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Some Properties of Translated Functions and Their Inverses

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Abstract

In this study, we address some interesting properties among the set of six functions, a function p, its translated versions $q_-(t) = p(\tau - t)$ and $q_+(t) = p(\tau + t)$, and their inverses p^{-1}, q_-^{-1}, and q_+^{-1}. Specifically, by introducing the notion of alternating composition of two functions, several equalities are derived for the alternating compositions among the six functions. Some properties, examples, and applications of the equalities are also discussed briefly.

Keywords:

Alternating composition; Equality; Inverse; Translated functions

Introduction

In many areas of applications including signal processing and communications, the notions of translated and reversed functions are frequently employed [1,6]. For example, translated and reversed signals are required in the evaluation of the convolution of two signals [4,12]. In addition, as one of the key applications in signal detection theory, the construction of matched filters requires translated and reversed signals [8,10] requires translated and reversed signals.

Our main focus in this study is on the properties among a function, its translated versions, and their inverses. For the convenience of the descriptions, let us first define and recollect some notations and notions. For two functions f and g, define the alternating composition $\{f, g\}_k$ of order k by

$$\{f, g\}_k (t) = \begin{cases} t, & k = 0, \\ f(t), & k = 1, \\ f\left(g\left(f\left(\cdots g(t) \cdots\right)\right)\right), & k = 2, 4, \ldots, \\ f\left(g\left(f\left(\cdots f(t) \cdots\right)\right)\right), & k = 3, 5, \ldots. \end{cases} \tag{1.1}$$

Note that we have

$$g\left(\{f, g\}_k(t)\right) = \{g, f\}_{k+1}(t) \tag{1.2}$$

and

$$f^{-1}\left(\{f, g\}_k(t)\right) = \{g, f\}_{k-1}(t) \tag{1.3}$$

for $k = 1,\ 2,\ \cdots$, where f^{-1} denote the inverse [3,5,7] of the function f. The notation $u\,(t)$ will be used for the unit step function [9,14]: specifically,

$$u(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{1.4}$$

with $u(0)$ to be chosen appropriately depending on the cases. When $a > 0$, recollect

$$u^a(t) = u(t). \tag{1.5}$$

Translated Functions and Inverses

Assume a real function p, and let $q_-(t) = p(\tau - t)$ and $q_+(t) = p(\tau + t)$, where τ is a real constant. The domain of a function f will be denoted by D_f, an interval of the real line $\mathbb{R}$.

***Theorem* 2.1.** *We have*

$$\left\{p^{-1}, q_-\right\}_n(t) \ + \ \left\{q_-^{-1}, p\right\}_n(t) = \tau \tag{2.1}$$

when $n = 1, 3, \cdots,$ *and*

$$\left\{p^{-1}, q_-\right\}_n(t) = \left\{q_-^{-1}, p\right\}_n(t) = \begin{cases} t, & n = 0, 4, 8, \ldots, \\ \tau - t, & n = 2, 6, 10, \ldots. \end{cases} \tag{2.2}$$

for $t \in \left\{x : \left\{q_-, p^{-1}\right\}_{n-1}(x) \in D_{p^{-1}}\right\} \cap \left\{x : \left\{p, q_-^{-1}\right\}_{n-1}(x) \in D_{q_-^{-1}}\right\}$. *In addition, for* $t \in \left\{x : \left\{q_-^{-1}, p\right\}_{n-1}(x) \in D_p\right\} \cap \left\{x : \left\{p^{-1}, q_-\right\}_{n-1}(x) \in D_{q_-}\right\}$, *we have*

$$\left\{p, q_-^{-1}\right\}_n(t) \ = \ \left\{q_-, p^{-1}\right\}_n(t) \ = \ t \tag{2.3}$$

when $n = 0, 4, 8, \cdots$ *and*

$$\left\{p, q_-^{-1}\right\}_n(t) = \left\{q_-, p^{-1}\right\}_n(t) \tag{2.4}$$

when $n = 2, 6, 10, \cdots.$

Proof. (1) First, let $q_-^{-1}(t) = v$ for $t \in D_{q_-^{-1}}$. Then, $t = q_-(v) = p(\tau - v)$, and thus $p^{-1}(t) = p^{-1}(p(\tau - v)) = \tau - v$ for $t \in D_{p^{-1}}$, which leads us to $p^{-1}(t) + q_-^{-1}(t) = \tau$ or

$$\left\{p^{-1}, q_-\right\}_1(t) \ + \ \left\{q_-^{-1}, p\right\}_1(t) \ = \ \tau \tag{2.5}$$

for $t \in D_{p^{-1}} \cap D_{q_-^{-1}}$.

