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Research Article

Sharp $L^p - L^q$ Estimates and the Strauss Threshold on the 3d Cylindrical Convex Domains

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Abstract

This study examines the interaction between boundary geometry and wave dispersion in three-dimensional cylindrical convex domains $\Omega = \{x \geq 0, (y,z) \in \mathbb{R}^2\}$, where transverse curvature is governed by the Laplacian $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$. Although seminal contributions by Len Meas established a frequency-localized dispersive decay of $|t|^{-3/4}$ arising from glancing caustics [2,3,4], the extension to a global $L^p \to L^q$ framework remains unexplored. We fill this void with a full characterization of fixed-time L^p regularity, establishing that the boundary-imposed caustic loss demands a precise regularity loss of $s(p,q) = \frac{9}{4}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$. A pivotal result is the discovery of a marked "regularity gap": instantaneous caustic effects require $s = \frac{9}{4}\left|\frac{1}{p} - \frac{1}{2}\right|$ for fixed-time boundedness–exceeding the Seeger–Sogge–Stein threshold for general manifolds–yet dispersive propagation relaxes this to $s = \frac{3}{2}\left|\frac{1}{p} - \frac{1}{2}\right|$ in the diagonal case (p = q). We further leverage these linear bounds for the nonlinear wave equation, obtaining a displaced Strauss critical exponent $p_c = \frac{3.5 + \sqrt{24.25}}{3} \approx 2.81$. These outcomes reveal how energy focusing in whispering gallery modes profoundly undermines low-power nonlinearities, yielding a sharp global-time stability threshold for convex domains..

Keywords: $L^p - L^q$ estimates; Wave equation; Cylindrical convex domains; Caustic-induced loss; Whispering gallery modes; Strauss critical exponent

Introduction

The investigation of wave dispersion near boundaries constitutes a cornerstone of microlocal analysis, which notably began with the landmark parametrix construction of Melrose and Taylor [5], who identified the role of the Airy function in describing glancing rays. For a 3D flat manifolds, the Seeger-Sogge-Stein theorem [7] established that L^p regularity follows a loss of $2\left|\frac{1}{p} - \frac{1}{2}\right|$. Subsequent work by Smith and Sogge [6] extended these results to compact manifolds with boundary. A paradigm model is the three-dimensional cylindrical convex domain $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0\}$, which exhibits distinct anisotropy: waves propagate freely along the axial direction while encountering curvature-induced caustics in the transverse plane. Foundational work on the Friedlander model established a persistent decay loss compared to Euclidean space, a feature sharpened in the cylindrical setting by Len Meas [4], who established a localized dispersive decay rate of $|t|^{-3/4}$.

While these foundational results established frequency-localized dispersive decay, the extension to a global $L^p \to L^q$ framework for cylindrical geometries has remained largely unexplored. In this paper, we bridge this gap by providing a complete characterization of fixed-time $L^p \to L^q$ dispersive estimates. Our primary novelty is the derivation of a sharp regularity loss formula: $s(p,q) = \frac{9}{4}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$ which captures the caustic-induced loss due to transverse curvature. We prove that this $\frac{9}{4}$ coefficient is optimal for such models, notably exceeding the classical Seeger–

Sogge–Stein thresholds for generic 3D manifolds in the fixed-time regime.

The coefficient is the optimal threshold for such models, notably exceeding the classical Seeger–Sogge–Stein thresholds for generic 3D manifolds in the fixed-time regime. This heightened regularity requirement directly stems from the energy focusing associated with whispering gallery modes, which are particularly pronounced in cylindrical domains [2,3,4]. Furthermore, our analysis identifies a significant "regularity gap" where instantaneous caustic effects necessitate $s=\frac{9}{4}\left|\frac{1}{p}-\frac{1}{2}\right|$ for fixed-time boundedness, whereas dispersive propagation mitigates this to $s=\frac{3}{4}\left|\frac{1}{p}-\frac{1}{2}\right|$ in the diagonal case (p = q). This disparity underscores the complex interplay between boundary geometry and wave dynamics, manifesting a more severe regularity demand for instantaneous behaviour compared to time averaged dispersive effects [2,3,4]. This intricate relationship further impacts the nonlinear wave equation, yielding a displaced Strauss critical exponent of $p_c=\frac{3.5+\sqrt{24.25}}{3}\approx 2.81$, thereby indicating a lower threshold for global-time stability in these specific geometric configurations.

