



Generalized Pizzetti Formula for the Maximal Operator Over the Unit Sphere in \mathbb{R}^n

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Abstract

We derive a Taylor's expansion with an explicit remainder for the spherical average over the unit sphere in \mathbb{R}^n , which is a generalization of the famous Pizzetti formula. Furthermore, we introduce an innovative technique applicable for examining the boundedness of specific integral operators. We illustrate our method by implementing it to furnish an intuitive proof of the boundedness of the well-known spherical maximal operator on $L^p(\mathbb{R}^n)$ using the neutrix calculus due to Van Der Corput, delta sequences as well as the Hardy-Littlewood maximal inequality.

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Background and Preliminaries

For $n \geq 2$, let \mathbb{R}^n be the n -dimensional Euclidean space and \mathbb{S}^{n-1} be the unit sphere of \mathbb{R}^n equipped with the surface measure $d\sigma$. Stein's spherical maximal operator [2] is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \left| \int_{\mathbb{S}^{n-1}} f(x-r\theta) d\sigma(\theta) \right|,$$

for $f \in L^p(\mathbb{R}^n)$, $\frac{1}{n} \leq p \leq \infty$. The operator $\mathcal{M}f$ is shown to be L^p -bounded for $p > \frac{n}{n-1}$ and $n \geq 2$. The case $n \geq 3$ was established by Stein, and Bourgain [4] demonstrated the case $n = 2$ with a complicated proof.

Let $\mathcal{D}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions [1] (or so-called the Schwartz space of test functions) with compact support in \mathbb{R}^n . The distribution $\delta(x-y)$ is defined over the space $\mathcal{D}(\mathbb{R}^n)$ as

$$\int_{\mathbb{R}^n} \phi(x) \delta(x-y) dx = \phi(y), \quad y \in \mathbb{R}^n,$$

where $\phi \in \mathcal{D}(\mathbb{R}^n)$.

It follows from [3, 5] that

$$\delta_k(x) = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} e^{-kx^2}, \quad x \in \mathbb{R},$$

is a delta sequence of \mathbb{R} , which implies that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \phi(x) \delta_k(x) dx = \phi(0),$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

The Hardy-Littlewood maximal operator M used in analysis is defined as

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy,$$

where f is a locally integrable function of \mathbb{R}^n and $|B(x,r)|$ denotes the volume of the ball $B(x,r)$ of any radius r at x . Let V_n be the volume of the unit ball. Then

$$A_r f(x) = \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x-r\theta) d\sigma(\theta) = \frac{1}{\sigma(\mathbb{S}^{n-1})} \mathcal{M}_r f(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$ and

$$\mathcal{M}_r f(x) = \int_{\mathbb{S}^{n-1}} f(x-r\theta) d\sigma(\theta).$$

Clearly, $A_r f(x)$ denotes the average value of $f(x)$ over the surface of ball $B(x,r)$ in \mathbb{R}^n . In particular, we obtain the well-known Pizzetti formula by setting $x = 0$. The neutrix limit due to Van Der Corput [7] is a method for discarding of unwanted infinite quantities from asymptotic expansions with wide applications in nonlinear operations of distributions and extending special functions [8]. Motivated by the generalization of Pizzetti formula, we shall prove the boundedness of the spherical maximal operator \mathcal{M} on $L^p(\mathbb{R}^n)$ using the neutrix limit and a distributional argument.

Taylor's expansion for $A_r f(x)$

Theorem 2. Let f be a Schwartz function and $r \geq 0$. Then

$$A_r f(x) = \sum_{j=0}^k \frac{1}{2^j j! n(n+2)\dots(n+2j-2)} \Delta^j f(x) r^{2j} + r^{2k+2} \sum_{|i|=2k+2} \frac{1}{i! \sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \partial^i f(x-\gamma r\theta) \theta^i d\sigma(\theta).$$

In particular, $A_0 f(x) = f(x)$.

$$|B(x,r)| = V_n r^n = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)} r^n,$$

$$V_n = \frac{1}{n} \sigma(\mathbb{S}^{n-1}),$$

where $\sigma(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in \mathbb{R}^n .