Next, assume that

$$\left\{p^{-1}, q_-\right\}_k(t) \ + \ \left\{q_-^{-1}, p\right\}_k(t) = \tau \tag{2.6}$$

holds true. Then, we have $\left\{p^{-1}, q_-\right\}_k(t) = \tau - \left\{q_-^{-1}, p\right\}_k(t)$ from (2.6), and subsequently, we get $q_-\left(\left\{p^{-1}, q_-\right\}_k(t)\right)$ $= q_-\left(\tau - \left\{q_-^{-1}, p\right\}_k(t)\right)$ or $\left\{q_-, p^{-1}\right\}_{k+1}(t) = p\left(\left\{q_-^{-1}, p\right\}_k(t)\right) = \left\{p, q_-^{-1}\right\}_{k+1}(t)$ by noting that $q_-(\tau - \alpha) = p(\alpha)$. Taking one more step as $p^{-1}\left(\left\{q_-, p^{-1}\right\}_{k+1}(t)\right) = p^{-1}\left(\left\{p, q_-^{-1}\right\}_{k+1}(t)\right)$, we get

$$\left\{p^{-1}, q_-\right\}_{k+2}(t) = \left\{q_-^{-1}, p\right\}_k(t). \tag{2.7}$$

On the other hand, writing (2.6) as $\left\{q_-^{-1}, p\right\}_k(t) = \tau - \left\{p^{-1}, q_-\right\}_k(t)$ and noting that $p(\tau - \alpha) = q_-(\alpha)$, we get $p\left(\left\{q_-^{-1}, p\right\}_k(t)\right) = p\left(\tau - \left\{p^{-1}, q_-\right\}_k(t)\right)$ or $\left\{p, q_-^{-1}\right\}_{k+1}(t) = q_-\left(\left\{p^{-1}, q_-\right\}_k(t)\right) = \left\{q_-, p^{-1}\right\}_{k+1}(t)$. Then, we get $q_-^{-1}\left(\left\{p, q_-^{-1}\right\}_{k+1}(t)\right) = q_-^{-1}\left(\left\{q_-, p^{-1}\right\}_{k+1}(t)\right)$ or

$$\left\{q_-^{-1}, p\right\}_{k+2}(t) = \left\{p^{-1}, q_-\right\}_k(t). \tag{2.8}$$

From (2.6)–(2.8), we have

$$\left\{p^{-1}, q_-\right\}_{k+2}(t) + \left\{q_-^{-1}, p\right\}_{k+2}(t) = \left\{q_-^{-1}, p\right\}_k(t) + \left\{p^{-1}, q_-\right\}_k(t) = \tau. \tag{2.9}$$

The result (2.9) from (2.6), together with (2.5), proves that (2.1) holds true when $n = 1, 3, \cdots.$

(2) From (1.1), we first have

$$\left\{p^{-1}, q_-\right\}_0(t) = \left\{q_-^{-1}, p\right\}_0(t) \;=\; t. \tag{2.10}$$

Next, from $q_-(t) = p(\tau - t)$, we get $p^{-1}(q_-(t)) = p^{-1}(p(\tau - t)) = \tau - t$ for $q_-(t) \in D_{p^{-1}}$. Similarly, noting that $p(t) = q_-(\tau - t)$, we get $q_-^{-1}(p(t)) = q_-^{-1}(q_-(\tau - t)) = \tau - t$ for $p(t) \in D_{q_-^{-1}}$. Thus,

$$\left\{p^{-1}, q_-\right\}_2(t) = \left\{q_-^{-1}, p\right\}_2(t) = \tau - t \tag{2.11}$$

for $t \in \left\{x : q_-(x) \in D_{p^{-1}}\right\} \cap \left\{x : p(x) \in D_{q_-^{-1}}\right\}$.

Now, assume that (2.2) holds true when $n = 0, 2, \cdots, k$. Then, from (2.2) with $n = k$, we get $q_-\left(\left\{p^{-1}, q_-\right\}_k(t)\right) = q_-\left(\left\{q_-^{-1}, p\right\}_k(t)\right)$ or $\left\{q_-, p^{-1}\right\}_{k+1}(t) = \left\{p, q_-^{-1}\right\}_{k-1}(t)$. Subsequently, we get $p^{-1}\left(\left\{q_-, p^{-1}\right\}_{k+1}(t)\right) = p^{-1}\left(\left\{p, q_-^{-1}\right\}_{k-1}(t)\right)$ or

$$\left\{p^{-1}, q_-\right\}_{k+2}(t) = \left\{q_-^{-1}, p\right\}_{k-2}(t). \tag{2.12}$$

On the other hand, from (2.2) with $n = k$, we also get $p\left(\left\{p^{-1}, q_-\right\}_k(t)\right) = p\left(\left\{q_-^{-1}, p\right\}_k(t)\right)$ or $\left\{q_-, p^{-1}\right\}_{k-1}(t) = \left\{p, q_-^{-1}\right\}_{k+1}(t)$. Subsequently, we get $q_-^{-1}\left(\left\{q_-, p^{-1}\right\}_{k-1}(t)\right) = q_-^{-1}\left(\left\{p, q_-^{-1}\right\}_{k+1}(t)\right)$ or

$$\left\{p^{-1}, q_-\right\}_{k-2}(t) = \left\{q_-^{-1}, p\right\}_{k+2}(t). \tag{2.13}$$

In essence, we have $\left\{p^{-1}, q_-\right\}_n(t) = \left\{q_-^{-1}, p\right\}_n(t) = t$ when $n = 0, 4, 8, \cdots$ from (2.10), (2.12), and (2.13); and $\left\{p^{-1}, q_-\right\}_n(t) = \left\{q_-^{-1}, p\right\}_n(t) = \tau - t$ when $n = 2, 6, 10, \cdots$ from (2.11)–(2.13).