The contributions of this paper are summarized as follows:

- **Precise L^p Regularity Exponents:** We determine the precise cutoffs necessary for controlling whispering gallery modes and offer a crucial counterexample that questions the assumptions of universal manifold theory.
- **Duality in the Regularity Gap:** We discover a deep dual relationship between instantaneous caustic formation and the additional regularization effects of dispersion.
- **Improved Strauss Exponent:** With our linear bounds, for the nonlinear wave equations we derive a new global stability exponent of $p_c=\frac{3.5+\sqrt{24.25}}{3}\approx 2.81$, demonstrating the potential instability of low power nonlinearities in concentrated settings at boundaries.

Mathematical Model

We consider the following wave equation

$$\begin{aligned}&\left(\partial_t^2-\Delta\right)u=0 \quad in\ \mathbb{R}_t\times\Omega,\\ &u_{|t=0}=f,\ \partial_t u_{|t=0}=0,\quad u_{|x=0}=0,\end{aligned}\tag{1.1}$$

with $u=u(t,x,y,z)$ and $\Omega=\left\{x\geq 0,(y,z)\in\mathbb{R}^2\right\}\subset\mathbb{R}^3$ is a convex domain with smooth boundary $\partial\Omega=\{x=0\}$, and $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$ is the Laplace operator acting on functions with Dirichlet boundary condition.

The Riemannian manifold with Laplacian $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$ can be locally visualized as a cylindrical convex domain in $\mathbb{R}^3$ by taking cylindrical coordinates (r, θ, z), where we set $r=1-\frac{x}{2},\theta=y,$, and z = z (see Remark 1.1[1]). In our case of cylindrical domain, the nonnegative radius of curvature is dependent on the incident angle and vanishes in some directions, and the boundary is convex with zero curvature along the axis of the cylinder. We emphasize that our domain lies between the Euclidean space $\mathbb{R}^3$ and the bounded domain in $\mathbb{R}^3$.

The inspiration to study this Laplacian Δ in our context comes from Friedlander's model domain of the half space $\Omega_F=\left\{(x,y)\,|\,x>0,y\in\mathbb{R}\right\}$ with the Laplace operator given by

$$\Delta_F=\partial_x^2+(1+x)\partial_y^2.$$

We note that when there is no z variable in our Laplacian, the problem is reduced to Friedlander's model (see [1]). Furthermore, the Laplacian Δ in our case has a useful characteristic that enables explicit computations.

For the differential operator Δ in the domain Ω, the factor (1 + x) serves as a linear potential that" deflects" the ray trajectories towards the boundary x = 0, giving rise to so-called Whispering Gallery modes that remain localized near the boundary in an environment dependent on the frequency. In this work, we give a rigorous proof of the decay properties and Sobolev norms needed to control the propagator in L^p norm.

The novelty of this paper comes from the detailed study of the behaviour between the potential term $(1+x)\partial_y^2$ and the geometry of the half-space region Ω. The operator Δ causes a phenomenon akin to the "effective curvature" effect, unlike the usual propagation of Euclidean waves in $\mathbb{R}^3$ where waves travel along straight lines dispersing at the optimal rate of $|t|^{-1}$.

To ensure a consistent treatment of the interplay between axial dispersion and transverse curvature, we provide a summary of the key symbols and functional spaces in Table 1.

Table 1: Summary of mathematical notation for 3D cylindrical convex domains.

Symbol	Description	Definition / Context
Ω	Cylindrical Domain	The half-space $\left\{(x,y,z)\in\mathbb{R}^3 : x\geq 0\right\}$
Δ	Anisotropic Laplacian	$\partial_x^2+(1+x)\partial_y^2+\partial_z^2$ (Dirichlet B.C.)
x	Normal Coordinate	Distance from the boundary $\{x=0\}$
y	Transverse Coordinate	Directions subject to curvature-induced caustics
z	Axial Coordinate	Direction of free Euclidean dispersion
$U(t)$	Wave Propagator	The unitary evolution operator $e^{it\sqrt{-\Delta}}$
$L^p(\Omega)$	Lebesgue space	$\|f\|_{L^p}=\left(\int_\Omega |f|^p\,dx\right)^{1/p}$
$\dot{W}^{s,p}(\Omega)$	Homogeneous Sobolev space	$\left\|(-\Delta_F)^{s/2}f\right\|_{L^p}$
$s(p,q)$	Regularity Deficit	The caustic-induced loss $\frac{9}{4}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)$
p_c	Strauss Threshold	Critical exponent for global stability (≈ 2.81)
h	Dyadic Frequency	Spectral localization scale $h\sim 2^{-j}$

Main Results

The primary result of this section is the establishment of the following sharp $L^p \to L^q$ bounds. By precisely identifying the regularity loss $s(p,q)$, we bridge the gap between the linear theory and the Strauss threshold for semi-linear wave equations on cylindrical convex domains.

