vertex cover number, or clique number. Thus, each one of these invariants satisfies Wei's duality and has a Tutte polynomial attached to it. We show how some families of graphs, say, cographs, triangle-free or bipartite graphs, may be characterized in terms of these demimatroids. Wei's numbers of these demimatroids could be used to study the resilience or robustness of a network.

Keywords:

Chromatic number; cograph; demimatroid; edge chromatic number; matching number; Tutte polynomial; Wei duality.
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Introduction

A demimatroid is a pair $M = (E, r)$, where E is a finite set and $r : 2^E \to \mathbb{Z}_+ := \{0, 1, \dots\}$ is a function such that:

(D1) $r(\emptyset) = 0$;

(D2) If $X \subseteq E$ and $e \in E$, then $r(X) \leq r(X \cup \{e\}) \leq r(X) + 1$.

If in addition, r is subadditive, that is,

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y),$$

then M is called a matroid. The function r is known as the rank function, and may be identified with the demimatroid itself. One readily verifies that a function $r : 2^E \to \mathbb{Z}_+$ is a demimatroid if and only if

(D$^{\text{i}}$) $r(\emptyset) = 0$;

(D$^{\text{ii}}$) If $A \subseteq B \subseteq E$, then

$$0 \leq r(B) - r(A) \leq |B| - |A|.$$

On the other hand, one observes that in a demimatroid (E, r), by repeatedly applying condition (D2), it results that if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$ and $r(A) \leq |A|$; but not vice versa; e.g. take $E = \{a, b\}$ with $r(A) = 0$ if $A \neq E$, and $r(E) = 2$.

With a demimatroid $M = (E, r)$ one can associate several other demimatroids, e.g. the dual demimatroid $M^* = (E, r^*)$ with

$$r^*(A) := |A| + r(E \setminus A) - r(E);$$

the nullity $M^\circ = (E, r^\circ)$ with

$$r^\circ(A) := |A| - r(A);$$

the supplement $M^\circledast = (E, r^\circledast)$ with

$$r^\circledast(A) = r(E) - r(E \setminus A);$$

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or its elongations [5,11]. When M is a matroid, its dual is also a matroid, but in general, the nullity and the supplement are only demimatroids. Apart from these kind of dualities, a demimatroid also has a Tutte polynomial attached to it, which is a two-variable polynomial encoding valuable information of the demimatroid itself (cf. Section 5).

For a demimatroid $M = (E, r)$, its rank is the integer $r(M) := r(E)$. The Wei numbers of a demimatroid (a concept coming from linear codes; cf. [9]) are defined as

$$\sigma_i(M) := \min\{|A| : r(A) = i\}, \qquad i = 0, \dots, k = r(M).$$

It is known that $\sigma_0(M) < \cdots < \sigma_k(M)$; [5,11]. One sometimes writes $\sigma_i(r)$ instead of $\sigma_i(M)$.

When $M = (E, r)$ is a matroid, it is well-known that the nullity of a subset $A \subseteq E$ is a measure of the "dependence" of A; in fact, it was proved in [10] that the nullity $r^\circ(A)$ equals the maximum number of non-redundant circuits contained in A. When M is just a demimatroid, there is no known such a nice interpretation. However, for the supplement demimatroid $s := r^{\circledast}$, its Wei numbers have a notable interpretation:

$$\sigma_i(s) = \min\{|A| : r(E \setminus A) = r(E) - i\}, \qquad i = 0, \dots, s(E).$$

Thus $\sigma_i(s)$ is the smallest number of elements of E that need to be removed to decrease the rank of M by i units. In this way, $\sigma_1(s)$ may be viewed as an indicator of the robustness of the rank $r(M)$ of the demimatroid, and, therefore, the totality of Wei numbers $\sigma_1(s), \dots, \sigma_k(s)$ could be used as a strong indicator of the robustness of the rank function; which may be useful when M arises from a network. Note that if s is the supplement of r, then r is the supplement of s; and thus, the Wei number $\sigma_1(r)$ is also an indicator of the robustness of the supplement rank function.