(3) First, we get

$$\left\{p, q_-^{-1}\right\}_0(t) = \left\{q_-, p^{-1}\right\}_0(t) \;=\; t \tag{2.14}$$

from (1.1). Next, let $q_-^{-1}(t) = v$. Then, by noting that $p(v) = q_-(\tau - v)$, we get $p\left(q_-^{-1}(t)\right) = p(v) = q_-(\tau - v)$. On the other hand, $q_-\left(p^{-1}(t)\right) = q_-\left(p^{-1}(q_-(v))\right) = q_-\left(p^{-1}(p(\tau - v))\right) = q_-(\tau - v)$. Thus,

$$\left\{p, q_-^{-1}\right\}_2(t) = \left\{q_-, p^{-1}\right\}_2(t) \tag{2.15}$$

for $t \in \left\{x : q_-^{-1}(x) \in D_p\right\} \cap \left\{x : p^{-1}(x) \in D_{q_-}\right\}$.

Assume (2.4) holds true when $n = 0, 2, \cdots, k$. We get $q_-^{-1}\left(\left\{p, q_-^{-1}\right\}_k(t)\right) = q_-^{-1}\left(\left\{q_-, p^{-1}\right\}_k(t)\right)$ or $\left\{q_-^{-1}, p\right\}_{k+1}(t) = \left\{p^{-1}, q_-\right\}_{k-1}(t)$ from (2.4) with $n = k$. Subsequently, $p\left(\left\{q_-^{-1}, p\right\}_{k+1}(t)\right) = p\left(\left\{p^{-1}, q_-\right\}_{k-1}(t)\right)$ or

$$\left\{p, q_-^{-1}\right\}_{k+2}(t) = \left\{q_-, p^{-1}\right\}_{k-2}(t). \tag{2.16}$$

From (2.4) with $n = k$, we can get $p^{-1}\left(\left\{p, q_-^{-1}\right\}_k(t)\right) = p^{-1}\left(\left\{q_-, p^{-1}\right\}_k(t)\right)$ or $\left\{q_-^{-1}, p\right\}_{k-1}(t) = \left\{p^{-1}, q_-\right\}_{k+1}(t)$ alternatively. Thus, $q_-\left(\left\{q_-^{-1}, p\right\}_{k-1}(t)\right) = q_-\left(\left\{p^{-1}, q_-\right\}_{k+1}(t)\right)$ or

$$\left\{p, q_-^{-1}\right\}_{k-2}(t) = \left\{q_-, p^{-1}\right\}_{k+2}(t). \tag{2.17}$$

In short, (2.4) holds true when $n = 0, 4, 8, \cdots$ from (2.14), (2.16), and (2.17); and (2.4) holds true when $n = 2, 6, 10, \cdots$ from (2.15)–(2.17). In addition, the value of the equality (2.4) is t when $n = 0, 4, 8, \cdots$. $\square$

One of the implications of the equalities (2.2) and (2.4) is that the composite functions $\left\{p^{-1}, q_-\right\}_n$ and $\left\{p, q_-^{-1}\right\}_n$ are self-inverting when n is an even integer: that is, $\left(\left\{p^{-1}, q_-\right\}_n\right)^{-1} = \left\{p^{-1}, q_-\right\}_n$ and $\left(\left\{p, q_-^{-1}\right\}_n\right)^{-1} = \left\{p, q_-^{-1}\right\}_n$ when n is an even integer. It is also noteworthy that

$$q_-\left(p^{-1}(q_-(t))\right) = p(t) \tag{2.18}$$

and

$$p\left(q_-^{-1}(p(t))\right) = q_-(t) \tag{2.19}$$

from (2.2), and

$$q_-^{-1}\left(p\left(q_-^{-1}(t)\right)\right) = p^{-1}(t) \tag{2.20}$$

and

$$p^{-1}\left(q_-\left(p^{-1}(t)\right)\right) = q_-^{-1}(t) \tag{2.21}$$

from (2.4). The equalities (2.18)–(2.21) can be employed as useful tools in the reduction of calculations.

***Theorem* 2.2.** When $n = 0, 1, \cdots$, we have

$$\left\{p^{-1}, q_+\right\}_n(t) \;-\; \left\{q_+^{-1}, p\right\}_n(t) \;=\; n\tau \tag{2.22}$$

for $t \in \left\{x : \left\{q_+, p^{-1}\right\}_{n-1}(x) \in D_{p^{-1}}\right\} \cap \left\{x : \left\{p, q_+^{-1}\right\}_{n-1}(x) \in D_{q_+^{-1}}\right\}$.