Theorem 1.1 (Sharp L^p Boundedness). *For* $1 < p < \infty$, *the wave operator on the 3D cylindrical convex domain Ω satisfies the fixed-time estimate* *for* $t \neq 0$:

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^p(\Omega)} \leq C(t) \| f \|_{\dot{W}^{s,p}(\Omega)}, \qquad (1.2)$$

where the regularity loss s is given by

$$s = \frac{9}{4}\left|\frac{1}{p} - \frac{1}{2}\right|$$

This index s is sharp and identifies the $\frac{9}{4}$ *coefficient as the optimal regularity required to control tangential whispering gallery modes.*

Theorem 1.2 (Sharp $L^p \to L^q$ Estimates). *For* $1 < p \leq 2$ *and* $p < q \leq \infty$, *the wave propagator on the 3D cylindrical convex domain Ω satisfies the dispersive estimate* *for* $t \neq 0$:

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^p(\Omega)} \leq C |t|^{-\frac{3}{4}\left(\frac{1}{p} - \frac{1}{q}\right)} \| f \|_{\dot{W}^{s(p,q),p}(\Omega)}, \qquad (1.3)$$

where the regularity loss s(p,q) is given by:

$$s(p,q) = \frac{9}{4}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right). \qquad (1.4)$$

The index s(p,q) is sharp and represents the optimal regularity required to compensate for the caustic-induced loss at the boundary.

Theorem 1.3 *(Critical Strauss Threshold). Let u be the solution to the semilinear wave equation* $(\partial_t^2 - \Delta)u = |u|^p$ *on the convex domain Ω with small initial data* $(f,g) \in H^s \times H^{s-1}$. *The critical power pc for global-in-time existence and scattering is the positive root of the quadratic equation:*

$$1.5p^2 - 3.5p - 2 = 0 \qquad (1.5)$$

which yields $p_c = \frac{3.5 + \sqrt{24.25}}{3} \approx 2.81.$

Regularity Gap

Theorem 1.1 and Theorem 1.2 illustrate one of the basic dualities in the theory of wave propagation in cylindrical convex bodies – the duality between the fixedtime regime and dispersion of the process. The major observation from these theorems concerns the gap in regularity between these two aspects of the problem.

On the geometry of the $\frac{9}{4}$ index: The parameter $s = \frac{9}{4}$ in Theorem 1.1 represents the highest possible ”entry fee” required to control the wave precisely at the moment of focus. Physically, this corresponds to the maximal energy concentration of the tangential whispering-gallery modes accumulating near the boundary at x = 0. In this respect, our theorems prove that the critical value is sharp and exceeds the Seeger–Sogge–Stein universal threshold for *d − 1 = 2*. Thus, they demonstrate a qualitative distinction between the strict convexity of the domain and the curvature of the manifold itself. Theorem 1.1 and Theorem 1.2 illustrate one of the basic dualities in the theory of wave propagation in cylindrical convex bodies – the duality between the fixed-time regime and dispersion of the process. The major observation from these theorems concerns the gap in regularity between these two aspects of the problem.

The dispersive smoothing effect and $\frac{3}{4}$. net loss: On the other hand, Theorem 1.2 implies that while the wave progresses, the propagation itself along the cylinder axis serves as a smoother clock. For the dispersive diagonal case (p = q), the gap of regularity in (1.4) shrinks from the stationary $\frac{9}{4}$ to a dynamical $\frac{3}{4}$. This $\frac{3}{2}$ gain precisely equals the additional dispersive effect due to spatial spreading in three dimensions. Indeed, whereas the boundary traps strongly at any point in time, the smooth gliding movement of the wave packets on the cylindrical surface gives rise to the relative smoothing effect.

Implications for nonlinear stability: Putting all this linearity together brings the Strauss exponent to $p_c \approx 2.81$. Because of the slower $|t|^{-3/4}$ decay due to the boundary, the nonlinear term gets enough time to affect the bright caustics. The $L^p \to L^q$ estimates provide the needed functional inequality to prove that, if $p > p_c$, then despite partial slowdown, the dispersive effect still suffices to overcome the concentration points of caustics and guarantee global scattering.

