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Research Article

Exponential Stability of Impulsive Stochastic Functional Deferential Equations with Delayed Impulsive*

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Abstract

Based on the research of exponential stability for impulsive stochastic functional deferential equations, this paper proposes and proves a novel generalized Halanay type inequality, which extending the applicability of the classical Halanay inequality. By integrating this new inequality with the Lyapunov functional method, this paper establishes sufficient criteria for the pth moment exponential stability of the system, providing a more general theoretical tool for analysing the stability of a broader class of non-autonomous impulsive stochastic delayed systems. The effectiveness of the theoretical results is verified through a numerical example.

Keywords: Generalized Halanay Inequality; Impulsive Stochastic Functional Differential Equations; Delayed Impulsive; Exponential Stability

Introduction

Impulsive stochastic functional differential equations (ISFDEs) constitute a class of dynamical models that simultaneously capture continuous-time evolutions with random perturbations dependent on past states and discrete-time impulsive effects. They play a vital role in numerous fields such as physics, biology, engineering control, and financial mathematics. Among the research on ISFDEs, stability analysis remains a core issue. To mention just a few, we refer to works like [1-4], and [5] for recent advances on various forms of exponential stability for impulsive stochastic delayed functional differential equations. Among the methodologies, the Lyapunov-Krasovskii functional method and the Razumikhin theorem are two classical and effective approaches.

However, constructing appropriate Lyapunov functionals often requires considerable skill and can be challenging. In contrast, the Halanay differential inequality, as an analytical technique that does not require explicit construction of Lyapunov functionals, directly utilizes differential inequalities via comparison principles to derive decay properties of solutions. It demonstrates unique advantages in handling stability of delay differential equations containing supremum functional terms. The standard form of the classical Halanay inequality is

$$u'(t) \le \lambda_1 u(t) + \lambda_2 \sup_{\theta\in[-\tau,0]} u(t+\theta), \quad -\lambda_1 > \lambda_2 > 0, \qquad (1.1)$$

and the generalized Halanay inequality takes the form

$$u'(t) \le \lambda_1(t)u(t) + \lambda_2(t) \sup_{\theta\in[-\tau,0]} u(t+\theta), \quad -\lambda_1(t) > \lambda_2(t) \ge 0, \qquad (1.2)$$

In [6], the coefficient of the delay term in (1.1) is extended to piecewise constant coefficients. In [7], the authors relax the condition in (1.2) to $-\lambda_1(t) > \lambda_2(t) \ge 0,$ while [8] provides a discrete version of the Halanay inequality. [9] extends the Halanay inequality to an integral inequality and applies it to fractional-order differential systems. These works generalize the Halanay inequality in various aspects, but all require that the negative of the instantaneous term coefficient is greater than or equal to the delay term coefficient. What would happen if, at certain moments, the negative of the instantaneous term coefficient is less than the delay term coefficient, or if the instantaneous term coefficient is sign-changing? The applicability of existing results remains limited.

Motivated by the above considerations, this paper establishes a new generalized Halanay-type integral inequality for a class of impulsive stochastic functional differential equations with time-varying delays and fixed impulsive moments. By combining this inequality with the Lyapunov functional method and systematically characterizing the cumulative effects of impulsive perturbations under the average dwell-time frame- work, we derive sufficient criteria for the pth moment exponential stability of the system. Importantly, these criteria do not require the instantaneous term coefficient to be always negative nor rely on pointwise comparisons between instantaneous and delay terms. Thus, this work provides a more general theoretical tool for stability analysis of impulsive stochastic functional differential equations.

The remainder of this paper is organized as follows. Section 2 formulates the problem and presents the derivation and proof of the generalized Halanay-type inequality. Section 3 states the main results, including exponential stability criteria for impulsive stochastic delayed functional differential equations. Section 4 provides an illustrative example. Finally, Section 5 concludes the paper with several remarks.

Preliminaries

We begin this section with the following notation. Let $R=(-\infty,+\infty)$, $R_+=[0,+\infty)$ and $\mathbb{N}$ be the set of positive integers. Let $C_1^2(R_+\times R^d, R)$ be the set of all functions $V(\cdot,\cdot): R_+\times R^d \to R,$ which are continuously differentiable on R_+ and twice continuously differentiable on R^d. For a constant τ > 0, let $PC([-\tau,0],R^d)$ be the Banach space of all piecewise continuous functions $\phi:[-\tau,0]\to R^d$, which are right continuous, endowed with the norm $\|\phi\| = \sup_{s\in[-\tau,0]}|\phi(s)|$. We work with a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual condition (i.e., it is right-continuous and $\mathcal{F}_0$ contains all the null sets).