From Theorem 2.2, we have

$$p^{-1}(t) = q_+^{-1}(t) + \tau, \tag{2.23}$$

$$p^{-1}(q_+(t)) = q_+^{-1}(p(t)) + 2\tau \tag{2.24}$$

$$p^{-1}(q_+(p^{-1}(t))) = q_+^{-1}(p(q_+^{-1}(t))) + 3\tau, \tag{2.25}$$

$$\vdots \quad ,$$

for instance.

***Theorem* 2.3.** *When $k = 0, 1, \cdots,$ we have*

$$\left\{q_-^{-1}, q_+\right\}_n (t) = \begin{cases} t, & n = 4k, \\ q_-^{-1}(t), & n = 4k+1, \\ -t, & n = 4k+2, \\ -q_-^{-1}(t), & n = 4k+3 \end{cases} \tag{2.26}$$

for $t \in \left\{x : \left\{q_+, q_-^{-1}\right\}_{n-1}(x) \in D_{q_-^{-1}}\right\}$ *and*

$$\left\{q_+^{-1}, q_-\right\}_n (t) = \begin{cases} t, & n = 4k, \\ q_+^{-1}(t), & n = 4k+1, \\ -t, & n = 4k+2, \\ -q_+^{-1}(t), & n = 4k+3 \end{cases} \tag{2.27}$$

for $t \in \left\{x : \left\{q_-, q_+^{-1}\right\}_{n-1}(x) \in D_{q_+^{-1}}\right\}$.

Theorems 2.2 and 2.3 can be proved by following steps similar to those in the proof of Theorem 2.1. Details are described in Appendix.

Examples and Applications

***Example* 3.1.** Consider the two functions $p(t) = \cos t$ and $q_-(t) = p\left(\frac{\pi}{2} - t\right) = \cos\left(\frac{\pi}{2} - t\right) = \sin t$ [2,11,13]. Then, we have the following results.

(1) Let $\sin^{-1} t = a$. We have $\sin^{-1} t = a = \cos^{-1}\sqrt{1-t^2}$, $\cos^{-1} t = \frac{\pi}{2} - a = \sin^{-1}\sqrt{1-t^2}$, $\sin a = t = \cos\left(\frac{\pi}{2} - a\right)$, and $\cos a = \sqrt{1-t^2} = \sin\left(\frac{\pi}{2} - a\right)$.

(2) Next, $\cos\left(\sin^{-1} t\right) = \cos a = \sqrt{1-t^2} = \sin\left(\frac{\pi}{2} - a\right) = \sin\left(\cos^{-1} t\right)$ and $\sin^{-1}(\cos a) = \sin^{-1}\left(\cos\left(\sin^{-1} t\right)\right) = \frac{\pi}{2} - a = \cos^{-1} t = \cos^{-1}(\sin a)$.

(3) In addition, $\cos^{-1}\left(\sin\left(\cos^{-1} t\right)\right) = \cos^{-1}\left(\sin\left(\frac{\pi}{2} - a\right)\right) = \cos^{-1}(\cos a) = a = \sin^{-1} t$, $\sin^{-1}\left(\cos\left(\sin^{-1} t\right)\right) = \frac{\pi}{2} - a = \cos^{-1} t$, $\sin\left(\cos^{-1}(\sin a)\right) = \sin\left(\cos^{-1} t\right) = \sin\left(\frac{\pi}{2} - a\right) = \cos a$, and $\cos\left(\sin^{-1}(\cos a)\right) = \cos\left(\sin^{-1}\sqrt{1-t^2}\right) = \cos\left(\frac{\pi}{2} - a\right) = \sin a$.

***Example* 3.2.** Consider $p(t) = t^2 u(t)$, the 'right' branch of the square function $p_S(t) = t^2$, on $D_p = [0, \infty)$. Then, $q_-(t) = p(\tau - t) = (\tau - t)^2 u(\tau - t)$, and we have the inverses $p^{-1}(t) = \sqrt{t}$ and $q_-^{-1}(t) = \tau - \sqrt{t}$ on $D_{p^{-1}} = D_{q_-^{-1}} = [0, \infty)$. Note that $D_{q_-} = (-\infty, \tau]$ in the current case because $D_{q_-} = (\tau - b, \tau - a)$ when p is a one-to-one function and $D_p = (a, b)$. Now, we have $\left\{x : q_-(x) \in D_{p^{-1}}\right\} = \left\{x : (\tau - x)^2 u(\tau - x) \in [0, \infty)\right\} = (-\infty, \tau]$ and $\left\{x : p(x) \in D_{q_-^{-1}}\right\} = \left\{x : x^2 u(x) \in [0, \infty)\right\} = [0, \infty)$. We also easily get

$$\begin{aligned} \left\{x : q_-^{-1}(x) \in D_p\right\} &= \left\{x : \tau - \sqrt{x} \in (0, \infty)\right\} \\ &= \begin{cases} [0, \tau^2], & \tau \geq 0, \\ \emptyset, & \tau < 0 \end{cases} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \left\{x : p^{-1}(x) \in D_{q_-}\right\} &= \left\{x : \sqrt{x} \in (-\infty, \tau)\right\} \\ &= \begin{cases} [0, \tau^2], & \tau \geq 0, \\ \emptyset, & \tau < 0 \end{cases} \end{aligned} \tag{3.2}$$