A simplicial complex Δ on a finite set E is an inclusion-closed family of subsets of E. Elements of Δ are the faces and maximal faces are the facets. For a simplicial complex Δ on the set E, the function

$$r(A) := \max\{|\sigma| : \sigma \in \Delta \text{ and } \sigma \subseteq A\}$$

is a demimatroid. This construction has been utilized to studying demimatroids on posets [3].

***Example* 1.1.** Let $G = (V, E)$ be a graph viewed as a simplicial complex whose facets are the edges of the graph. Let r be the demimatroid on V determined by this complex. This demimatroid has rank 2. In this case $\sigma_1(s)$ is the smallest number of vertices that need to be removed to decrease by 1 the rank of r, i.e. it equals the vertex covering number of the graph.

***Example* 1.2.** For a connected graph $G = (V, E)$, let Δ be the simplicial complex whose facets are the spanning trees of G and let r denote the demimatroid on E determined by this complex (the graphic matroid of G). If G has n vertices, then this demimatroid has rank $n - 1$. In this case $\sigma_1(s)$ equals the edge cut number of the graph.

On the set of edges of a graph, one may define several demimatroids, say matching number, edge chromatic number or the degree (see next section). For these demimatroids, $\sigma_1(s)$ will be an indicator of the robustness of its rank function. For example, when r is the edge chromatic number, $\sigma_1(r)$ is known in the literature as the edge chromatic stability index of the graph (cf. [1,2]). As another example, when r is the degree of a graph, $\sigma_1(r)$ equals the minimum number of edges that need to be removed to decrease the degree of the graph by 1; this is related to the study of the hubs (i.e. vertices with many neighbors) of a network. See also [4,15]. All these examples suggest that demimatroids on graphs are worthy of study.

Here, our main aim is to define several demimatroids on the edge or vertex set of a graph (Sections 2 and 3 respectively) and characterize some families of graphs using these demimatroids (Section 4). For a lightly lecture on matroids consider [12]. For references on demimatroids consult [5,11], and for a general theory of graphs see [7].

Demimatroids on the edge set of a graph

Let $G = (V, E)$ be a (simple) graph. For a subset A of edges of the graph, we denote by V_A its vertex set and by C_A its set of connected components; from now on, we will identify the set of edges A with the graph (V_A, A). Any graph $G = (V, E)$ has associated in a canonical way the matroid $M = (E, r)$, where

$$r(A) := |V_A| - |C_A|$$

for all $A \subseteq E$. This is known as the graphic matroid of G [13, Sec. 1.3]. In this section we introduce some new demimatroids defined on the edge set of a graph. We begin with the following.

Matching number

A set of edges in a graph is a matching if no two edges have a common vertex. The matching number of a graph is the maximum size of a matching.

***Proposition* 2.1.** The pair (E, r), with $r(A)$ the matching number of A, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of edges and e an edge not in A. If M is a matching in A, then M is also a matching in $A \cup e$. Hence $r(A) \leq r(A \cup e)$. If M is a matching in $A \cup e$ of size greater than $r(A)+1$, then $M \setminus e$ is a matching of size at least $r(A)+1$ in A, which is not possible. Then $r(A \cup e) \leq r(A)+1$.

In general (E, r) is not a matroid: $G = \{12, 23, 34\}$ (where ij denotes an edge connecting vertices i and j). Set $A = \{12, 23\}$ and $B = \{23, 34\}$. Hence

$$r(A \cup B) + r(A \cap B) = 2 + 1 = 3, \qquad r(A) + r(B) = 1 + 1 = 2.$$

Edge chromatic number

A proper edge coloring of a graph is the assignment of colors to the edges of the graph verifying that adjacent edges receive distinct colors. The edge chromatic number of a graph is the minimum number of colors needed for a proper edge coloring.