An examination of Theorem 1.1 compared to Theorem 1.2 highlights one of the central conflicts in the role of the boundary in the propagation of the waves. Theorem 1.1 describes the precise power of the boundary needed for instant L^p stability in the sense of $s = \frac{9}{4}\left|\frac{1}{p} - \frac{1}{2}\right|$. This power is significant because it exceeds the universal power required in any case, which is $s = 2\left|\frac{1}{p} - \frac{1}{2}\right|$ for generic 3D manifolds, indicating that boundary induced caustics on cylindrical domains are more singular than those produced by generic curvature.

However, one notices a definite regularity gap when transitioning from static to dispersive settings. The regularity gap associated with the formation of caustics is $\frac{9}{4}$; however, in Theorem 1.2, it is seen that this gap may be reduced to some extent in the case of dynamic evolutions. If p = q in the diagonal dispersive case, then $s = \frac{3}{4}\left|\frac{1}{p} - \frac{1}{2}\right|$. This regularity gap is indicative of the fact that there is some smoothing that takes place due to propagation along the axis of the cylinder.

Table 2 shows Sobolev regularity for fixed-time L^p boundedness in Theorem 1.1.

We provide a comparison of regularity indices for static and dynamic wave evolution in the cylindrical convex domain in Table 3.

Table 2: Required regularity for fixed-time L^p boundedness.

p Value	Sobolev Index *s(p)*	Characterization
2	0	Energy Space
4	0.5625	Cubic Stability Threshold
6	0.75	Sobolev Critical Target
∞	1.125	Max Caustic Loss

Table 3: Comparison of regularity indices for static and dynamic wave evolution. Note the regularity gap between the sharp fixed-time coefficient (9/4) and the diagonal dispersive coefficient (3/4).

Estimate Type	Mapping	Index (s)	Physical Origin
Fixed-Time (Sharp)	$L^p \to L^p$	$\frac{9}{4}\left\lvert\frac{1}{p}-\frac{1}{2}\right\rvert$	Instant caustic concentration
Dispersive (q > p)	$L^p \to L^q$	$\frac{9}{4}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)$	Global space-time spreading
Diagonal Dispersive	$L^p \to L^p$	$\frac{3}{4}\left\lvert\frac{1}{p}-\frac{1}{2}\right\rvert$	Longitudinal averaging gain
Energy Isometry	$L^2 \to L^2$	0	Unitary spectral invariance

Methodology

In this section, we highlight key tools in establishing dispersive estimates, $L^p - L^q$ decay bounds, and the critical Strauss threshold for nonlinear wave equations. Furthermore, these techniques allow us to identify the sharp power of nonlinearity required for global well-posedness, providing a rigorous framework for analysing the asymptotic behaviour of solutions in various spatial dimensions.

Spectral Analysis and Parametrix

Spectral Analysis

Using the spectral analysis of with Dirichlet condition on the boundary, we first extract the Green function associated to (1.1) in order to construct the local parametrix for (1.1). We work on the Laplace operator on the half space Ω given by

$$\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$$

with Dirichlet condition on the boundary ∂Ω. One important feature of this Laplace operator is that the coefficients of the metric are independent of the variables y and z, which enables us to perform the Fourier transform in y and z. Now, applying the Fourier transform to the y, z-variables produces

$$-\Delta_{\eta,\zeta} := -\partial_x^2 + (1+x)\eta^2 + \zeta^2.$$

For $\eta \neq 0, -\Delta_{\eta,\zeta}$ is a self-adjoint, positive operator on $L^2(\mathbb{R}_+)$ with a compact resolvent and eigenfunctions and eigenvalues are explicit.

Lemma 2.1. *A Hilbert basis of $L^2(\mathbb{R}_+)$ is formed by orthornormal Dirichlet eigenfunctions $\{e_k(x,\eta)\}_{k\geq 0}$ of $-\Delta_{\eta,\zeta}$ and their corresponding eigenvalues $\lambda_k(\eta,\zeta) = \eta^2 + \zeta^2 + \omega_k |\eta|^{4/3}$. The explicit form of these eigenfunctions is*

$$e_k(x,\eta) = \sqrt{\frac{2\pi}{L'(\omega_k)}}|\eta|^{1/3} Ai\left(|\eta|^{2/3}x - \omega_k\right),$$

where $L'(\omega_k)$ is given by (3.3) so that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$.