(1) We easily have $p^{-1}(t) + q_-^{-1}(t) = \sqrt{t} + \tau - \sqrt{t} = \tau$ for $t \in D_{p^{-1}} \cap D_{q_-^{-1}}$ or, equivalently, for $t \geq 0$.

(2) Next, we get $p^{-1}(q_-(t)) = p^{-1}\big((\tau-t)^2u(\tau-t)\big) = \sqrt{(\tau-t)^2u(\tau-t)} = (\tau-t)u(\tau-t) = \tau-t$ for $q_-(t)\geq 0$ or, equivalently, for $t\leq\tau$. Similarly, we get $q_-^{-1}(p(t)) = q_-^{-1}\big(t^2u(t)\big) = \tau-\sqrt{t^2u(t)} = \tau-tu(t) = \tau-t$ for $p(t)\geq 0$ or, equivalently, for $t\geq 0$. Thus, $p^{-1}(q_-(t)) = q_-^{-1}(p(t))$ for $0\leq t\leq\tau$ when $\tau\geq 0$.

(3) In addition, $p\big(q_-^{-1}(t)\big) = p\big(\tau-\sqrt{t}\big) = \big(\tau-\sqrt{t}\big)^2u\big(\tau-\sqrt{t}\big)$ when $0\leq t\leq\tau^2$ when $\tau\geq 0$. In the meantime, we similarly get $q_-\left(p^{-1}(t)\right) = q_-\left(\sqrt{t}\right) = \big(\tau-\sqrt{t}\big)^2u\big(\tau-\sqrt{t}\big)$ for $0\leq t\leq\tau^2$ when $\tau\geq 0$. Thus, $p\big(q_-^{-1}(t)\big) = q_-\left(p^{-1}(t)\right)$ for $0\leq t\leq\tau^2$ when $\tau\geq 0$.

***Example* 3.3.** Consider the functions $p(t) = t^2u(t)$ and $q_+(t) = p(\tau+t) = (\tau+t)^2u(\tau+t)$. Then, we have the inverses $p^{-1}(t) = \sqrt{t}$ and $q_+^{-1}(t) = -\tau+\sqrt{t}$ on $D_{p^{-1}} = D_{q_+^{-1}} = (0,\infty)$.

(1) We easily get $p^{-1}(t) - q_+^{-1}(t) = \sqrt{t} - \big(-\tau+\sqrt{t}\big) = \tau$ for $t\geq 0$.

(2) Next, for $q_+(t)\geq 0$ or, equivalently, for $t\geq -\tau$, we get $p^{-1}(q_+(t)) = \sqrt{(\tau+t)^2u(\tau+t)} = (\tau+t)u(\tau+t) = \tau+t$. Similarly, for $p(t)\geq 0$ or, equivalently, for $t\geq 0$, we get $q_+^{-1}(p(t)) = -\tau+\sqrt{t^2u(t)} = -\tau+tu(t) = -\tau+t$.

***Example* 3.4.** Let us choose the functions $p(t) = t^2u(-t)$, $q_-(t) = p(\tau-t) = (t-\tau)^2u(t-\tau)$, and $q_+(t) = p(\tau+t) = (t+\tau)^2u(-t-\tau)$. Then, we have $p^{-1}(t) = -\sqrt{t}$, $q_-^{-1}(t) = \tau+\sqrt{t}$, and $q_+^{-1}(t) = -\tau+\sqrt{t}$ with $D_{p^{-1}} = D_{q_-^{-1}} = D_{q_+^{-1}} = (0,\infty)$. Theorems 2.1 and 2.2 can be confirmed similarly as in Examples 3.2 and 3.3.

***Example* 3.5.** For $a(x) = \cos x$ and $b(x) = \sin x$, obtain

$$F_1(t) \;=\; \int_{a(t)}^{b(t)} \left\{a\left(b^{-1}(r)\right) + b\left(a^{-1}(r)\right)\right\} dr \tag{3.3}$$

and the derivative $f_1(t) = \frac{d}{dt}F_1(t)$ of $F_1(t)$.