***Proposition* 2.2.** The pair (E, r), with $r(A)$ the edge chromatic number of A, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of edges and e an edge not in A. If $A \cup e$ has an edge coloring of minimal size k, then the induced coloring on the edges of A is also an edge coloring. So $r(A) \leq r(A \cup e)$. To the second inequality, just note that if A has an edge coloring of size k, then $A \cup e$ has an edge coloring of size $k+1$, so $r(A \cup e) \leq r(A)+1$.

In general (E, r) is not a matroid: $G = \{12, 13, 23, 34, 35, 45\}$. Set $A = \{12, 23, 34, 45\}$ and $B = \{12, 13, 35, 45\}$. Hence

$$r(A \cup B) + r(A \cap B) = 4 + 1 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Edge list chromatic number

A graph is k-edge-list-colorable if there is a proper coloring of its edges for any list of k colors assigned to each edge. The edge-list-chromatic number of a graph is the minimum k for which the graph is k-edge-list-colorable.

***Proposition* 2.3.** The pair (E, r), with $r(A)$ the edge list chromatic number of A, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of edges and e an edge not in A. Say $r(A \cup e) = t$. Assign to any edge in A a list of t colors. To the edge e assign the same list as any edge in A. For this assignment, there exists an edge list coloring of $A \cup e$, which, when restricted to A, results in an edge coloring of A. Thus $r(A) \leq r(A \cup e)$.

To the second inequality, set $r(A) = k$ and to each edge $f \in A \cup e$ assign a list B_f of $k+1$ colors. Choose one of the colors in B_e, say c. Eliminate this color from all the lists B_f with $f \neq e$. If necessary, eliminate one element from B_f, so that for each $f \neq e$ the new list has k colors. Because $r(A) = k$, there exist a proper edge coloring from the new list for A. Finally, with this proper edge coloring and the assignment of c to the edge e, we obtain a proper edge coloring from the original list. Therefore $r(A \cup e) \leq k+1$.

In general (E, r) is not a matroid: $G = \{12, 23, 34, 14, 45, 56, 67, 47\}$. Set $A = \{12, 23, 34, 47, 67, 56\}$ and $B = \{23, 12, 14, 45, 56, 67\}$. It is well-known that for bipartite graphs the edge chromatic number and the edge list chromatic number coincide [7]. Hence

$$r(A \cup B) + r(A \cap B) = 4 + 2 = 6, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Degree

The degree of a graph is the maximum degree of any of its vertices.

***Proposition* 2.4.** The pair (E, r), with $r(A)$ the degree of A, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of edges and e an edge not in A. In this case it is clear that $r(A) \leq r(A \cup e) \leq r(A)+1$.

In general (E, r) is not a matroid: $G = \{12, 23, 14, 15\}$. Set $A = \{12, 23, 14\}$ and $B = \{12, 23, 15\}$. Hence

$$r(A \cup B) + r(A \cap B) = 3 + 2 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Vertex cover number

A set of vertices of a graph is a vertex cover of the graph if any edge is incident with at least one vertex in the set. The vertex cover number of the graph is the minimum size of a vertex cover.

***Proposition* 2.5.** The pair (E, r), with $r(A)$ the vertex cover number of A, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of edges and e an edge not in A. If B is a vertex cover of $A \cup e$, then it induces a vertex cover of A. So $r(A) \le r(A \cup e)$. To the second inequality, if B is a vertex cover of A, then adding, if necessary, a vertex to B, we obtain a vertex cover of $A \cup e$. So $r(A \cup e) \le r(A) + 1$.

In general (E, r) is not a matroid: $G = \{12, 23, 34\}$. Set $A = \{12, 23\}$ and $B = \{23, 34\}$. Hence

$$r(A \cup B) + r(A \cap B) = 2 + 1 = 3, \qquad r(A) + r(B) = 1 + 1 = 2.$$

Induced independence number

A set of vertices of a graph $G = (V, E)$ is an independent set if no two vertices in the set form an edge of the graph. The independence number of a graph is the maximum size of an independent set. For $A \subseteq E$ denote by $G[A]$ the subgraph of G induced by A.

