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Existence and Uniqueness of Solution to Dirichlet Problem for General Linear Elliptic Equation of the Second Order in Perforated Domain

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Abstract

In this paper, we consider the homogeneous Dirichlet problem for a general second-order linear elliptic equation with limit integrability exponents of the lower-order coefficients in a bounded perforated domain of $\mathbb{R}^n$. Existence and uniqueness theorems are proved for this problem in the case $n > 2$.

Keywords:

Perforated Domain, Linear Elliptic Equation, Unique Solvability, Sobolev Limit Exponent

Introduction

This paper discusses the unique solvability of general second-order elliptic equations in perforated domains in the case where the coefficients of the lower-order terms satisfy the limiting Sobolev integrability exponents.

Perforated materials and porous media are complex objects for mathematical modeling. In recent decades, many works have appeared related to homogenization of processes in such media (see, for example, recent papers [1–6]). The question remains about the existence and uniqueness of solutions to boundary value problems in such media for general equations. This paper closes this question for linear second-order elliptic equations of general form.

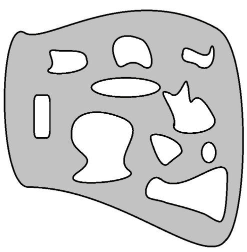

Figure 1: Perforated domain.

Let $D_\varepsilon \subset \mathbb{R}^n$, where $n > 2$, be a bounded perforated domain with arbitrary distribution of cavities. We assume that the boundary of the domain satisfies the Lipschitz condition (see Fig. 1). The construction can be carried out in more detail as follows. Consider a fixed bounded domain D. Let B be a union of sets in the space $\mathbb{R}^n$. We apply homothety to

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this union, reducing it by $N \in \mathbb{N}$ times. Let us denote $\varepsilon = \frac{1}{N}$, then we get the domain B_ε. Next, we consider the domain $D_\varepsilon = D \setminus B_\varepsilon$.

In this paper, we consider an operator

$$\mathcal{L}u := \operatorname{div}(A(x)\nabla u) + \operatorname{div}(a(x)\,u) + b(x)\cdot\nabla u + c(x)u,$$

considered in a perforated bounded domain D_ε. Here $A(x) = \{A_{ij}(x)\}$ is a measurable symmetric matrix satisfying the uniform ellipticity condition

$$\alpha|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \le \alpha^{-1}|\xi|^2 \quad \text{for a.a. } x \in D \text{ and all } \xi \in \mathbb{R}^n,\ \alpha = const > 0. \tag{1}$$

Vector functions $a(x) = (a^1(x), \dots, a^n(x))$, $b(x) = (b^1(x), \dots, b^n(x))$, and the coefficient $c(x)$ satisfy the following conditions:

$$a_j(x) \in L_n(D), \qquad b_j(x) \in L_n(D), \quad j = 1, \dots n, \qquad c(x) \in L_{\frac{n}{2}}(D). \tag{2}$$

We introduce the Sobolev space of functions $\overset{\circ}{W}{}_2^1(D_\varepsilon)$ as the completion of infinitely differentiable functions in a bounded domain D_ε vanishing in a vicinity of the boundary of the domain, with respect to the norm

$$\| u \|_{\overset{\circ}{W}{}_2^1(D_\varepsilon)} = \left(\int\limits_{D_\varepsilon} |\nabla v|^2\, dx\right)^{1/2}.$$

This expression is a norm by virtue of the well-known Friedrichs inequality. In this paper, we study the Dirichlet problem

$$\mathcal{L}u = \operatorname{div} f, \qquad f = (f_1, \dots f_n), f_j \in L_2(D),\ u \in \overset{\circ}{W}{}_2^1(D_\varepsilon), \tag{3}$$

in a bounded domain $D_\varepsilon \subset \mathbb{R}^n$, $n > 2$.