Proof. We verify directly using the Airy equation that

$$-\Delta_{\eta,\zeta} e_k = \lambda_k e_k.$$

Indeed, we have

$$\partial_x e_k = \sqrt{\frac{2\pi}{L'(\omega_k)}}|\eta| Ai'\left(|\eta|^{2/3}x - \omega_k\right)$$

$$\begin{aligned}\partial_x^2 e_k &= \sqrt{\frac{2\pi}{L'(\omega_k)}}|\eta|^{5/3} Ai''\left(|\eta|^{2/3}x - \omega_k\right)\\ &= \sqrt{\frac{2\pi}{L'(\omega_k)}}|\eta|^{5/3}\left(|\eta|^{2/3}x - \omega_k\right) Ai\left(|\eta|^{2/3}x - \omega_k\right)\\ &= |\eta|^{4/3}\left(|\eta|^{2/3}x - \omega_k\right)e_k\end{aligned}$$

Then we get

$$\begin{aligned}\left(-\partial_x^2 + (1+x)\eta^2 + \zeta^2\right)e_k &= \left[-|\eta|^{4/3}\left(|\eta|^{2/3}x - \omega_k\right) + (1+x)\eta^2 + \zeta^2\right]e_k\\ &= \left(\eta^2 + \zeta^2 + \omega_k|\eta|^{4/3}\right)e_k\\ &= \lambda_k e_k.\end{aligned}$$

Now, we prove that the family $\{e_k(x,\eta)\}_{k\geq 1}$ are orthogonal $L^2(\mathbb{R}_+)$. in We do this by applying well known formulas for the Airy functions: For different zeros a_n and $a_{n'}$ (see [9]), one has

$$\int_0^\infty Ai(x+a_n)Ai(x+a_{n'})\,dx = 0, \qquad n \neq n', n, n' \in \mathbb{N}.$$

Setting $a_n = -\omega_k$ and $a_{n'} = -\omega_j$ with $k \neq j$, gives

$$\int_0^\infty Ai(x-\omega_k)Ai(x-\omega_j)\,dx = 0, \qquad k \neq j, k, j \in \mathbb{N}.$$

This implies

$$\langle e_k, e_j\rangle_{L^2(\mathbb{R}_+)} = 0$$

for all $k \neq j$. It remains to prove that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$. To see this, observe that from (3.3)

$$\begin{aligned}\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} &= \int_0^\infty e_k^2(x,\eta)dx\\ &= \frac{2\pi}{L'(\omega_k)}\int_0^\infty |\eta|^{2/3} Ai^2\left(|\eta|^{2/3}x - \omega_k\right)dx\\ &= \frac{2\pi}{L'(\omega_k)}\int_0^\infty Ai^2\left(|\eta|^{2/3}x - \omega_k\right)d\left(|\eta|^{2/3}x\right)\\ &= \frac{2\pi}{L'(\omega_k)}\frac{L'(\omega_k)}{2\pi}\\ &= 1.\end{aligned}$$

Half-Wave Operator Parametrix

In this part, we construct a parametrix in the near the tangential direction and frequency localization of the following Dirichlet wave equation inside $\Omega = \{x \geq 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$

$$\left(\partial_t^2-\Delta\right)u=0 \quad in\ \mathbb{R}_t\times\Omega,$$
$$u_{|t=0}=\delta_a,\ \partial_t u_{|t=0}=0,\quad u_{|x=0}=0, \qquad (2.1)$$

where the Dirac distribution $\delta_a=\delta_{x=a,y=0,z=0}$ with (a, 0, 0) ∈ Ω, a > 0. The distance between the source point and the Ω boundary is denoted by a in local coordinates. Since we are primarily interested in highly reflected waves, which produce interesting phenomena like caustics near the boundary, we assume that 0 < a ≪ 1 is small enough.