***Proposition* 2.6.** The pair (E, r), with $r(A)$ the independence number of $G[A]$, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let $\emptyset \subsetneq A \subsetneq A' \subseteq E$. Note that if x and y are nonadjacent vertices in $G[A]$, it is because the edge $\{x, y\}$ does not belong to E. Thus x and y are also nonadjacent vertices in $G[A']$. Then $r(A) \le r(A')$.

Let $A \subset E$ be a nonempty proper subset, and $e \in E \setminus A$. If B is an independent set of $G[A \cup e]$, then B contains at most one vertex of e. Thus $B \setminus x$ is an independent set of $G[A]$. Therefore $r(A \cup e) \le r(A) + 1$.

In general (E, r) is not a matroid: $G = \{14, 24, 34, 15, 25, 35\}$. Set $A = \{14, 24, 25\}$ and $B = \{24, 25, 35\}$. Hence

$$r(A \cup B) + r(A \cap B) = 3 + 2 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Edge cover number

A set of edges in a graph is an edge cover if every vertex of the graph belongs to at least one edge in the set. The edge cover number is the minimum size of an edge cover.

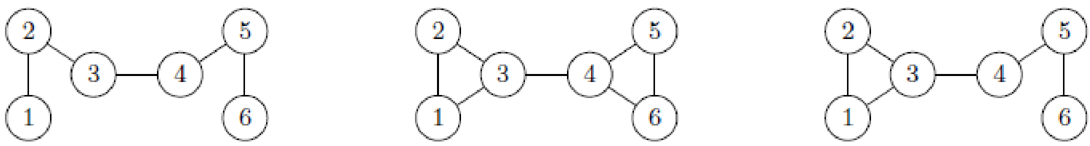

Figure 1: The graphs P_6, $2K_3 + e$, $K_3 + P_3$.

***Proposition* 2.7.** The pair (E, r), with $r(A)$ the edge cover number of A, is a demimatroid if and only if the graph has no P_6, $2K_3 + e$ and $K_3 + P_3$ as subgraphs.

Proof. Assume $P_6 = \{12, 23, 34, 45, 56\}$ is a subgraph of G. Thus $r(\{12, 23, 45, 56\}) = 4$ and $r(\{12, 23, 45, 56\} \cup \{34\}) = 3$, which is not possible. The other two cases can be treated similarly.

Conversely, clearly $r(\emptyset) = 0$. Let A be a nonempty proper subset of edges and e an edge not in A. Choose $B \subseteq A \cup e$ to be a minimum edge cover of $A \cup e$. If $e \notin B$, then B is an edge cover of A, and in this case $r(A) \le r(A \cup e)$. So we may assume $e \in B$. If e is disjoint from A, then $B \setminus e$ is an edge cover of A, so $r(A) \le r(A \cup e) - 1$. If $e = uv$ is a pendant edge of A, and no edge of $B \setminus e$ covers u, then $(B \setminus e) \cup xu$, with $xu \in A \setminus B$, is an edge cover of A. So, in this case, $r(A) \le r(A \cup e)$.

Consider now the case that both ends of e belong to A. Say $e = uv$. If there is an edge xu in B, then, by the minimality of B, v does not belong to any edge in $B \setminus e$. Let $yv \in A \setminus B$. Then $(B \setminus e) \cup yv$ is an edge cover of A of size $r(A \cup e)$. So $r(A) \le r(A \cup e)$. Hence we may assume that e is the only edge in B that contains u or v.

Consider an edge $xu \in A$. There is an edge $xy \in B$. If $yv \in A$, then we delete two edges of B and add two new edges to obtain an edge cover of A with the same size as B, so $r(A) \leq r(A \cup e)$. So xy belongs to a P_2 or a triangle adjacent to u. Analogously, we have a P_2 or a triangle adjacent to v. This implies that the graph has as a subgraph one of the forbidden graphs, which is not possible. Then $r(A) \leq r(A \cup e)$.