Theorem 1. (Hardy-Littlewood's inequality see [6]) For $n \geq 1$, $1 < p \leq \infty$, and $f \in L^p(\mathbb{R}^n)$, there is a constant $C_{p,n}$ such that

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|f\|_{L^p(\mathbb{R}^n)}.$$

In the following, we deduce a Taylor expansion with respect to r for the operator

Proof. Let $i = (i_1, i_2, \dots, i_n)$ be an n -tuple of nonnegative integers. We define

$$|i| = i_1 + i_2 + \dots + i_n, \quad i! = i_1! i_2! \dots i_n!,$$

$$x^i = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n},$$

$$\partial^i f = \partial_1^{i_1} \dots \partial_n^{i_n} f = \frac{\partial^{|i|} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}.$$

Using Taylor's formula for $\theta \in \mathbb{S}^{n-1}$, we have

$$f(x-r\theta) = f(x) + \sum_{|i|=1} \frac{\partial^i f(x)}{i!} (-r\theta)^i + \dots + \sum_{|i|=2j} \frac{\partial^i f(x)}{i!} (-r\theta)^i + \sum_{|i|=2j+1} \frac{\partial^i f(x)}{i!} (-r\theta)^i + \sum_{|i|=2j+2} \frac{\partial^i f(x-\gamma r\theta)}{i!} (-r\theta)^i$$

for $j = 0, 1, \dots$ and $\gamma \in (0, 1)$.

Clearly,

$$r^{2j+1} \sum_{|i|=2j+1} \frac{\partial^i f(x)}{i!} \int_{\mathbb{S}^{n-1}} (-\theta)^i d\sigma(\theta) = 0$$

due to the cancellations over the unit sphere \mathbb{S}^{n-1} . Therefore,

$$\begin{aligned}
A_r f(x) &= f(x) + r^2 \sum_{|i|=2} \frac{\partial^i f(x)}{i!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} (-\theta)^i d\sigma(\theta) \\
&+ \dots + r^{2j} \sum_{|i|=2j} \frac{\partial^i f(x)}{i!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} (-\theta)^i d\sigma(\theta) \\
&+ r^{2j+2} \sum_{|i|=2j+2} \frac{1}{i!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \partial^i f(x - \gamma r \theta) (-\theta)^i d\sigma(\theta).
\end{aligned}$$

Thus,

$$\begin{aligned}
A_r f(x) &= f(x) + r^2 \sum_{|i|=1} \frac{\partial^{2i} f(x)}{(2i)!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \theta^{2i} d\sigma(\theta) \\
&+ \dots + r^{2j} \sum_{|i|=j} \frac{\partial^{2i} f(x)}{(2i)!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \theta^{2i} d\sigma(\theta) \\
&+ r^{2j+2} \sum_{|i|=j+1} \frac{1}{(2i)!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \partial^{2i} f(x - \gamma r \theta) \theta^{2i} d\sigma(\theta).
\end{aligned}$$

Applying the following formulas from [9]

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \theta^{2i} d\sigma(\theta) &= \frac{2\Gamma\left(\frac{1}{2} + i_1\right)\Gamma\left(\frac{1}{2} + i_2\right)\dots\Gamma\left(\frac{1}{2} + i_n\right)}{\Gamma\left(|i| + \frac{n}{2}\right)}, \\
\Gamma\left(\frac{1}{2} + i_1\right) &= \frac{(2i_1)!\sqrt{\pi}}{4^{i_1} i_1!}, \quad \Gamma\left(\frac{1}{2} + i_2\right) = \frac{(2i_2)!\sqrt{\pi}}{4^{i_2} i_2!}, \dots, \Gamma\left(\frac{1}{2} + i_n\right) = \frac{(2i_n)!\sqrt{\pi}}{4^{i_n} i_n!} \\
\Delta^j &= \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)^j = \sum_{|i|=j} \frac{j!}{i_1! \dots i_n!} \left(\frac{\partial^2}{\partial x_1^2}\right)^{i_1} \dots \left(\frac{\partial^2}{\partial x_n^2}\right)^{i_n},
\end{aligned}$$