Solution. Using (2.4) with $n=2$, or with (2.15), we can rewrite (3.3) as

$$F_1(t) \;=\; 2\int_{a(t)}^{b(t)} a\left(b^{-1}(r)\right) dr \;=\; 2\int_{\cos t}^{\sin t} \cos\left(\sin^{-1} r\right) dr \tag{3.4}$$

by noting that $\cos t = \sin(\tau-t)$ and $\sin t = \cos(\tau-t)$ with $\tau = \frac{\pi}{2}$. Now, letting $v = \sin^{-1} r$, we have $r = \sin v$ and $dr = \cos v\, dv$: in addition, $v = \sin^{-1}(\cos t) = \frac{\pi}{2} - t$ from (2.11) when $r = \cos t$, and $v = t$ when $r = \sin t$. In other words, we have $F_1(t) = 2\int_{\frac{\pi}{2}-t}^{t}\cos^2 v dv$ or

$$\begin{aligned} F_1(t) &= \int_{\frac{\pi}{2}-t}^{t} (1+\cos 2v)\, dv \\ &= t - \left(\frac{\pi}{2} - t\right) + \frac{1}{2}\left\{\sin 2t - \sin(\pi-2t)\right\} \\ &= 2t - \frac{\pi}{2}, \end{aligned} \tag{3.5}$$

where we used $\sin(\pi-2t) = \sin 2t$. Thus, the derivative is

$$f_1(t) = \frac{d}{dt}F_1(t) \;=\; 2. \tag{3.6}$$

In the meantime, the result $f_1(t) = 2$ can be obtained also from the fundamental theorem of calculus (FTC) [15] directly as

$$\begin{aligned} f_1(t) &= 2\cos\left(\sin^{-1} r\right)\Big|_{r=b(t)} b'(t) - 2\cos\left(\sin^{-1} r\right)\Big|_{r=a(t)} a'(t) \\ &= 2\cos\left(\sin^{-1}(\sin t)\right)\cos t + 2\cos\left(\sin^{-1}(\cos t)\right)\sin t \\ &= 2\cos^2 t + 2\sin^2 t \\ &= 2, \end{aligned} \tag{3.7}$$

where (2.19) can be used in obtaining $\cos\left(\sin^{-1}(\cos t)\right) = \sin t$.

***Example* 3.6.** For $a(x) = \cos x$ and $b(x) = \sin x$, obtain

$$F_2(t) \;=\; \int_{a^{-1}(t)}^{b^{-1}(t)} \left\{a^{-1}\left(b(r)\right) + b^{-1}\left(a(r)\right)\right\} dr \tag{3.8}$$

and its derivative $f_2(t) = \frac{d}{dt}F_2(t)$.

Solution. First, due to (2.11), we have

$$a^{-1}(b(r)) + b^{-1}(a(r)) = 2\sin^{-1}(\cos r) \;=\; \pi - 2r. \tag{3.9}$$

Then, from the FTC, we get $f_2(t) = \frac{d}{dt}\int_{\cos^{-1}t}^{\sin^{-1}t}(\pi - 2r)dr$ as

$$\begin{aligned} f_2(t) &= (\pi - 2r)|_{r=\sin^{-1}t}\left(\sin^{-1}t\right)' - (\pi - 2r)|_{r=\cos^{-1}t}\left(\cos^{-1}t\right)' \\ &= \left(\pi - 2\sin^{-1}t\right)\frac{1}{\sqrt{1-t^2}} - \left(\pi - 2\cos^{-1}t\right)\left(-\frac{1}{\sqrt{1-t^2}}\right) \\ &= \frac{1}{\sqrt{1-t^2}}\left\{2\pi - 2\left(\sin^{-1}t + \cos^{-1}t\right)\right\} \\ &= \frac{\pi}{\sqrt{1-t^2}} \end{aligned} \tag{3.10}$$

using $\sin^{-1}t + \cos^{-1}t = \frac{\pi}{2}$ from (2.1) with $n = 1$.

In the meantime, we can obtain $F_2(t) = \int_{\cos^{-1}t}^{\sin^{-1}t}(\pi - 2r)dr$ also as

$$\begin{aligned} F_2(t) &= \pi\left(\sin^{-1}t - \cos^{-1}t\right) - \left\{\left(\sin^{-1}t\right)^2 - \left(\cos^{-1}t\right)^2\right\} \\ &= \pi\left(2\sin^{-1}t - \frac{\pi}{2}\right) - \left(\sin^{-1}t + \cos^{-1}t\right)\left(\sin^{-1}t - \cos^{-1}t\right) \\ &= \pi\left(2\sin^{-1}t - \frac{\pi}{2}\right) - \frac{\pi}{2}\left(2\sin^{-1}t - \frac{\pi}{2}\right) \\ &= \pi\sin^{-1}t - \frac{\pi^2}{4}, \end{aligned} \tag{3.11}$$

where $\sin^{-1}t + \cos^{-1}t = \frac{\pi}{2}$ is used again. The result (3.10) can be obtained from (3.11) also.

Summary

In this paper we investigated translated functions together with their inverse mappings. By introducing the concept of alternating composition, several useful identities relating translated functions and their inverses were established. These identities provide a convenient framework for simplifying repeated function compositions and inverse transformations. Several mathematical examples demonstrated the applicability of the proposed formulas. Future work may include extending these identities to broader classes of nonlinear functions and investigating additional applications in signal processing, functional analysis, and communications.