Let us now prove that $r(A \cup e) \leq r(A) + 1$. Fix an edge cover B of A. If e is disjoint of A or e is a pendant edge of A, then $B \cup e$ is an edge cover of $A \cup e$, so $r(A \cup e) \leq r(A) + 1$. In the other case, B is an edge cover of $A \cup e$, and then $r(A \cup e) \leq r(A) \leq r(A) + 1$.

Demimatroids on the vertices of a graph

Let $G = (V, E)$ be a graph, and for a subset A of V denote by $G[A]$ the induced subgraph determined by A.

Chromatic number

A proper k-coloring of the vertices of a graph is an assignment of at most k colors to the vertices verifying that adjacent vertices receive distinct colors. The chromatic number of the graph is the minimum k such that the graph has a k-coloring.

***Proposition* 3.1.** The pair (V, r), with $r(A)$ the chromatic number of $G[A]$, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of vertices and v a vertex not in A. If $L : V_{A\cup v} \to \{1, \dots, k\}$ is a proper coloring of $G[A \cup v]$, then L induces a proper coloring of $G[A]$. Hence $r(A) \leq r(A \cup v)$. If L is a proper coloring of $G[A]$ with $r(A)$ colors, adding one more color, if necessary, we obtain a proper coloring of $G[A \cup v]$. Therefore $r(A \cup v) \leq r(A) + 1$.

In general (V, r) is not a matroid: $G = \{12, 23, 13, 34\}$. Set $A = \{1, 3, 4\}$ and $B = \{2, 3, 4\}$. Hence

$$r(A \cup B) + r(A \cap B) = 3 + 2 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Matching number

***Proposition* 3.2.** The pair (V, r), with $r(A)$ the matching number of $G[A]$, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of vertices and v a vertex not in A. If B is a matching in $G[A]$, then B is also a matching in $G[A \cup v]$. Hence $r(A) \leq r(A \cup v)$. If B is a matching in $G[A \cup v]$ of size greater than $r(A) + 1$, then there is at most one edge in B containing v. Therefore $G[A]$ has a matching of size at least $r(A) + 1$, which is not possible. Then $r(A \cup v) \leq r(A) + 1$.

In general (V, r) is not a matroid: $G = \{12, 23, 34, 14\}$. Set $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Hence

$$r(A \cup B) + r(A \cap B) = 2 + 1 = 3, \qquad r(A) + r(B) = 1 + 1 = 2.$$

Independence number

***Proposition* 3.3.** The pair (V, r), with $r(A)$ the independence number of $G[A]$, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of vertices and v a vertex not in A. If B is an independent subset of $G[A]$, then B is also an independent set in $G[A \cup v]$. Hence $r(A) \leq r(A \cup v)$. If B is an independent set of $G[A \cup v]$ of size greater than $r(A) + 1$, then necessarily A would have an independent set of size at least $r(A) + 1$, which is not possible. Therefore $r(A \cup v) \leq r(A) + 1$.

In general (V, r) is not a matroid: $G = \{13, 23, 34, 45\}$. Set $A = \{1, 3, 4, 5\}$ and $B = \{2, 3, 4, 5\}$. Hence

$$r(A \cup B) + r(A \cap B) = 3 + 2 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

Clique number

A set of vertices of a graph is a clique if any two vertices are adjacent. The clique number is the maximum size of a clique.

***Proposition* 3.4.** The pair (V, r), with $r(A)$ the clique number of $G[A]$, is a demimatroid.

Proof. Trivially $r(\emptyset) = 0$. Let A be a nonempty proper set of vertices and v a vertex not in A. If B is a clique of $G[A]$, then B is also a clique of $G[A \cup v]$. Hence $r(A) \leq r(A \cup v)$. If B is a clique of $G[A \cup v]$ of size greater than $r(A) + 1$, then necessarily $G[A]$ would have a clique of size at least $r(A) + 1$, which is not possible. Therefore $r(A \cup v) \leq r(A) + 1$.