we arrive at

$$\begin{aligned}
&r^{2j} \sum_{|i|=j} \frac{\partial^{2i} f(x)}{(2i)!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \theta^{2i} d\sigma(\theta) \\
&= r^{2j} \sum_{|i|=j} \frac{\partial^{2i} f(x)}{(2i)!} \frac{1}{\sigma(\mathbb{S}^{n-1})} \frac{2\Gamma\left(\frac{1}{2} + i_1\right)\dots\Gamma\left(\frac{1}{2} + i_n\right)}{\Gamma\left(|i| + \frac{n}{2}\right)} \\
&= \frac{2\pi^{n/2}}{2^{2j} j! \sigma(\mathbb{S}^{n-1}) \Gamma\left(j + \frac{n}{2}\right)} \Delta^j f(x) r^{2j} \\
&= \frac{\Gamma(n/2)}{2^{2j} j! \Gamma\left(j + \frac{n}{2}\right)} \Delta^j f(x) r^{2j}.
\end{aligned}$$

This implies that

$$A_r f(x) = \Gamma(n/2) \sum_{i=0}^j \frac{1}{2^{2i} i! \Gamma\left(i + \frac{n}{2}\right)} \Delta^i f(x) r^{2i} + r^{2j+2} \sum_{|i|=2j+2} \frac{1}{i! \sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \partial^i f(x - \gamma r \theta) \theta^i d\sigma(\theta)$$

In particular, we derive that for $x = 0$

$$A_r f(0) = \Gamma(n/2) \sum_{i=0}^j \frac{1}{2^{2i} i! \Gamma\left(i + \frac{n}{2}\right)} \Delta^i f(0) r^{2i} + r^{2j+2} \sum_{|i|=2j+2} \frac{1}{i! \sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \partial^i f(x - \gamma r \theta) \theta^i d\sigma(\theta).$$

which is the well-known Pizzetti formula [1] by noting that

$$\frac{\Gamma(n/2)}{2^{2i} i! \Gamma\left(i + \frac{n}{2}\right)} = \frac{1}{2^{2i} i! n(n+2)\dots(n+2i-2)}.$$

This completes the proof.

$\psi(x, x) = 1$ for $x \in \mathbb{R}$, we have

Boundedness of the operator \mathcal{M}

Theorem 3. Let f be a Schwartz function and $p > \frac{n}{n-1}$ for $n \geq 2$. Then \mathcal{M} is a bounded operator over the space $L^p(\mathbb{R}^n)$ in the sense of neutrix limit, and

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \approx C(p, n) \|f\|_{L^p(\mathbb{R}^n)},$$

where the constant $C(p, n) > 0$ depends on p and n .

Proof. Let $f \in \mathcal{D}(\mathbb{R}^n)$ and $r > 0$. Then $\mathcal{M}_r f(x)$ defined in Equation (1.1) is a $C^\infty(\mathbb{R}^+)$ function of r with compact support for every $x \in \mathbb{R}^n$. Given a $C^\infty(\mathbb{R}^2)$ function $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$ with

$$\mathcal{M}f(x) = \int_0^\infty \psi(r, t) \left(\int_{\mathbb{S}^{n-1}} f(x - t\theta) d\sigma(\theta) \right) \delta(t - r) dt, \tag{3.1}$$

by noting that $\psi(r, t) \int_{\mathbb{S}^{n-1}} f(x - t\theta) d\sigma(\theta)$

is a function of t in $\mathcal{D}(\mathbb{R}^+)$. In particular, we choose $\psi(r, t) = 1_{[r, r+\eta)}(t)$, $\eta > 0$, then equation (3.1) would take the form

$$\mathcal{M}_r f(x) = \int_r^{r+\eta} \left(\int_{\mathbb{S}^{n-1}} f(x - t\theta) d\sigma(\theta) \right) \delta(t - r) dt. \tag{3.2}$$

Using the delta sequence given above, we define

$$\mathcal{M}_r^k f(x) = \int_r^{r+\eta} \left(\int_{\mathbb{S}^{n-1}} f(x - t\theta) d\sigma(\theta) \right) \delta(t - r) dt. \tag{3.3}$$

for $k = 1, 2, \dots$

Making the variable change $y = t\theta$ with $\theta \in \mathbb{S}^{n-1}$, we get

$$\mathcal{M}_r^k f(x) = \int_{r \leq |y| < r+\eta} f(x - y) \frac{\delta_k(|y| - r)}{|y|^{n-1}} dy. \tag{3.4}$$

From Hölder's inequality, we come to

$$\begin{aligned} |\mathcal{M}_r^k f(x)| &\leq \left(\int_{r \leq |y| < r+\eta} f(x-y) \frac{|f(x-y)|^p}{|y|^{p(n-1)}} dy \right)^{1/p} \\ &\quad \times \left(\int_{r \leq |y| < r+\eta} \delta_k^q (|y|-r) dy \right)^{1/q}, \end{aligned} \quad (3.5)$$

where $1/p + 1/q = 1$ with $p, q > 1$.