The topic of our analysis is the following impulsive stochastic functional differential equations:

$$\begin{cases} dx(t) = f(t,x(t),x_t)dt + g(t,x(t),x_t)dw(t), & t\ne t_k,\ t\ge t_0, \\ \Delta x(t_k) = I_k\left(t_k^-, x(t_k^-), x(t_k^- + \theta)\right), & k\in\mathbb{N}_+, \\ x(t_0+\theta) = \phi(\theta), & \theta\in[-\tau,0] \end{cases} \qquad (2.1)$$

where $x(t)\in\mathbb{R}^d, x_t := x(t-\tau(t))$ is regarded as a PC-valued stochastic process, and $\tau(t)$ is a time varying delay function satisfying $0\le\tau(t)\le\tau.$ Both $f:[t_0,+\infty)\times\mathbb{R}^d\times PC([-\tau,0],R^d)\to\mathbb{R}^d$ *and* $g:[t_0,+\infty)\times\mathbb{R}^d\times PC([-\tau,0],R^d)\to\mathbb{R}^{d\times m}$ are continuous functionals, $w(t)\in\mathbb{R}^m$ is an $\{\mathcal{F}_t\}-$adapted Brownian motion defined on a complete probability space

$(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$. $\Delta x(t_k) = x(t_k) - x(t_k^-)$,

where $x(t_k^-) = \lim_{h\to0^-} x(t_k+h)$. $I_k\left(t_k^-, x(t_k^-), x_{t_k^-}\right):[t_0,+\infty)\times\mathbb{R}^d\times PC([-\tau,0],R^d)\to\mathbb{R}^d$

represents the impulsive perturbation of x at t_k. The fixed moments of impulsive times t_k satisfy $0\le t_0<t_1<\ldots<t_k$ and $t_k\to\infty$ (as k → ∞). We further assume that there is a lower bound for impulsive intervals, that is, $t_{k+1}-t_k>\tau, k\in\mathbb{N}$.

It can be seen that Eq. (2.1) has memory and includes a finite-time segment of past states. Such a consideration allows us to treat the situation that the future behaviour depends on the present as well as the past state of a system. Now, we present definitions and lemmas which are important to obtain our results.

Definition 2.1. *For function* $V\in C_1^2([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}^+)$, *we define an operator* $\mathcal{L}$ *associated with system (2.1) as*

$$\mathcal{L}V(t,x_t) = V_t(t,x_t) + V_x(t,x_t)f(t,x,x_t) + \frac{1}{2}\mathrm{trace}\left[g^T(t,x,x_t)V_{xx}(t,x_t)g(t,x,x_t)\right],$$

$$\text{where } V_t(x)=\frac{\partial V(t,x)}{\partial t},\ V_x(t,x)=\left(\frac{\partial V(t,x)}{\partial x_1},\dots,\frac{\partial V(t,x)}{\partial x_n}\right),\ \text{ and } V_{xx}(t,x)=\left(\frac{\partial^2 V(t,x)}{\partial x_2\partial x_3}\right)_{n\times n}.$$

Definition 2.2. *For any sequence of impulsive instants* $\{t_k\}_{k\in\mathbb{N}}$, *if the following inequality is satisfied for* $T_a>0, N_0>0$,

$$N(t,s)\le\frac{t-s}{T_a}+N_0, \qquad (2.2)$$

where $N(t,s)$ *stands for the frequency of impulses during the interval* $(s,t]$, *then* $T_a>0$ *and* $N_0>0$ *are named average dwell time and the elasticity number, respectively.*

Definition 2.3. *For* $p\ge 1$, *system (2.1) is called exponentially stable in pth moment, if for any initial value* $\phi\in C_0^b([t_0-\tau,t_0];\mathbb{R}^n)$, *there are constants* $M,\alpha>0$ *such that for* $t\ge t_0$,

$$\mathbb{E}\|x(t)\|^p\le M\|\phi\|^p e^{-\alpha(t-t_0)}. \qquad (2.3)$$

Lemma 2.1. *Let* $u:[t_0-\tau,\infty)\to[0,\infty)$ *be a continuous differentiable function. Assume that for all* $t\ge t_0$,

$$\dot u(t)\le\lambda_1(t)u(t)+\lambda_2(t)\sup_{\theta\in[-\tau(t),0]}u(t+\theta),$$

where $\tau(t):[t_0,\infty)\to[0,\tau]$ *is continuous,* $\lambda_1(t)\in C([t_0,\infty),\mathbb{R})$ *and* $\lambda_2(t)\in C([t_0,\infty),\mathbb{R}_+)$. *Define*

$$\rho(t):=\exp\left(\sup_{\theta\in[-\tau(t),0]}\int_t^{t+0}(\lambda_1(\upsilon))d\upsilon\right).$$

$H(t)=\lambda_1(t)+\lambda_2(t)\rho(t)$, *then for every* $t\ge t_0$,

$$u(t)\le\bar u(t_0)e^{\int_{t_0}^t H(s)ds},$$

where $\bar u(t_0)=\sup_{-\tau\le\theta\le 0}u(t_0+\theta)$.