The solution to problem (3) is understood to be a function $u \in \overset{\circ}{W}{}_2^1(D_\varepsilon)$ for which the integral identity

$$\int\limits_{D_\varepsilon} (A(x)\nabla u \cdot \nabla\varphi)\, dx + \int\limits_{D_\varepsilon} (a(x)\cdot\nabla\varphi)u\, dx - \int\limits_{D_\varepsilon} (b(x)\cdot\nabla u)\varphi\, dx - \int\limits_{D_\varepsilon} c(x)u\varphi\, dx = \int\limits_{D_\varepsilon} (f\cdot\nabla\varphi)\, dx \tag{4}$$

holds for all test functions $\varphi \in \overset{\circ}{W}{}_2^1(D_\varepsilon)$. When considering the question of the unique solvability of the Dirichlet problem (3), we need an additional condition of the form

$$\int\limits_{D_\varepsilon} (c(x)\varphi - (a(x)\cdot\nabla\varphi))\, dx \leqslant 0, \tag{5}$$

satisfied on non-negative functions $\varphi \in \overset{\circ}{W}{}_2^1(D_\varepsilon)$.

In proving the uniqueness of solutions to problem (3) under conditions (1), (2) and (5), we rely on the estimate from Theorem 1 (see [7]) of the form

$$\left(\int\limits_{D_\varepsilon} |\nabla u|^2 dx\right)^{\frac{1}{2}} \leqslant C(\alpha, n, \|a\|_{L_n(D)}, \|b\|_{L_n(D)}, D_\varepsilon)\left(\int\limits_{D_\varepsilon} |f|^2\, dx\right)^{\frac{1}{2}}, \tag{6}$$

in which the constant $C(\alpha, n, \|a\|_{L_n(D)}, \|b\|_{L_n(D)}, D)$ from (6) is represented explicitly.

The following assertion holds.

Theorem 1. *If conditions* (1), (2), (5) *are satisfied and* $f \in \big(L_2(D)\big)^n$, *then the Dirichlet problem* (3) *is uniquely solvable and the solution satisfies the estimate* (6).

We will use the following representation of the lower-order coefficients $a, b \in (L_n(D_\varepsilon))^n$ and $c \in (L_{\frac{n}{2}}(D_\varepsilon))^n$ of the equation under consideration

$$\begin{aligned} a = \breve{a} + \widehat{a}, &\quad \breve{a} \in (L_\infty(D))^n, \quad \widehat{a} \in (L_n(D))^n, \quad \| \widehat{a} \|_{L_n(D)} \leqslant \theta_1, \\ b = \breve{b} + \widehat{b}, &\quad \breve{b} \in (L_\infty(D))^n, \quad \widehat{b} \in (L_n(D))^n, \quad \| \widehat{b} \|_{L_n(D)} \leqslant \theta_1, \\ c = \breve{c} + \widehat{c}, &\quad \breve{c} \in (L_\infty(D))^n, \quad \widehat{c} \in (L_{\frac{n}{2}}(D))^n, \quad \| \widehat{c} \|_{L_{\frac{n}{2}}(D)} \leqslant \theta_2 \end{aligned} \tag{7}$$

with sufficiently small constants $\theta_1, \theta_2 \in (0, 1)$, which are determined in further analysis.

Unique solvability of the stated problem

In this section, the unique solvability of the Dirichlet problem (3) is proved in an arbitrary bounded perforated domain D_ε.

The proof of the existence of a solution to the Dirichlet problem (3) is based on auxiliary assertions, which are established below. First, we need estimates of the bilinear form associated with the operator $\mathcal{L}$ from (3), which has the form

$$\mathcal{L}(u,v) = \int\limits_{D_\varepsilon} (A\nabla u \cdot \nabla v)\, dx + \int\limits_{D_\varepsilon} (a \cdot \nabla v) u\, dx - \int\limits_{D_\varepsilon} (b \cdot \nabla u) v\, dx - \int\limits_{D_\varepsilon} cuv\, dx \tag{8}$$

and is defined on functions $u, v \in \overset{\circ}{W}{}^1_2(D_\varepsilon)$.

Lemma 1. *If the coefficients of the operator $\mathcal{L}$ from (3) satisfy conditions (1) and (2), then*

$$\mathcal{L}(u,u) \geq \frac{\alpha}{2} \int\limits_{D_\varepsilon} |\nabla u|^2\, dx - C \int\limits_{D_\varepsilon} u^2\, dx, \tag{9}$$

where $C = C(\alpha, a, b, c, n)$.