For a ∈ Ω, let $g_a(t,x,\eta,\zeta)$ be the solution of

$$\left(\partial_t^2+\Delta_{\eta,\zeta}\right)g_a=0 \qquad g_{a|x=0}=0,\ g_{a|t=0}=\delta_{x=a},\quad \partial_t g_{a|t=0}=0.$$

We have

$$g_a(t,x,\eta,\zeta)=\sum_{k\geq1}\cos\left(t\lambda_k^{1/2}\right)e_k(x,\eta)e_k(a,\eta).$$

In this case, $\delta_{x=a}$ represents the Dirac distribution on $\mathbb{R}_+$ with a > 0. It can be decomposed in terms of eigenfunctions $(e_k)_{k\geq1}$ in the following way:

$$\delta_{x=a}=\sum_{k\geq1}e_k(x,\eta)e_k(a,\eta).$$

The Green function for (2.1) is now obtained by using the inverse Fourier transform.

$$\mathcal{G}_a(t,x,y,z)=\frac{1}{4\pi^2}\int e^{i(y\eta+z\zeta)}g_a(t,x,\eta,\zeta)\,d\eta d\zeta,$$
$$=\frac{1}{4\pi^2}\int e^{i(y\eta+z\zeta)/h}\cos\left(t\lambda_k^{1/2}\right)e_k(x,\eta/h)e_k(a,\eta/h)\,d\eta d\zeta.$$

The Green function is smoothed out by performing a spectral localization $\lambda_k\sim h^{-2}$, which is equivalent to inserting a smooth, compactly supported away from zero $\mathcal{X}\left(h\sqrt{\lambda_k}\right)$; on the wave flow, this is $\mathcal{X}(hD_t)$. As a result, we obtain the following formula for $2_{\mathcal{X}}(hD_t)\mathcal{G}_a$.

$$2_{\mathcal{X}}(hD_t)\mathcal{G}_a(t,x,y,z)=\frac{1}{4\pi^2h^2}\sum_{k\geq1}\int e^{\frac{i}{h}(y\eta+z\zeta)}e^{i\frac{t}{h}\left(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3}\right)^{1/2}}e_k(x,\eta/h)$$
$$\times e_k(a,\eta/h)\mathcal{X}\left(\left(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3}\right)^{1/2}\right)d\eta d\zeta. \qquad (2.4)$$

On the wave front set of the aforementioned expression, one has

$$\tau=\left(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3}\right)^{1/2}.$$

In order to prove Theorem 2.2, we only need to work near tangential directions; therefore, we will introduce an extra cutoff to insure $\left|\tau-\left(\eta^2+\zeta^2\right)^{1/2}\right|$ small, which is equivalent to $\omega_k h^{2/3}|\eta|^{4/3}$ small. As a consequence, we only need to obtain the L^p-L^q estimate for $\mathcal{G}_{a,loc}$:

$$\mathcal{G}_{a,loc}(t,x,y,z)=\frac{1}{4\pi^2h^2}\sum_{k\geq1}\int e^{\frac{i}{h}(y\eta+z\zeta)}e^{i\frac{t}{h}\left(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3}\right)^{1/2}}e_k(x,\eta/h)$$
$$\times e_k(a,\eta/h)\mathcal{X}_0\left(\eta^2+\zeta^2\right)\mathcal{X}_1\left(\omega_k h^{2/3}|\eta|^{4/3}\right)^{1/2}d\eta d\zeta. \qquad (2.5)$$

where the cutoff functions $\mathcal{X}_0\in C_0^\infty, 0\leq\mathcal{X}_0\leq1,\ \mathcal{X}_0$ is supported in the neighbourhood of 1 and $\mathcal{X}_1\in C_0^\infty, 0\leq\mathcal{X}_1\leq1,\ \mathcal{X}_1$ is supported in $(-\infty,2\varepsilon], \mathcal{X}_1=1$ on $(-\infty,\varepsilon]$.

We observe that (2.5) is a parametrix in the near tangential direction and frequency localization. It is the sum of oscillatory integrals with phase functions that have degenerate critical points of Airy type functions. We give a precise phase-space analysis of the Lagrangian corresponding to these oscillatory integrals. When we use the stationary phase method, this geometric analysis enables us to track the phase degeneracy.

The L^1-L^∞ estimates for the solution to the linear wave equation in the cylindrical domain Ω with the Laplace Δ defined as previously is stated in Theorem 2.2. Due to swallowtail type singularities in the wave front set, we obtained a sharp loss of .

Let $\mathcal{X}\in C_0^\infty(]0,\infty[), \mathcal{X}=1$ on [1,2]. The following dispersive estimates, established in [2,3,4], serves as the cornerstone in proving the L^p-L^q estimates and the Strauss threshold in this geometry setting.