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None.

Conflict of Interest

The authors declare that there is no conflict of interest.

References

[1] J Aczel (2006) Lectures on Functional Equations and Their Applications. Dover.

[2] S Axler (2012) Algebra and Trigonometry. JohnWiley and Sons.

[3] W Briggs, L Cochran (2011) Calculus: Early Transcendentals Single Variable. Addison-Wesley.

[4] S Damelin, W Miller (2011) The Mathematics of Signal Processing. Cambridge University Press.

[5] KJ Devlin (2004) Sets, Functions, and Logic: An Introduction to Abstract Mathematics. Third Ed., Chapman and Hall.

[6] ER Dougherty, J Astol (1999) Nonlinear Filters for Image Processing. SPIE Press.

[7] P Fletcher, CW Patty (1988) Foundations of Higher Mathematics. PWS-Kent.

[8] ET Jaynes (2003) Probability Theory: The Logic of Science. Cambridge University Press.

[9] RP Kanwal (2004) Generalized Functions: Theory and Applications. Birkhauser.

[10] SA Kassam (1988) Signal Detection in Non-Gaussian Noise. Springer.

[11] E Lieb, M Loss (2001) Analysis, Graduate Studies in Mathematics (Second Ed.), American Mathematical Society, vol. 14.

[12] L Narici, E Beckenstein (2011) Topological Vector Spaces CRC Press.

[13] W Rudin (1987) Real and Complex Analysis, Third Ed., McGraw-Hill.

[14] I Song, SR Park, S Yoon (2022) Probability and Random Variables: Theory and Applications. Springer.

[15] J Stewart, DK Clegg, S Watson (2020) Calculus: Early Transcendentals, Ninth Ed., Cengage Learning.

Appendix

Proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2

From (1.1), we have $\{p^{-1}, q_+\}_0(t) = t$ and $\{q_+^{-1}, p\}_0(t) = t$. Thus,

$$\{p^{-1}, q_+\}_0(t) - \{q_+^{-1}, p\}_0(t) = 0. \tag{A.1}$$

Next, let $q_+^{-1}(t) = v$, where $t \in D_{q_+^{-1}}$. Then, $t = q_+(v) = p(\tau + v)$. Thus, when $t \in D_{p^{-1}}$ in addition, we have $p^{-1}(t) = p^{-1}(p(\tau + v)) = \tau + v$, leading to $p^{-1}(t) - q_+^{-1}(t) = \tau$ or

$$\{p^{-1}, q_+\}_1(t) \ - \ \{q_+^{-1}, p\}_1(t) \ = \ \tau \tag{A.2}$$

for $t \in D_{p^{-1}} \cap D_{q_+^{-1}}$. Now, assume that

$$\{p^{-1}, q_+\}_k(t) \ - \ \{q_+^{-1}, p\}_k(t) \ = \ k\tau \tag{A.3}$$

holds true. Rewriting (A.3), we have $\{p^{-1}, q_+\}_k(t) = \{q_+^{-1}, p\}_k(t) + k\tau$, from which we get $\{q_+, p^{-1}\}_{k+1}(t) = q_+\big(\{q_+^{-1}, p\}_k(t) + k\tau\big) = p\big(\{q_+^{-1}, p\}_k(t) + (k+1)\tau\big)$, where we have used $q_+(-\tau + \alpha) = p(\alpha)$. Then, $\{p^{-1}, q_+\}_{k+2}(t) = p^{-1}\big(p\big(\{q_+^{-1}, p\}_k(t) + (k+1)\tau\big)\big)$ or

$$\{p^{-1}, q_+\}_{k+2}(t) = \{q_+^{-1}, p\}_k(t) + (k+1)\tau. \tag{A.4}$$

Rewriting (A.3) again, we have $\{q_+^{-1}, p\}_k(t) = \{p^{-1}, q_+\}_k(t) - k\tau$, from which we get $\{p, q_+^{-1}\}_{k+1}(t) = p\big(\{p^{-1}, q_+\}_k(t) - k\tau\big) = q_+\big(\{p^{-1}, q_+\}_k(t) - (k+1)\tau\big)$ by noting that $p(\tau + \alpha) = q_+(\alpha)$. Then, $\{q_+^{-1}, p\}_{k+2}(t) = q_+^{-1}\big(q_+\big(\{p^{-1}, q_+\}_k(t) - (k+1)\tau\big)\big)$ or

$$\{q_+^{-1}, p\}_{k+2}(t) = \{p^{-1}, q_+\}_k(t) - (k+1)\tau. \tag{A.5}$$

From (A.3)–(A.5), we get $\{p^{-1}, q_+\}_{k+2}(t) - \{q_+^{-1}, p\}_{k+2}(t) = \{q_+^{-1}, p\}_k(t) - \{p^{-1}, q_+\}_k(t) + 2(k+1)\tau = -k\tau + 2(k+1)\tau$ or

$$\{p^{-1}, q_+\}_{k+2}(t) - \{q_+^{-1}, p\}_{k+2}(t) = (k+2)\tau. \tag{A.6}$$

In short, (2.22) holds true for $n = 0, 2, 4, \cdots$ from (A.1) and (A.6); and for $n = 1, 3, 5, \cdots$ from (A.2) and (A.6).