Proof. Define the comparison function *w(t)* as the solution of the scalar delay differential inequality:

$$\dot w(t)\le\lambda_1(t)w(t)+\lambda_2(t)\sup_{\theta\in[-\tau(t),0]}w(t+\theta),\quad w(t_0)=\bar u(t_0). \qquad (2.4)$$

By standard comparison arguments, we have

$$u(t)\le w(t),\qquad t\ge t_0,$$

We next derive an upper estimate for *w*. Using the variation-of-constants formula for (2.4), for $t\ge t_0$,

$$w(t)=w(t_0)e^{\int_{t_0}^t\lambda_1(s)ds}+\int_{t_0}^t e^{\int_s^t\lambda_1(\upsilon)d\upsilon}\lambda_2(s)\sup_{\theta\in[-\tau(s),0]}w(s+\theta)ds. \qquad (2.5)$$

Then, by (2.5), for any $\bar\tau\in[0,\tau(t)]\subset[0,\tau]$,

$$\begin{aligned}
w(t)&=w(t_0)e^{\int_{t_0}^t\lambda_1(s)ds}+\int_{t_0}^{t-\bar\tau}e^{\int_s^t\lambda_1^*(\upsilon)d\upsilon}\lambda_2(s)\sup_{\theta\in[-\tau(s),0]}w(s+\theta)ds\\
&\quad+\int_{t-\bar\tau}^t e^{\int_s^t\lambda_1^*(\upsilon)d\upsilon}\lambda_2(s)\sup_{\theta\in[-\tau(s),0]}w(s+\theta)ds\\
&=e^{\int_{t-\bar\tau}^t\lambda_1^*(s)ds}\left(w(t_0)e^{\int_{t_0}^{t-\bar\tau}\lambda_1^*(s)ds}+\int_{t_0}^{t-\bar\tau}e^{\int_s^{t-\bar\tau}\lambda_1^*(\upsilon)d\upsilon}\lambda_2(s)\sup_{\theta\in[-\tau(s),0]}w(s+\theta)ds\right)\\
&\quad+\int_{t-\bar\tau}^t e^{\int_s^t\lambda_1^*(\upsilon)d\upsilon}\lambda_2(s)\sup_{\theta\in[-\tau(s),0]}w(s+\theta)ds\\
&\ge e^{\int_{t-\bar\tau}^t\lambda_1^*(s)ds}w(t-\bar\tau),
\end{aligned}$$

Where $\lambda_1^*(t)=\begin{cases}\lambda_1(t), & t\ge t_0,\\ 0, & t_0-\tau\le t<t_0,\end{cases}$, thus $\sup_{\bar\tau\in[0,\tau(t)]}\left(e^{\int_{t-\bar\tau}^t\lambda_1^*(s)ds}w(t-\bar\tau)\right)\le w(t)$, then we claim that for every $t\ge t_0$,

$$\sup_{\theta\in[-\tau(t),0]}w(t+\theta)\le\rho(t)w(t). \qquad (2.6)$$

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By (2.4) and (2.6),

$\overline{w}(t) \le \lambda_1(t)w(t) + \lambda_2(t)\rho(t)w(t) \le H(t)w(t)$, hence

$$w(t) \le w(t_0)e^{\int_{t_0}^{t} H(s)ds} = \overline{u}(t_0)e^{\int_{t_0}^{t} H(s)ds}, \quad t \ge t_0.$$

Then $u(t) \le \overline{u}(t_0)e^{\int_{t_0}^{t} H(s)ds}, \quad t \ge t_0$. The proof is complete.

Remark 2.1 The key estimate (2.6) is the same mechanism that appears in generalized Halanay inequalities: The factor $\rho(t)$ characterizes the control ability of the cumulative effect of the instantaneous term coefficient $\lambda_1(\cdot)$ over the time-delay interval on the contribution of the time-delay term.

Remark 2.2 While the classical Halanay inequality mainly targets deterministic constant delay systems, Lemma 2.1 in this paper is designed from the outset to accommodate extensions to stochastic and impulsive scenarios, and the setting of time-varying delays and coefficients allows the lemma to be applied to a broader class of non-autonomous stochastic impulsive delay systems.

Main Results

Theorem 3.1. *Assume there exist* $V(t,x) \in C_1^2([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}^+)$, $\lambda_1(t) \in C([t_0,+\infty),\mathbb{R})$ *and* $\lambda_2(t) \in C([t_0,+\infty),\mathbb{R}^+)$, *positive constants* $c_1, c_2, \mu_{1k}, \mu_{2k}, a_1, a_2$ *and* T_a, *such that the following conditions hold:*

(H_1) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$.

(H_2) *For* $t \ne t_k$ *and* $\phi \in PC_{\mathcal{F}_t}^p([-\tau,0],\mathbb{R}^n)$,

$$\mathbb{E}\mathcal{L}V(t,\phi) \le \lambda_1(t)\mathbb{E}V(t,\phi(0)) + \lambda_2(t)\sup_{-\tau(t)\le\theta\le 0}\mathbb{E}V(t+\theta,\phi(\theta)).$$

(H_3) *For all* $(t_k,\phi) \in [t_0,\infty)\times PC_{\mathcal{F}_t}^p([-\tau,0],\mathbb{R}^n)$,

$$EV(t_k,\phi(0)+I_k(t_k,\phi(\theta))) \le \mu_{1k}EV(t_k^-,\phi(0)) + \mu_{2k}\sup_{-\tau(t)\le\theta\le 0}EV(t_k^-+\theta,\phi(\theta)).$$

(H_4) $H(t) = \lambda_1(t) + \lambda_2(t)\rho(t)$, *where* $\rho(t) := \exp\left(\sup_{\theta\in[-\tau(t),0]}\int_t^{t+0}(\lambda_1(\upsilon))d\upsilon\right)$, *and for any* $t > s \ge t_0, -a_2(t-s) \le \int_s^t H(w)dw \le -a_1(t-s) + b$.