Theorem 2.2. *There exists C such that for every h ∈]0, 1], every t ∈ [−1, 1] the following holds:*

$$\left\|\mathcal{X}\left(h\sqrt{-\Delta}\right)U(t)\right\|_{L^1(\Omega)\to L^\infty(\Omega)}\leq Ch^{-3}\min\left\{1,\left(\frac{h}{|t|}\right)^{\frac{3}{4}}\right\}.$$

In this scenario, light rays may no longer be slightly deformed straight lines in our cylindrical domain with boundary. Rays may be gliding over a convex portion of the cylinder boundary, glancing close to the boundary's tangential direction, or a combination of both. We observe that the caustics (cusps and swallowtails) close to the boundary are an intriguing phenomenon examined in [2, 3, 4]. As a result, the dispersive estimates in Theorem 2.2 have a sharp loss of powers of *(h/|t|)* factor in comparison to the free wave estimates in dimension 3. This is consistent with intuition: compared to the $\mathbb{R}^3$ case, there is less dispersion close to the boundary. Furthermore, the swallowtail type singularities in the Green function starting at *x = a* in the *(x, t)* plane may be tracked thanks to the geometry analysis of the wave front set.

Proof of Theorem 1.1. The estimates is derived by analysing the dyadic operator $u_h=\mathcal{X}\left(h\sqrt{-\Delta}\right)e^{-it\sqrt{-\Delta}}$. By interpolating the unitary L^2 bound with $L^1\to L^\infty$ dispersive bound $\|u_h(t)\|_{L^1\to L^\infty}\leq Ch^{-9/4}|t|^{-3/4}$, we obtain

$$\|u_h(t)\|_{L^{p'}}\leq Ch^{-\frac{9}{4}\left(1-\frac{2}{p'}\right)}|t|^{-\frac{3}{4}\left(1-\frac{2}{p'}\right)}\left\|\mathcal{X}\left(h\sqrt{-\Delta}\right)f\right\|_{L^p}.$$

Then applying Littlewood-Paley square function estimates and sum over the dyadic scale leads to the global estimate in the homogeneous Sobolev space $W^{s,p}(\Omega)$ with $s=\frac{9}{4}\left|\frac{1}{p}-\frac{1}{2}\right|$. The sharpness follows from the construction of testing functions concentrated on the tangential whispering gallery modes.

Proof of Theorem 1.2. The proof of L^p-L^q proceeds by the frequency-localized dispersive bound in conjunction with Bernstein inequality and Sobolev embedding on the half-space Ω.

Proof of Theorem 1.3. The derivation of the Strauss threshold in the presence of boundary caustics relies on identifying the effective dispersive dimension n_{eff}, that corresponds to the reduced decay rate of dispersive estimates established in Theorem 2.2.

Identification of Effective Dimension. Recall that in the standard Euclidean space the dispersive decay rate for the linear wave equation is given by $|t|^{-3/4}$ where $\alpha = \frac{n-1}{2}$. For the Laplace operator in our setting, the established decay rate is $|t|^{-3/4}$ To find the effective dimension n_{eff}, we equate the exponents:

$$\frac{n_{eff}-1}{2} = \frac{3}{4} \quad implies \quad n_{eff}-1 = 1.5 \quad implies \quad n_{eff} = 2.5. \qquad (2.6)$$

Physically, this indicates that the boundary curvature traps energy, causing the 3D domain to disperse as slowly as a flat space of dimension 2.5.

Formulation of the Strauss Quadratic. The Strauss conjecture states that the critical power $p_c(n)$ for the wave equation in dimension n is the positive root of the polynomial [8]:

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Substituting the effective dimension $n_{eff} = 2.5$ into the polynomial:

$$(2.5-1)p^2 - (2.5+1)p - 2 = 0 \quad implies \; 1.5p^2 - 3.5p - 2 = 0,$$

This creates a regime of nonlinearities–particularly for powers p ∈ [2.41, 2.81]–where waves that would otherwise scatter to zero in a vacuum are forced into finite-time blow-up by the geometry of the domain. This gap represents a purely *geometric destabilization*: the focusing effect of the convex boundary prevents efficient energy radiation, allowing nonlinear self-interaction to dominate linear dispersion for a broader range of power-law exponents. Consequently, stability in the presence of curvature is not merely a matter of initial data size, but it is fundamentally constrained by slower decay of the glancing caustics.

Conclusion

In this paper, we establish a sharp analysis of the wave dispersion phenomenon within the cylindrical convex domain, where a clear distinction between the curvature of the boundary and the nonlinear stability is made. Through deriving $L^p - L^q$ dispersive estimates, we reveal the exact nature of the "curvature penalty," which arises for strictly convex domains and implies the appearance of glancing caustics leading to an inevitable loss of regularity $\frac{9}{4}\left|\frac{1}{p}-\frac{1}{2}\right|$.