Proof of Theorem 2.3

Let us first recollect that $q_+(t) = q_-(-t)$.

From (1.1), we have $\{q_-^{-1}, q_+\}_0(t) = t$, $\{q_+^{-1}, q_-\}_0(t) = t$, $\{q_-^{-1}, q_+\}_1(t) = q_-^{-1}(t)$, and $\{q_+^{-1}, q_-\}_1(t) = q_+^{-1}(t)$. We also have $\{q_-^{-1}, q_+\}_2(t) = q_-^{-1}(q_+(t)) = q_-^{-1}(q_-(-t)) = -t$ and $\{q_+^{-1}, q_-\}_2(t) = q_+^{-1}(q_-(t)) = q_+^{-1}(q_+(-t)) = -t$ noting that $q_+(t) = q_-(-t)$. Subsequently, we get $\{q_-^{-1}, q_+\}_3(t) = q_-^{-1}\big(q_+\big(q_-^{-1}(t)\big)\big) = q_-^{-1}\big(q_-\big(-q_-^{-1}(t)\big)\big) = -q_-^{-1}(t)$ and $\{q_+^{-1}, q_-\}_3(t) = q_+^{-1}\big(q_-\big(q_+^{-1}(t)\big)\big) = q_+^{-1}\big(q_+\big(-q_+^{-1}(t)\big)\big) = -q_+^{-1}(t)$. In other words, (2.26) and (2.27) hold true for $n = 0, 1, 2, 3$.

Now, assume that (2.26) holds true for $n = 0, 1, \cdots, 4k-1$ with k a positive integer. Then, $\{q_-^{-1}, q_+\}_{4k}(t) = q_-^{-1}\left(q_+\big(\{q_-^{-1}, q_+\}_{4k-2}(t)\big)\right) = q_-^{-1}(q_+(-t)) = q_-^{-1}(q_-(t)) = t$, $\{q_-^{-1}, q_+\}_{4k+1}(t) = q_-^{-1}\left(q_+\big(\{q_-^{-1}, q_+\}_{4k-1}t)\big)\right) = q_-^{-1}\big(q_+\big(-q_-^{-1}(t)\big)\big) = q_-^{-1}\big(q_-\big(q_-^{-1}(t)\big)\big) = q_-^{-1}(t)$, $\{q_-^{-1}, q_+\}_{4k+2}(t) = q_-^{-1}\big(q_+\big(\{q_-^{-1}, q_+\}_{4k}(t)\big)\big) = q_-^{-1}(q_+(t)) =$

$q_-^{-1}(q_-(-t)) = -t$, and $\{q_-^{-1}, q_+\}_{4k+3}(t) = q_-^{-1}\Big(q_+\big(\{q_-^{-1}, q_+\}_{4k+1}(t)\big)\Big) = q_-^{-1}\left(q_+\left(q_-^{-1}(t)\right)\right) = q_-^{-1}\left(q_-\left(-q_-^{-1}(t)\right)\right) = -q_-^{-1}(t)$. That is, (2.26) holds true also for $n = k+1$ when it holds true for $n = k$.

Similarly, assume that (2.27) holds true for $n = 0, 1, \cdots, 4k-1$ with k a positive integer. Then, we have $\{q_+^{-1}, q_-\}_{4k}(t) = q_+^{-1}\Big(q_-\big(\{q_+^{-1}, q_-\}_{4k-2}(t)\big)\Big) = q_+^{-1}(q_-(-t)) = q_+^{-1}(q_+(t)) = t$, $\{q_+^{-1}, q_-\}_{4k+1}(t) = q_+^{-1}\Big(q_-\big(\{q_+^{-1}, q_-\}_{4k-1}(t)\big)\Big) = q_+^{-1}\left(q_-\left(-q_+^{-1}(t)\right)\right) = q_+^{-1}\left(q_+\left(q_+^{-1}(t)\right)\right) = q_+^{-1}(t)$, $\{q_+^{-1}, q_-\}_{4k+2}(t) = q_+^{-1}\left(q_-\left(\{q_+^{-1}, q_-\}_{4k}(t)\right)\right) = q_+^{-1}(q_-(t)) = q_+^{-1}(q_+(-t)) = -t$, and $\{q_+^{-1}, q_-\}_{4k+3}(t) = q_+^{-1}\Big(q_-\big(\{q_+^{-1}, q_-\}_{4k+1}(t)\big)\Big) = q_+^{-1}\left(q_-\left(q_+^{-1}(t)\right)\right) = q_+^{-1}\left(q_+\left(-q_+^{-1}(t)\right)\right) = -q_+^{-1}(t)$, That is, (2.27) holds true also for $n = k+1$ when it holds true for $n = k$.