In general (V, r) is not a matroid: $G = \{12, 13, 23, 34\}$. Set $A = \{1, 3, 4\}$ and $B = \{2, 3, 4\}$. Hence

$$r(A \cup B) + r(A \cap B) = 3 + 2 = 5, \qquad r(A) + r(B) = 2 + 2 = 4.$$

A demimatroid characterization of some families of graphs

Cographs

A cograph is a graph defined recursively as: (i) K_1 is a cograph, (ii) if G is a cograph, then so is its graph complement, and (iii) if G and H are cographs, then so is their graph union $G \cup H$.

The class of cograph graphs has been widely studied. There are many characterizations of them, e.g. for a graph G are equivalent: (a) G is cograph; (b) G is a graph all of whose connected induced subgraphs have diameter at most 2; (c) G contains a dominating induced C_4 or a dominating vertex [14]; (d) G is P_4-free. In this section we give a new characterization in terms of demimatroids.

Let $G = (V, E)$ be a graph. For a subset A of edges in E, define

$$I_G(A) := \{B \subseteq E : B \text{ is a connected induced subgraph of } G \text{ and } B \subseteq A\}.$$

Define also

$$r(A) := |V_A| - |C_A|$$

and

$$s(A) := \max\{r(B) : B \in I_G(A)\}.$$

***Proposition* 4.1.** The pair (E, s) is a demimatroid if and only if G is P_4-free.

Proof. If $P_4 = \{12, 23, 34\}$ is an induced subgraph of G, then $s(\{12, 34\}) = 1$ and $s(\{12, 34\} \cup \{23\}) = 3$, which is not possible, hence G must be P_4-free.

Conversely, trivially $s(\emptyset) = 0$. Let A be a nonempty proper subset of E and $e = uv \in E \setminus A$. If $B \in I_G(A)$, then $B \in I_G(A \cup e)$, hence $s(A) \leq s(A \cup e)$. Let $B \in I_G(A \cup e)$ with $s(A \cup e) = r(B)$. If $B \subseteq A$, then $s(A \cup e) \leq s(A) + 1$. Thus assume $e \in B$.

If e is a pendant edge of B, then $C = B \setminus e \in I_G(A)$ and $r(C \cup e) \leq r(C) + 1 \leq s(A) + 1$.

If e is not a pendant edge and $C = B \setminus e$ is connected, write $e = uv$. The argument in the paper shows that, using the P_4-free condition, one can delete an endpoint of e and obtain a connected induced subgraph in $I_G(A)$ whose rank is one less. Therefore $s(A \cup e) \leq s(A) + 1$.

If e joins two components of C, the existence of the corresponding edges in the two components would induce a P_4, a contradiction.

$(P_6, 2K_3 + e, K_3 + P_3)$-free graphs

Let $G = (V, E)$ be a graph. Let $M = (E, r)$ be the matching demimatroid on the edges of G, i.e. for a subset A of edges in E, $r(A)$ equals the matching number of A. Define

$$s(A) := \max\{r(B) : B \in I_G(A)\}.$$

***Theorem* 4.1.** The pair (E, s) is a demimatroid if and only if G is $(P_6, 2K_3 + e, K_3 + P_3)$-free.

Proof. Assume $\{12, 23, 34, 45, 56\}$ is an induced subgraph of G. Thus $s(\{12, 23, 45, 56\}) = 1$ and $s(\{12, 23, 45, 56\} \cup \{34\}) = 3$, which is not possible. So G is P_6-free. The other two cases may be treated similarly.

Conversely, clearly $s(\emptyset) = 0$. Let A be a nonempty proper subset of E and $e \in E \setminus A$. Choose $B \in I_G(A)$ with $s(A) = r(B)$. Thus $B \in I_G(A \cup e)$ and $s(A) \leq s(A \cup e)$. Fix $B \in I_G(A \cup e)$ with $s(A \cup e) = r(B)$.