(H_5) $\sum_{i=0}^{N(t,t_0)}\ln\upsilon_i < (a_1T_a - a_2\tau)N(t,t_0)$, *where* $\upsilon_0 = 1$, $\upsilon_k = \max\{\mu_{1k} + \mu_{2k}e^{a_2\tau}, 1\}$, $a_1T_a - a_2\tau > 0$, $T_a > 0$ *is the average dwell time between every implusive and* $N(t,t_0)$ *stands for the frequency of impulses during the interval* $(t_0,t]$.

Then, the trivial solution of System (2.1) is pth moment exponentially stable.

Proof. Let $x(t) = x(t,t_0,\phi)$ represent the solution of system (2.1). By using It?'s formula, for $t \in [t_{k-1},t_k), k \in \mathbb{Z}^+$, we get

$$dV(t,x(t)) = \mathcal{L}V(t,x_t)dt + V_x(t,x(t))g(t,x(t),x_t)dw(t).$$

Choosing a sufficiently small $\Delta t > 0$, let $\Delta t > 0$, $t + \Delta t \in (t_{k-1},t_k)$, we obtain

$$\mathbb{E}V(t+\Delta t, x(t+\Delta t)) - \mathbb{E}V(t,x(t)) = \int_t^{t+\Delta t}\mathbb{E}\mathcal{L}V(s,x_s)ds,$$

which implies

$$D^+\mathbb{E}V(t,x(t)) = \mathbb{E}\mathcal{L}V(t,x_t) \le \lambda_1(t)\mathbb{E}V(t,\phi(0)) + \lambda_2(t)\sup_{-\tau(t)\le\theta\le 0}\mathbb{E}V(t+\theta,\phi(\theta)), t \in [t_{k-1},t_k), \quad (3.1)$$

where D^+ denotes the upper right-hand Dini derivative. Combine with Lemma 2.1 and (H_4), we get

$$\mathbb{E}V(t,x(t)) \le \mathbb{E}\overline{V}(t_0)e^{\int_{t_0}^{t} H(s)ds}, \quad t \in [t_0,t). \quad (3.2)$$

By conditions (H_3), (H_4), (H_5) and (3.2), for t = t_1, we have

$$\begin{aligned}\mathbb{E}V\left(t_1,x\left(t_1\right)\right)&\le \mu_{11}EV\left(t_1^-,\phi\left(0\right)\right)+\mu_{21}\sup_{-\tau\left(t_1\right)\le\theta\le 0}EV\left(t_1^-+\theta,\phi\left(\theta\right)\right)\\&\le \mu_{11}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}+\mu_{21}\sup_{-\tau\left(t_1\right)\le\theta\le 0}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1+\theta}H(s)ds}\\&\le \mu_{11}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}+\mu_{21}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}\cdot\sup_{-\tau\left(t_1\right)\le\theta\le 0}e^{\int_{t_1}^{t_1+\theta}H(s)ds}\\&\le \mu_{11}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}+\mu_{21}e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}\\&\le \upsilon_1\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_1}H(s)ds}.\end{aligned}$$

Hence, we have

$$\mathbb{E}V\left(t,x\left(t\right)\right)\le\upsilon_1\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds},\ t\in\left[t_1-\tau\left(t\right),t_1\right].$$

Similar with the derivation of $\left[t_0,t_1\right)$, for $t\in\left[t_1,t_2\right)$, we get

$$\begin{aligned}\mathbb{E}V\left(t,x\left(t\right)\right)&\le\mathbb{E}\bar{V}\left(t_1\right)e^{\int_{t_1}^{t}H(s)ds}\\&\le\upsilon_1\mathbb{E}\bar{V}\left(t_0\right)\sup_{-\tau\left(t_1\right)\le\theta\le 0}e^{\int_{t_0}^{t_1+\theta}H(s)ds}e^{\int_{t_1}^{t}H(s)ds}\\&\le\upsilon_1\mathbb{E}\bar{V}\left(t_0\right)\sup_{-\tau\left(t_1\right)\le\theta\le 0}e^{\int_{t_1}^{t_1+\theta}H(s)ds}e^{\int_{t0}^{t}H(s)ds}\\&\le\upsilon_1e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t0}^{t}H(s)ds}.\qquad(3.3)\end{aligned}$$