By switching to polar coordinates, we have

$$\begin{aligned} \left(\int_{r \leq |y| < r+\eta} \delta_k^q (|y|-r) dy \right)^{1/q} &= \left(\int_r^{r+\eta} \delta_k^q (t-r) t^{n-1} dt \right)^{1/q} \\ &= \left(\int_0^\eta \delta_k^q (t) (t+r)^{n-1} dt \right)^{1/q} \\ &= \left(\int_0^\infty \delta_k^q (t) (t+r)^{n-1} dt \right)^{1/q}. \end{aligned}$$

Noticing that

$$\begin{aligned} (t+r)^{n-1} &= \sum_{i=0}^{n-1} \binom{n-1}{i} r^{n-1-i} t^i, \quad \text{and} \\ \delta_k(t) &= \left(\frac{k}{\pi} \right)^{\frac{1}{2}} e^{-kt^2}, \quad t \in \mathbb{R} \end{aligned}$$

we have

$$\begin{aligned} &\left(\int_{r \leq |y| < r+\eta} \delta_k^q (|y|-r) dy \right)^{1/q} \\ &\leq \left(\sum_{i=0}^{n-1} \left(\frac{\binom{n-1}{i} r^{n-1-i}}{\pi^{q/2} q^{\frac{i+1}{2}}} \right) \left(\int_0^\infty e^{-t^2} t^i dt \right) k^{\frac{q-i-1}{2}} \right)^{1/q} \\ &\leq \sum_{i=0}^{n-1} \left(\frac{\binom{n-1}{i} r^{n-1-i}}{\pi^{q/2} q^{\frac{i+1}{2}}} \right)^{1/q} \left(\int_0^\infty e^{-t^2} t^i dt \right)^{1/q} k^{\frac{q-i-1}{2q}}, \end{aligned} \quad (3.6)$$

using the fact

$$(a_1 + a_2 + \dots + a_m)^\alpha \leq a_1^\alpha + a_2^\alpha + \dots + a_m^\alpha,$$

where $a_1, \dots, a_m \geq 0$ and $0 < \alpha \leq 1$.

Applying the neutrix limit to (3.6), we get, in the neutrix sense, that

$$\begin{aligned} & \mathcal{N} - \lim_{k \rightarrow \infty} \left(\int_{r \leq |y| < r+\eta} \delta_k^q (|y|-r) dy \right)^{1/q} \\ & \sim \mathcal{N} - \lim_{k \rightarrow \infty} \sum_{i=0}^{n-1} \left(\frac{\binom{n-1}{i} r^{n-1-i}}{\pi^{q/2} q^{\frac{i+1}{2}}} \right)^{1/q} \left(\int_0^\infty e^{-t^2} t^i dt \right)^{1/q} k^{\frac{q-i-1}{2q}}, \end{aligned}$$

where \mathcal{N} is the neutrix having domain $N = \{1, 2, \dots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$k^\lambda \ln^{m-1} k, \ln^m k, \quad (\lambda > 0, m = 1, 2, \dots)$$

and all functions of k that converge to zero in the normal sense as k tends to infinity [7].

Observe that the neutrix limit of the right-hand side (3.6) is zero unless $q \in \{2, 3, \dots, n\}$. Since p and q are conjugate numbers, we shall choose q so that p is minimized. By choosing $q = n$, we imply that