If $B \subseteq A$, then $s(A \cup e) \leq s(A) + 1$. Otherwise $e \in B$. If e is a pendant edge of B, then $C = B \setminus e \in I_G(A)$ and, since (E, r) is a demimatroid,

$$s(A \cup e) = r(B) = r(C \cup e) \leq r(C) + 1 \leq s(A) + 1.$$

If e is not a pendant edge and $C = B \setminus e$ is connected, let M be a maximum matching in B. Necessarily there is an edge in M containing one of the vertices u or v, say u. Set $D = B \setminus u$. Then $s(A) \geq r(D) \geq r(B) - 1$, and hence

$$1 + s(A) \geq r(B) = s(A \cup e).$$

If C has two components, the forbidden induced subgraphs arise unless the previous case applies. This completes the proof.

Triangle-free graphs

Let $G = (V, E)$ be a graph. Let r be the demimatroid on E given by edge chromatic number, i.e. for a subset A of edges in E, $r(A)$ equals the edge chromatic number of A. Define

$$s(A) := \max\{r(B) : B \in I_G(A)\}.$$

***Proposition* 4.2.** The pair (E, s) is a demimatroid if and only if G is triangle-free.

Proof. Say $\{12, 13, 23\}$ is an induced triangle of G. Then $s(\{12, 13\}) = 1$ and $s(\{12, 13\} \cup \{23\}) = 3$, which is not possible.

Conversely, choose $B \in I_G(A)$ with $s(A) = r(B)$. If $B \subseteq A$ there is nothing to prove. Otherwise $e \in B$. If e is pendant, deleting it gives the desired bound. If e is not pendant and $B \setminus e$ is connected, an edge coloring argument allows deletion of an endpoint without increasing the value. If e joins two components, taking the component with larger edge chromatic number gives $s(A \cup e) \leq s(A) + 1$.

Degree

Let $G = (V, E)$ be a graph. Let r be the demimatroid on E given by the degree of the graph, i.e. for a subset A of edges in E, $r(A)$ equals the maximum degree of an edge of A. Define

$$s(A) := \max\{r(B) : B \in I_G(A)\}.$$

***Proposition* 4.3.** The pair (E, s) is a demimatroid.

Proof. Clearly $s(\emptyset) = 0$. It is also clear that $s(A) \leq s(A \cup e)$ for any subset of edges. Let $B \in I_G(A \cup e)$ with $s(A \cup e) = r(B)$. If $B \subseteq A$, then $s(A \cup e) \leq s(A) + 1$. If e is a pendant edge of B, deleting it either leaves the maximum degree unchanged or decreases it by one. If e is not a pendant edge and $B \setminus e$ is connected, delete the endpoint that does not attain the largest degree; if one endpoint does attain it, delete the other endpoint. If $B \setminus e$ has two components, keep the larger component containing e and proceed as above.

***Proposition* 4.4.** Demimatroids r and s above coincide if and only if the graph is triangle-free.

Proof. If G contains an induced triangle, taking A as two of the edges gives $r(A) = 2$ while $s(A) = 1$. Conversely, assume that G is triangle-free. Let x be the vertex in A with the largest degree, adjacent to $x_1, \ldots, x_d$. The subgraph induced by $x, x_1, \ldots, x_d$ has degree d and is connected. Since G is triangle-free, this induced subgraph is contained in A. Hence $s(A) = r(A)$ for all A.

Tutte polynomial

The Tutte polynomial of a demimatroid $M = (E, r)$ is defined as

$$T_M(x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

References for the Tutte polynomial of a matroid are [6,13,16]. For the case of demimatroids, one may consult [11].