When t = t_2, by (H_3), (H_4), (H_5) and (3.3), we obtain

$$\begin{aligned}\mathbb{E}V\left(t_2,x\left(t_2\right)\right)&\le\mu_{12}EV\left(t_2^-,\phi\left(0\right)\right)+\mu_{22}\sup_{-\tau\left(t_2\right)\le\theta\le 0}EV\left(t_2^-+\theta,\phi\left(\theta\right)\right)\\&\le\mu_{12}\upsilon_1e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_2}H(s)ds}+\mu_{22}\upsilon_1e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)\sup_{-\tau\left(t_2\right)\le\theta\le 0}e^{\int_{t_0}^{t_2+\theta}H(s)ds}\\&\le\left(\mu_{12}+\mu_{22}e^{a_2\tau}\right)\upsilon_1e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_2}H(s)ds}\\&\le\upsilon_1\upsilon_2e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t_2}H(s)ds}.\end{aligned}$$

For $t\in\left[t_2-\tau,t_2\right]$, we get

$$\mathbb{E}V\left(t,x\left(t\right)\right)\le\upsilon_1\upsilon_2e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds},$$

By using mathematical induction, we can suppose that when $k\ge 2$, for $t\in\left[t_{k-1},t_k\right)$,

$$\mathbb{E}V\left(t,x\left(t\right)\right)\le\upsilon_1\upsilon_2e^{a_2\tau}\mathbb{E}\bar{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds}$$

Then, combine with (H_3), (H_4) and (H_5), for t = t_k,

$$\begin{aligned}\mathbb{E}V\left(t_k,x\left(t_k\right)\right)&\le \mu_{1k}EV\left(t_k^-,\phi(0)\right)+\mu_{2k}\sup_{-\tau(t_k)\le\theta\le0}EV\left(t_k^-+\theta,\phi(\theta)\right)\\&\le \mu_{1k}\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t_k}H(s)ds}+\mu_{2k}\sup_{-\tau(t_k)\le\theta\le0}\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t_k+\theta}H(s)ds}\\&\le\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t_k}H(s)ds}\left(\mu_{1k}+\mu_{2k}e^{\int_{t_0}^{t_k+\theta}H(s)ds}\right)\\&\le\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t_k}H(s)ds}.\end{aligned}$$

Thus, we get

$$\mathbb{E}V\left(t,x(t)\right)\le\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t_k}H(s)ds},\qquad t\in\left[t_k-\tau,t_k\right].$$

Combine with Lemma 2.1 and (H_4), we obtain

$$\begin{aligned}\mathbb{E}V\left(t,x(t)\right)&\le\mathbb{E}\overline{V}\left(t_k\right)e^{\int_{t_k}^{t}H(s)ds}\\&\le\prod_{i=1}^{k-1}\upsilon_i e^{(k-1)a_2\tau}\mathbb{E}\overline{V}\left(t_0\right)\sup_{-\tau(t_k)\le\theta\le0}e^{\int_{t_0}^{t_k+\theta}H(s)ds}e^{\int_{t_k}^{t}H(s)ds}\\&\le\prod_{i=1}^{k-1}\upsilon_i e^{ka_2\tau}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds},\qquad t\in\left[t_k,t_{k+1}\right).\end{aligned}\tag{3.4}$$

By $\left(H_5\right)$ $\sum_{i=0}^{N(t,t_0)}\ln\upsilon_i<\left(a_1T_a-a_2\tau\right)N\left(t,t_0\right)$, we know there exist a const $c\in\left(\frac{a_2\tau}{T_a},a_1\right)$, such that

$\sum_{i=0}^{N(t,t_0)}\ln\upsilon_i\le\left(cT_a-a_2\tau\right)N\left(t,t_0\right)$. Thus, by (2.2), (3.4) and (H_4) for $t\in\left[t_0,+\infty\right)$, we have

$$\begin{aligned}\mathbb{E}V\left(t,x(t)\right)&\le e^{\left(\sum_{i=0}^{N(t_0,t)}\ln\upsilon_i+N(t_0,t)a_2\tau\right)}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds}\\&\le e^{\left(\left(cT_a-a_2\tau\right)N(t,t_0)+a_2\tau N(t,t_0)\right)}\mathbb{E}\overline{V}\left(t_0\right)e^{\int_{t_0}^{t}H(s)ds}\\&\le e^{cT_a\left[\frac{t-t_0}{T_a}+N_0\right]}\mathbb{E}\overline{V}\left(t_0\right)e^{-a_1(t-t_0)+b}\\&\le e^{cT_aN_0+b}\mathbb{E}\overline{V}\left(t_0\right)e^{-(a_1-c)(t-t_0)}\\&\le e^{a_1T_aN_0+b}\mathbb{E}\overline{V}\left(t_0\right)e^{-(a_1-c)(t-t_0)},\end{aligned}$$

By (H_1), $$\mathbb{E}\left\|x(t)\right\|^p\le\frac{c_2}{c_1}e^{a_1T_aN_0+b}\left\|\phi\right\|^p e^{-(a_1-c)(t-t_0)},$$

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the trivial solution of System (2.1) is pth moment exponentially stable.