$$\begin{aligned} & \mathcal{N} - \lim_{k \rightarrow \infty} \left(\int_{r \leq |y| < r+\eta} \delta_k^n (|y|-r) dy \right)^{1/n} \\ & \sim \left(\frac{1}{\pi^{n/2} n^{\frac{n}{2}}} \right)^{1/n} \left(\int_0^\infty e^{-t^2} t^{n-1} dt \right)^{1/n} = \left(\frac{1}{\pi^{n/2} n^{\frac{n}{2}}} \right)^{1/n} \left(\frac{1}{2} \Gamma(n/2) \right)^{1/n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left(\int_{r \leq |y| < r+\eta} \frac{|f(x-y)|^p}{|y|^{p(n-1)}} dy \right)^{1/p} \leq \left(\frac{1}{r^{p(n-1)}} \int_{r \leq |y| < r+\eta} |f(x-y)|^p dy \right)^{1/p} \\ & = \left(\frac{2^{(n-1)p}}{(2r)^{p(n-1)}} \int_{r \leq |y| < r+\eta} |f(x-y)|^p dy \right)^{1/p} \\ & \leq 2^{n-1} \left(\frac{1}{(2r)^{p(n-1)}} \int_{|y| \leq 2r} |f(x-y)|^p dy \right)^{1/p}, \end{aligned}$$

where we choose η such that $0 < \eta < r$.

From $p = \frac{n}{n-1}$, we obtain that

$$\begin{aligned} & \left(\int_{r \leq |y| < r+\eta} \frac{|f(x-y)|^p}{|y|^{p(n-1)}} dy \right)^{1/p} \leq 2^{n-1} \left(\frac{1}{(2r)^n} \int_{|y| \leq 2r} |f(x-y)|^{\frac{n}{n-1}} dy \right)^{\frac{n-1}{n}} \\ & = 2^{n-1} \left(\frac{V_n}{V_n (2r)^n} \int_{|y| \leq 2r} |f(x-y)|^{\frac{n}{n-1}} dy \right)^{\frac{n-1}{n}} \\ & \leq 2^{n-1} V_n^{\frac{n-1}{n}} \left(M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}. \end{aligned}$$

Thus,

$$|\mathcal{M}_r f(x)| \sim C_n \left(M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

where

$$C_n = 2^{n-1} V_n^{\frac{n-1}{n}} \left(\frac{1}{\pi^{n/2} n^2} \right)^{1/n} \left(\frac{1}{2} \Gamma(n/2) \right)^{1/n}.$$

Hence,

$$\mathcal{M}f(x) = \sup_{r>0} |\mathcal{M}_r f(x)| \sim C_n \left(M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

which claims that

$$\|\mathcal{M}f(x)\|_{L^p(\mathbb{R}^n)} \approx C_n \left\| \left(M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \right\|_{L^p(\mathbb{R}^n)}.$$

Considering that for $p > \frac{n}{n-1}$ and using Theorem 1, we get

This completes the proof.

Conclusion

We have established the boundedness of Stein's spherical maximal function by employing the delta sequence via the method of negligible functions. The outcome in this new context aligns with the conventional result. It appears that the approach introduced here could potentially offer an initial estimate of the range of values for p where an operator is bounded, as it disregards any factors exhibiting infinite growth. In addition, we also have presented a Taylor's expansion with an explicit integral remainder for the function over the unit sphere in \mathbb{R}^n base on the surface integrals, Gamma functions and Laplace operators, which has never been investigated before.

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Competing interests

The authors declare that they have no competing interests.

Author contributions

C.L. drafted the manuscript. All authors conceived of the study, participated in its design and coordination, read and approved the final version.

Data Availability

No data were used to support this study.

References

- Gelfand IM, Shilov GE (1964) Generalized Functions. Academic Press, New York Vol. I.
- Stein EM (1976) Maximal functions: I. Spherical means, Proc Nat Acad Sci U.S.A. 73: 2174–2175.
- Li C, Li CP (2014) On defining the distributions δ^k and $(\delta')^k$ by fractional derivatives. Appl Math Comput 246: 502–513.
- Bourgain J (1986) Averages in the plane over convex curves and maximal operators. J Analyse Math 47: 69–85.
- Koh EL, Li C (1992) On the Distributions δ^k and $(\delta')^k$. Math Nachr 157: 243–248.
- Melas AD (2003) The best constant for the centered Hardy–Littlewood maximal inequality. Ann Math 157: 647–688.
- Van Der Corput JG (1959- 60) Introduction to the neutrix calculus. J Analyse Math 7: 291–398.
- Fisher B (1982) On defining the convolution of distributions. Math Nachr 106: 261– 269.
- Li C (2020) An example of the generalized fractional Laplacian on \mathbb{R}^n . Contemp Math Art 1: 215.