Proof. By the uniform ellipticity condition (1), we have

$$\mathcal{L}(u,u) \geq \alpha \int\limits_{D_\varepsilon} |\nabla u|^2\, dx - \Big| \int\limits_{D_\varepsilon} (a \cdot \nabla u) u\, dx \Big| - \Big| \int\limits_{D_\varepsilon} (b \cdot \nabla u) u\, dx \Big| - \Big| \int\limits_{D_\varepsilon} cu^2\, dx \Big|. \tag{10}$$

First, we estimate the second and third terms on the right-hand side of identity (8). Both terms are estimated identically, and we will only give the estimate for the first of them. From (7) we have

$$\int\limits_{D_\varepsilon} (a \cdot \nabla u) u\, dx = \int\limits_{D_\varepsilon} (\breve{a} \cdot \nabla u) u\, dx + \int\limits_{D_\varepsilon} (\widehat{a} \cdot \nabla u) u\, dx. \tag{11}$$

For the first integral on the right-hand side in (11) we have

$$\Big| \int\limits_{D_\varepsilon} (\breve{a} \cdot \nabla u) u\, dx \Big| \leqslant C \|\breve{a}\|_{L_\infty(D)} \int\limits_{D_\varepsilon} |\nabla u|\, |u|\, dx$$

and from Cauchy inequality with $\delta > 0$ we find

$$\Big| \int\limits_{D_\varepsilon} (\breve{a} \cdot \nabla u) u\, dx \Big| \leqslant \delta \int\limits_{D_\varepsilon} |\nabla u|^2\, dx + C(\delta) \|\breve{a}\|^2_{L_\infty(D)} \int\limits_{D_\varepsilon} u^2\, dx. \tag{12}$$

We estimate the second integral on the right-hand side (11) using Hölder inequality, by virtue of which

$$\Big| \int\limits_{D_\varepsilon} (\widehat{a} \cdot \nabla u) u\, dx \Big| \leqslant \|\widehat{a}\|_{L_n(D_\varepsilon)} \Big(\int\limits_{D_\varepsilon} |\nabla u|^2\, dx \Big)^{1/2} \Big(\int\limits_{D_\varepsilon} |u|^{\frac{2n}{n-2}}\, dx \Big)^{\frac{n-2}{2n}}. \tag{13}$$

By Sobolev inequality

$$\Big(\int\limits_{D_\varepsilon} |u|^{\frac{2n}{n-2}}\, dx \Big)^{\frac{n-2}{2n}} \leqslant C(n) \Big(\int\limits_{D_\varepsilon} |\nabla u|^2\, dx \Big)^{1/2}. \tag{14}$$

From here and from (13) taking into account (7), we get

$$\Big| \int\limits_{D_\varepsilon} (\widehat{a} \cdot \nabla u) u\, dx \Big| \leqslant C(n)\theta_1 \int\limits_{D_\varepsilon} |\nabla u|^2\, dx. \tag{15}$$

From (12) and (15) we find

$$\Big| \int\limits_{D_\varepsilon} (a \cdot \nabla u) u\, dx \Big| \leqslant \delta \int\limits_{D_\varepsilon} |\nabla u|^2\, dx + C(\delta) \|\breve{a}\|^2_{L_\infty(D)} \int\limits_{D_\varepsilon} u^2\, dx + C(n)\theta_1 \int\limits_{D_\varepsilon} |\nabla u|^2\, dx. \tag{16}$$

Similarly for the third term

$$\Big| \int\limits_{D_\varepsilon} (b \cdot \nabla u) u\, dx \Big| \leqslant \delta \int\limits_{D_\varepsilon} |\nabla u|^2\, dx + C(\delta) \|\breve{b}\|^2_{L_\infty(D)} \int\limits_{D_\varepsilon} u^2\, dx + C(n)\theta_1 \int\limits_{D_\varepsilon} |\nabla u|^2\, dx. \tag{17}$$

Now let us estimate the last term in (10). By (7), we have

$$\int\limits_{D_\varepsilon} cu^2\,dx = \int\limits_{D_\varepsilon} \check{c}u^2\,dx + \int\limits_{D_\varepsilon} \widehat{c}u^2\,dx. \tag{18}$$