Remark 3.1 (H_4) allows H(t) to be positive on finite intervals, while requiring long-term dominance of negative average. On the other hand, different time-dependent parameters $\{\mu_{1k}\},\{\mu_{2k}\}$ were adopted to draw the impulsive strength and delayed impulsive strength, which is more general and has a broader range of applications.

Remark 3.2 In [10], the authors proposed $\dot{q}(t)\le -\lambda_1(t)q(t)+\lambda_2(t)\sup_{-\tau\le\theta\le 0} q(t+\theta)$, and $\lambda_1(t)$ and $\lambda_2(t)\in C([t_0,+\infty),\mathbb{R}^+)$ with $\mu(t)=\lambda_1(t)-\lambda_2(t)\ge 0$, which is a strict condition. In this paper, this article does not require the traditional Halanay condition, $\lambda_1(t)$ could be sign-changed, and we allow the negative of the instantaneous terms coefficient could less than delayed terms coefficient in certain intervals, the stability of the system no longer relies on pointwise comparison between $\lambda_1(t)$ and $\lambda_2(t)$, but depends on the integral behaviour of the overall composite function H(t) which modulated by the time-varying application factor $\rho(t)$. So Lemma 2.1 is one generalized Halanay inequality, in which the time-varying coefficient $\lambda_1(t)$ could be sign-changed.

Remark 3.3 Unlike the approach in [6], where the delayed terms coefficient gains $\lambda_2(t)$ is piecewise constant and may exceed the decay rate only over sparse intervals E, Lemma 2.1 establishes a generalized Halanay inequality under continuous and time-varying coefficients. Notably, the piecewise-constant case studied in Mazenc et al. [6] can be viewed as a special case of our results. In Lemma 1 of [11], two constant coefficients are used to control the Lyapunov functional operator. In this paper, we employ two time-varying coefficients to control the Lyapunov functional operator, and the stability of the system is governed by functional-type constraints. This distinction implies that the conclusions of this paper are more general compared to those existed results.

Corollary 3.1. *Assume that condition* $(H_1)-(H_2)$ *are satisfied and* (H_5^*) *If* $\{\mu_{1k}\},\{\mu_{2k}\}$ *in* (H_3) *are constant sequence,*

$$\ln\upsilon < a_1T_a - a_2\tau, \text{ where } \upsilon=\max\{\mu_{1k}+\mu_{2k}e^{a_2\tau},1\}.$$

Then, the trivial solution of System (2.1) is pth moment exponentially stable.

Remark 3.4 By assuming that 1k and 2k are constant sequences, it simplifies the analysis of the cumulative effect of impulsive perturbations, eliminating the need to estimate the frequency of impulses through the average dwell time and the elasticity number. This simplification makes Corollary 3.1 more suitable for engineering scenarios where impulsive strengths are relatively uniform.

Theorem 3.1 presented earlier has established the basic criterion for the pth moment exponential stability of impulsive stochastic functional differential equations. However, in practical scenarios, the future behaviour of systems often depends on multiple independent time-varying delay components. To adapt to such complex delay distribution characteristics and further expand the application scope of the stability criterion, the following corollary is proposed. It decomposes the single supremum constraint of the delay term in Theorem 3.1 into a weighted sum of a finite number of independent time-varying delay terms.

Theorem 3.2. *Assume there exist* $V(t,x)\in C_1^2([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}^+)$, $\lambda_0(t)\in C([t_0,+\infty),\mathbb{R})$ *and* $\lambda_i(t)\in C([t_0,+\infty),\mathbb{R}^+)$, $i=1,2,\ldots,n_0$, *positive constants* $c_1,c_2,\mu_{1k},\mu_{2k},a_1,a_2,b$ *and* T_a, *such that the following conditions hold:*

(H_1) $c_1\|x\|^p\le V(t,x)\le c_2\|x\|^p$.

(H_2) *For* $t\ne t_k$ *and* $\phi\in PC^p_{\mathcal{F}_t}([-\tau,0],\mathbb{R}^n)$,

$$\mathbb{E}\mathcal{L}V(t,\phi)\le\lambda_0(t)\mathbb{E}V(t,\phi(0))+\sum_{i=1}^{n_0}\lambda_i(t)\mathbb{E}V(t-\tau_i(t),\phi(-\tau_i)).$$

where x_t *appear in the form* $x(t-\tau_1(t)),\ldots,x(t-\tau_{n_0}(t))$, *and* $\tau_i(t)\in[0,\tau]$.

(H_3) *For all* $(t_k,\phi)\in[t_0,\infty)\times PC^p_{\mathcal{F}_t}([-\tau,0],\mathbb{R}^n)$,

$$EV(t_k,\phi(0)+I_k(t_k,\phi(\theta)))\le\mu_{1k}EV(t_k^-,\phi(0))+\mu_{2k}\sup_{-\tau\le\theta\le 0}EV(t_k^-+\theta,\phi(\theta)).$$

(H_4) $H(t)=\lambda_0(t)+\sum_{i=1}^{n_0}\lambda_i(t)\rho(t)$, *where* $\rho(t):=\exp\left(\sup_{\theta\in[-\tau,0]}\int_t^{t+0}(\lambda_0(\upsilon))d\upsilon\right)$, *and for any*

$t>s\ge t_0, -a_2(t-s)\le\int_s^t H(w)dw\le -a_1(t-s)+b$.