Let G be a graph with n vertices and let m_k denote the number of k-edge matchings. The matching polynomial of G is defined as

$$m(G; x) := \sum_{k \geq 0} m_k x^k.$$

It is well known that the matching polynomial is not an evaluation of the Tutte polynomial associated to the graphic matroid; however, if $T(G; x, y)$ is the Tutte polynomial associated to the matching demimatroid of the graph $G = (V, E)$, one can check that

$$m(G; x) = x^{r(E)} T\left(\frac{1}{x} + 1, 1\right).$$

It is worth mentioning that if G and H are disjoint graphs, we have

$$m(G \sqcup H; x) = m(G; x) + m(H; x).$$

Thus, from this property, one may verify directly that, for the matching demimatroid,

$$T(G \sqcup H; x, y) = T(G; x, y)T(H; x, y).$$

In a matroid $M = (E, r)$, an independent set is a subset A of E such that $r(A) = |A|$. The maximal ones of these are called the bases, and an important property of matroids is that all bases are equicardinal, which is precisely the rank of the matroid. If $T(M; x, y)$ is the Tutte polynomial of M, then it is well-known that $T(M; 1, 1)$ counts the number of bases of M.

When M is only a demimatroid, but not a matroid, the situation is entirely different; we may define independent sets and bases in the same fashion, but could happen in this case that $T(M; 1, 1) = 0$.

***Example* 5.1.** Consider the graph G given in Figure 2. Let M be the demimatroid on the edges of G determined by the edge chromatic number (Proposition 2.2). We have that $r(M) = 4$, however $T(1, 1) = 0$, that is, there is not bases with four elements. There are six 3-sets A for which $r(A) = 3$ (the two triangles and those determined by the vertices of degree 3). Each of these are bases of the demimatroid. There is a 2-set B for which $r(B) = 2$ (the one determined by the only vertex of degree two). This 2-set is also a base of the demimatroid.

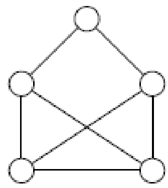

Figure 2: A graph without “bases” of size 4

k-Demimatroids

A possible generalization of the notion of a demimatroid is that of a k-demimatroid. However, this has not been sufficiently studied yet. In this section we give a pair of examples of 2-demimatroids defined on the edge set of a graph. See [8] for a different point of view.

A k-demimatroid is a pair $M = (E, r)$, where E is a finite set and $r : 2^E \to \mathbb{Z}_+$ is a function such that:

(K1) $r(\emptyset) = 0$;

(K2) If $A \subseteq E$ and $e \in E$, then

$$r(A) \leq r(A \cup \{e\}) \leq r(A) + k.$$

Chromatic number

Let $G = (V, E)$ be a graph, $A \subseteq E$ and V_A the vertices of A, i.e. those vertices which are in at least one edge in A. A proper vertex coloring of A is an assignment of colors to the vertices of A verifying that any two adjacent vertices receive distinct colors. The vertex chromatic number of A is the minimum number of colors in a proper vertex coloring of A.

***Proposition* 6.1.** The pair (E, r), with $r(A)$ the vertex chromatic number of A, is a 2-demimatroid.

Proof. Clearly $r(\emptyset) = 0$. Let A be a nonempty proper subset of edges and e an edge not in A. Let $L : V_A \to \{1, \ldots, k\}$ be a proper vertex coloring of $A \cup e$. Since L induces a vertex coloring of A, we get $r(A) \leq r(A \cup e)$. To the second inequality, let L be a vertex coloring of A with $r(A)$ colors. Say $e = uv$. If e is disjoint or pendant with respect to A, then $r(A \cup e) = r(A)$. In the other case, reassigning to v a new color, we obtain a vertex coloring of $A \cup e$ with $r(A) + 1$ colors. Then $r(A \cup e) \leq r(A) + 1$.

Number of vertices

Let $G = (V, E)$ be a graph. For a subset A of edges, denote by

$$r(A) := |V_A|$$

its number of vertices.

***Proposition* 6.2.** The pair (E, r), with $r(A)$ the number of vertices of A, is a 2-demimatroid.

Proof. Clearly $r(\emptyset) = 0$. Let A be a nonempty proper subset of edges and e an edge not in A. In this situation, condition (K2) is immediate.

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