It is clear that

$$\left|\int\limits_{D_\varepsilon} \check{c}u^2\,dx\right| \leqslant \|\check{c}\|_{L_\infty(D)} \int\limits_{D_\varepsilon} u^2\,dx. \tag{19}$$

We estimate the second integral on the right-hand side (18) using the Hölder and Sobolev inequality (14), by virtue of which

$$\left|\int\limits_{D_\varepsilon} \widehat{c}u^2\,dx\right| \leqslant \|\widehat{c}\|_{L_{\frac{n}{2}}(D)} \Bigg(\int\limits_{D_\varepsilon} |u|^{\frac{2n}{n-2}}\,dx\Bigg)^{\frac{n-2}{n}} \leqslant C(n)\theta_2 \int\limits_{D_\varepsilon} |\nabla u|^2\,dx. \tag{20}$$

Finally, after appropriately choosing δ, θ_1, and θ_2 from (10), (16), (17), (19), and (20), we arrive at the estimate (9). Lemma is proved.

Lemma 2. *If the coefficients of the operator $\mathcal{L}$ from (3) satisfy conditions (1) and (2), then for a fixed $u \in \overset{\circ}{W}{}^1_2(D_\varepsilon)$ the mapping $v \mapsto \mathcal{L}(u,v)$, where the form $\mathcal{L}(u,v)$ is defined in (8), is a bounded linear functional on $\overset{\circ}{W}{}^1_2(D_\varepsilon)$ and the estimate*

$$|\mathcal{L}(u,v)| \leqslant C\|u\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \|v\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \tag{21}$$

holds true, where $C = C(\alpha, a, b, c, n)$.

Proof. By the uniform ellipticity condition (1), we have

$$\left|\int\limits_{D_\varepsilon} (A\nabla u\cdot\nabla v)\,dx\right| \leqslant \alpha^{-1}\|u\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \|v\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)}. \tag{22}$$

Based on conditions (2) and (5), we estimate the second and third terms of the form (8) using Hölder inequality:

$$\begin{aligned} \left|\int\limits_{D_\varepsilon} (a\cdot\nabla v)u\,dx\right| &\leqslant \|v\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \int\limits_{D_\varepsilon} |a|^2u^2\,dx \leqslant \\ &\leqslant \|v\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \Bigg(\int\limits_{D} |a|^n\,dx\Bigg)^{2/n} \Bigg(\int\limits_{D_\varepsilon} |u|^{\frac{2n}{n-2}}\,dx\Bigg)^{\frac{n-2}{2n}} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \left|\int\limits_{D_\varepsilon} (b\cdot\nabla u)v\,dx\right| &\leqslant \|u\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \int\limits_{D_\varepsilon} |b|^2v^2\,dx \leqslant \\ &\leqslant \|u\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \Bigg(\int\limits_{D} |b|^n\,dx\Bigg)^{2/n} \Bigg(\int\limits_{D_\varepsilon} |v|^{\frac{2n}{n-2}}\,dx\Bigg)^{\frac{n-2}{2n}}. \end{aligned} \tag{24}$$

Proceeding similarly with the fourth term, we obtain

$$\left|\int\limits_{D_\varepsilon} cuv\,dx\right| \leqslant \Bigg(\int\limits_{D_\varepsilon} |c|^{\frac{n}{2}}\,dx\Bigg)^{2/n} \Bigg(\int\limits_{D_\varepsilon} |u|^{\frac{2n}{n-2}}\,dx\Bigg)^{\frac{n-2}{2n}} \Bigg(\int\limits_{D_\varepsilon} |v|^{\frac{2n}{n-2}}\,dx\Bigg)^{\frac{n-2}{2n}}. \tag{25}$$

By the Sobolev embedding theorem (14), from (22), (23), (24), and (25) we arrive at (21). Lemma is proved.

Proof of Theorem 1.