(H_5) $\sum_{i=0}^{N(t,t_0)}\ln\upsilon_i<(a_1T_a-a_2\tau)N(t,t_0)$, *where* $\upsilon_0=1$, $\upsilon_k=\max\{\mu_{1k}+\mu_{2k}e^{a_2\tau},1\}$, $k\in\mathbb{Z}_+, a_1T_a-a_2\tau>0$, $T_a>0$ *is the average dwell time between every implusive and* $N(t,t_0)$ *stands for the frequency of impulses during the interval* $(t_0,t]$.

Then, the System (2.1) is pth moment exponentially stable.

Proof. Let $x(t)=x(t;t_0,\phi)$ be the solution of system (2.1), by its formula and condition (H_2), we have

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$$D^{+}\mathbb{E}V\big(t,x(t)\big)\le \lambda_0(t)\mathbb{E}V\big(t,x(t)\big)+\sum_{i=1}^{n_0}\lambda_i(t)\mathbb{E}V\big(t-\tau_i(t),x(t-\tau_i(t))\big),$$

where D^{+} denotes the upper right-hand Dini derivative. Since for each *i*, we have $\mathbb{E}V\big(t-\tau_i(t),x(t-\tau_i(t))\big)\le \sup_{\theta\in[-\tau,0]}\mathbb{E}V\big(t+\theta,x(t+\theta)\big)$, define $\tilde{\lambda}_1(t)=\sum_{i=0}^{n_0}\lambda_i(t)$, we obtain

$$D^{+}\mathbb{E}V\big(t,x(t)\big)\le \lambda_0(t)\mathbb{E}V\big(t,x(t)\big)+\tilde{\lambda}_1(t)\sup_{\theta\in[-\tau,0]}\mathbb{E}V\big(t+\theta,x(t+\theta)\big).$$

By Lemma 2.1, we have $u(t)\le \bar{u}(t_0)e^{\int_{t_0}^{t}H_1(s)ds}$, $t\in[t_0,t_1)$, where

$(H_1)(t)=\lambda_0(t)+\tilde{\lambda}_1(t)\rho(t),\quad \rho(t)=\exp\Big(\sup_{\theta\in[-\tau,0]}\int_t^{t+\theta}\lambda_0(\upsilon)d\upsilon\Big).$ Following exactly the same inductive procedure as in the proof of Theorem 3.1, we obtain for any $t\in[t_0,+\infty)$,

$$\mathbb{E}\left\|x(t)\right\|^{p}\le \frac{c_2}{c_1}e^{cT_aN_0+b}\left\|\phi\right\|^{p}e^{-(a_1-c)(t-t_0)}.$$

Remark 3.5 Corollary 3.1 decomposes the time-delay dependence of the system into a finite number of independent time-varying delay terms $\mathbb{E}V\big(t-\tau_i(t),\phi(-\tau_i)\big)(i=1,2,\ldots,n_0)$, which is more consistent with complex time-delay distribution scenarios such as multi-dimensional delays and piecewise delays in practical engineering and physical systems, significantly enhancing the applicability of the conclusion to non-uniform delay systems.

Examples

Example 4.1. *We give the following impulsive stochastic functional differential equation:*

$$\begin{cases} dx(t)=\left[\frac{1}{2}+\frac{3}{4}\sin(t)\right]x(t)dt+\sqrt{\frac{1}{50\pi}\int_{-\tau}^{0}x^{2}(t+\theta)d\theta dw(t)}, & t\ne t_k, t\ge 0,\\ x(t_k)=0.6x(t_k^{-})+0.01x(t_k^{-}+\theta), & k\in\mathbb{Z}^{+},\\ x(\theta)=\phi(\theta), & -\tau\le\theta\le 0, \end{cases}\tag{4.1}$$

where $\tau=\frac{\pi}{2},\phi(\theta)=10.$ *Let* $V\big(t,x(t)\big)=x^{2}(t)$, *then*

$$\begin{aligned}\mathbb{E}\mathcal{L}V(t,x_t)&=\left[-1+\frac{3}{2}\sin(t)\right]\mathbb{E}x^{2}(t)+\frac{1}{50\pi}\int_{-\tau}^{0}\mathbb{E}x^{2}(t+\theta)d\theta\\ &\le\left[-1+\frac{3}{2}\sin(t)\right]\mathbb{E}V\big(t,x(t)\big)+\frac{1}{100}\sup_{-\tau\le\theta\le 0}\mathbb{E}V\big(t+\theta,x(t+\theta)\big).\end{aligned}\tag{4.2}$$

In addition, when $t=t_k$, *we have*

$$\begin{aligned}\mathbb{E}V\big(t_k,x(t_k)\big)&=\mathbb{E}\left[0.6x(t_k^{-})+0.01x(t_k^{-}+\theta)\right]^{2}\\ &\le 2\times 0.6^{2}\times\mathbb{E}x^{2}(t_k^{-})+2\times 0.01^{2}\times\mathbb{E}x^{2}(t_k^{-}+\theta)\\ &\le 0.72\mathbb{E}V\big(t_k^{-},x(t_k^{-})\big)+0.0002\sup_{-\tau\le\theta\le 0}\mathbb{E}V\big(t_k^{-}+\theta,x(t_k^{-}+\theta)\big).\end{aligned}\tag{4.3}$$

We let $\lambda_1(t)=-1+\frac{3}{2}\sin(t),\ \lambda_2(t)=\frac{1}{100},\ \mu_{1k}=0.71,\ \mu_{2k}=0.0002,\ t_k-t_{k-1}=5,\ p=2,$ *and* $c_1=c_2=1,$ *then* $\rho(t)=\exp\left(\sup_{\theta\in\left[\frac{\pi}{2},0\right]}\int_t^{t+\theta}\big(\lambda_1(\upsilon)\big)d\upsilon\right)\le e^{\frac{\pi}{2}+\frac{3}{2}}\approx 21.56,\ H(t)=\lambda_1(t)+\lambda_2(t)\rho(t)$ $\le -0.7844+\frac{3}{2}\sin t,$ *and for any* $t>s\ge t_0,\ -2.2844(t-s)\le\int_s^t H(w)dw\le -0.7844(t-s)+3.$

Thus $a_1=0.7844,\ a_2=2.2844,\ b=3,\ \upsilon_k=\max\left\{\mu_{1k}+\mu_{2k}e^{a_2\tau},1\right\}=1,$ *then* $\sum_{i=0}^{N(t,t_0)}\ln\upsilon_i=0$ $<(a_1T_a-a_2\tau)N(t,t_0)\approx 3337N(t,t_0).$ *By Theorem* 3.1, *we know that system* (4.1) *is exponentially stable.*

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Remark 4:1 The function 1(t) $\lambda_1(t)$ is positive on some intervals, indicating that the instantaneous terms coefficient is not strictly negative. Hence, existing results in the literature [10,11] do not apply to this case.

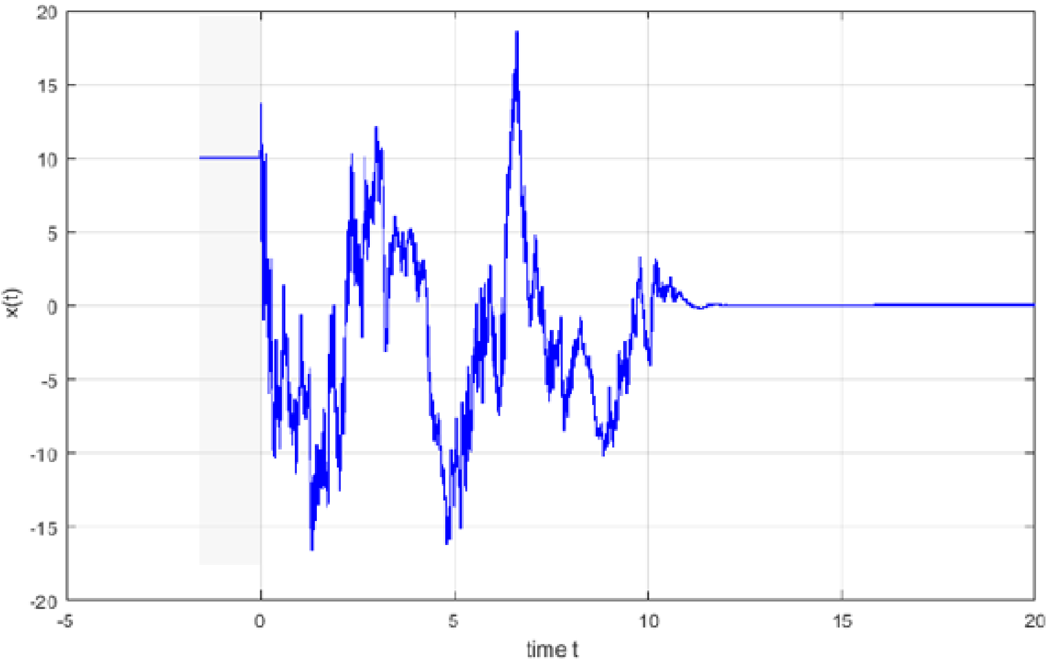

Figure 1: The trajectory of state x(t) of system (4.1).

Conclusion

This paper establishes a new generalized Halanay-type inequality for the analysis of exponential stability of a class of impulsive stochastic functional differential equations with time-varying delays. Unlike classical results, the proposed inequality does not require coefficient functions to maintain fixed sign patterns or satisfy pointwise dominance conditions between instantaneous and delayed terms. This generalization significantly extends the applicability of the Halanay method to systems where the instantaneous term coefficient may change signs or even be locally dominated by the delay term coefficient, providing a more versatile theoretical framework for stability analysis of impulsive stochastic systems with delays. Future work may focus on extending these results to systems with infinite delay structures or Markovian switching parameters.

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