For $\sigma > \sigma_0 > 0$, we define the operator $\mathcal{L}_\sigma$ by the formula $\mathcal{L}_\sigma u = \mathcal{L}u - \sigma u$. From the estimate (9) of Lemma 1 it follows that the form corresponding to this operator

$$\mathcal{L}_\sigma(u,u) = \int\limits_{D_\varepsilon} (A\nabla u\cdot\nabla u)\,dx + \int\limits_{D_\varepsilon} (a\cdot\nabla u)u\,dx - \int\limits_{D_\varepsilon} (b\cdot\nabla u)u\,dx - \int\limits_{D_\varepsilon} cu^2\,dx + \sigma\int\limits_{D_\varepsilon} u^2\,dx$$

is coercive for sufficiently large σ , that is,

$$\mathcal{L}_{\sigma_0}(u,u) \geq \alpha/2 \int\limits_{D_\varepsilon} |\nabla u|^2\,dx.$$

Here $\sigma_0 = \sigma_0(\alpha, a, b, c, n)$.

Note that with this choice $\sigma = \sigma_0$ the bilinear form

$$\mathcal{L}_{\sigma_0}(u,v) = \int\limits_{D_\varepsilon} (A\nabla u \cdot \nabla v)\, dx + \int\limits_{D_\varepsilon} (a \cdot \nabla v) u\, dx - \int\limits_{D_\varepsilon} (b \cdot \nabla u) v\, dx - \int\limits_{D_\varepsilon} cuv\, dx + \sigma_0 \int\limits_{D_\varepsilon} uv\, dx \tag{26}$$

is bounded. This follows from the estimate (21) and the estimate

$$\int\limits_{D_\varepsilon} uv\, dx \leqslant \|u\|_{L_2(D_\varepsilon)} \|v\|_{L_2(D_\varepsilon)} \leqslant C(n,D) \|u\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)} \|v\|_{\overset{\circ}{W}{}^1_2(D_\varepsilon)},$$

following from the Friedrichs inequality. Thus, the operator $\mathcal{L}_{\sigma_0}$ is bounded and coercive in the Hilbert space $H = \overset{\circ}{W}{}^1_2 (D_\varepsilon)$.

Let H^{-1} be the dual space of H. We define the operator $\mathfrak{I}_u : H \to H^{-1}$ by the formula

$$\mathfrak{I} = \mathfrak{I}_u v = \int\limits_{D_\varepsilon} uv\, dx, \quad v \in H. \tag{27}$$

We show that the mapping $\mathfrak{I}_u$ is compact. To this end, we note that the mapping $\mathfrak{I}_u$ can be represented as a composition

$$\mathfrak{I}_u = \mathfrak{I}_1 \circ \mathfrak{I}_2. \tag{28}$$

Here $\mathfrak{I}_2 : H \to L_2(D_\varepsilon)$ is a natural embedding. By the Kondrashov–Sobolev embedding compactness theorem [8, Theorem 7.22] the operator $\mathfrak{I}_2$ is compact, and the mapping $\mathfrak{I}_1 : L_2(D_\varepsilon) \to H^{-1}$ is defined by the formulae (27) and (28). Since the operator $\mathfrak{I}_1$ is continuous and the operator $\mathfrak{I}_2$ is compact, it follows that the operator $\mathfrak{I}$ is compact.

The equation $\mathcal{L}u = l$ for $u \in H$, where the l is a functional in the space H^{-1} dual to H, is equivalent to the equation $\mathcal{L}_{\sigma_0} u + \sigma_0 \mathfrak{I}_u u = l$. By the Lax–Milgram lemma (see [9]), the inverse operator $\mathcal{L}_{\sigma_0}^{-1}$ defines a continuous one-to-one mapping of H^{-1} onto H. Therefore, applying this operator to the previous equation, we obtain the equivalent equation

$$u + \sigma_0 \mathcal{L}_{\sigma_0}^{-1} \mathfrak{I}_u u = \mathcal{L}_{\sigma_0}^{-1} l. \tag{29}$$

The mapping $T = -\sigma_0 \mathcal{L}_{\sigma_0}^{-1} \mathfrak{I}_u$ is also compact due to the compactness of $\mathfrak{I}$. Therefore, by the Fredholm alternative (see, for example, Theorem 5.11 [8, 5.9], the existence of a function $u \in H$ satisfying equation (29) is a consequence of the uniqueness in H of the trivial solution to the equation $\mathcal{L}u = 0$.

The unique solvability of the Dirichlet problem (3) follows from the a priori estimate (6), by virtue of which the solution is unique. The theorem is proved.

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