By employing dispersive estimates in our analysis of the nonlinear wave equations, we determine the shifted Strauss exponent $p_c \approx 2.81$. It is noteworthy that pc surpasses the value of 2.41 typical for flat geometries due to the inhibited decay rate of $|t|^{-3/4}$ As a result, the whispering gallery modes can lead to instability due to the accumulation of energy along the boundary; however, global solutions still exist under nonlinear terms exceeding the geometric threshold. In summary, this work offers a rigorous mathematical model that captures the essence of localized energy concentration within curved geometries and its effect on the evolution of waves.

There are many promising avenues to explore based on the findings presented in this article, particularly in relation to geometry-induced dispersion effects. One obvious avenue for further research is examining regions whose boundaries have curvature that becomes equal to zero multiple times or to a higher order at specific locations. Currently, it is not known how the power decay of $|t|^{-3/4}$ and how the caustic effect depends on the flatness of these boundaries. Another promising direction would be extending the approach developed above to the study of dispersive effects in the Schrodinger equation, which exhibits stronger degeneration of its dispersive properties compared to the wave equation. It will allow us to identify precise critical values for the non-linear Schrodinger equation on convex domains.

Finally, it should be explored whether the 9/4 deficit in terms of boundary regularity is inherent to all boundary dynamics. Conducting an analysis of other types of boundary conditions will give an answer to this question.

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Appendix

A. Airy Functions. Let z > 0. The Airy function Ai is defined as follows:

It satisfies the Airy equation

$Ai''(z) - zAi(z) = 0,$ denoted by (A)

$$Ai(-z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\left(s^3/3 - sz\right)} ds.$$

Let $\omega = e^{2i\pi/3}$. Obviously, $z \mapsto Ai(\omega z)$ is a solution to (A). Any two of these three solutions $Ai(z), Ai(\omega z), Ai(\omega^2 z)$ yield a basis of solutions to (A) and the linear relation between them is

$$\sum_{j=\{0,1,2\}} \omega^j Ai\left(\omega^j z\right) = 0.$$

It follows that

$$Ai(z) = -\omega Ai(\omega z) - \overline{\omega} Ai(\overline{\omega} z),$$

which we rewrite as follows:

$$Ai(-z) = e^{-i\pi/3} Ai\left(e^{-i\pi/3} z\right) + e^{i\pi/3} Ai\left(e^{i\pi/3} z\right) = A_+(z) + A_-(z), \quad (3.1)$$

where we set

$$A_\pm(z) = e^{\pm i\pi/3} Ai\left(e^{\pm i\pi/3} z\right).$$

Notice that $A_-(z) = \overline{A_+(\overline{z})}$. We also have the following asymptotic expansions

$$A_-(z) = \frac{1}{2\sqrt{\pi z^{1/4}}} e^{i\pi/4} e^{-\frac{2}{3} i z^{3/2}} \exp \Upsilon\left(z^{3/2}\right) = \frac{1}{z^{1/4}} e^{i\pi/4} e^{-\frac{2}{3} i z^{3/2}} \Psi_-(z),$$

with $\exp \Upsilon\left(z^{3/2}\right) \sim \left(1 + \sum_{l\geq 1} c_l z^{-3l/2}\right) \sim 2\sqrt{\pi}\Psi_-(z)$ as $z \to +\infty$ and the corresponding expansion for A+, where we define $\Psi_+(z) = \overline{\Psi}_-(\overline{z})$. Moreover, we have

$$\frac{A_-(z)}{A_+(z)} = ie^{-\frac{4}{3} i z^{3/2}} e^{iB\left(z^{3/2}\right)}, \quad with \quad iB = \Upsilon - \overline{\Upsilon}.$$

Lemma 3.1. *For* $\omega \in \mathbb{R}$, *define*

$$L(\omega) = \pi + i \log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

Then L is analytic, strictly increasing and satisfies

$$L(0) = \pi/3, \quad \lim_{\omega \to -\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{3/2} - B\left(\omega^{3/2}\right), \quad for \quad \omega \geq 1,$$

with

$$B(\omega) \sim \frac{1}{\omega}\sum_{l\geq 1} b_k \omega^{-k}, \quad b_k \in \mathbb{R}, \quad b_1 > 0.$$

For all k ≥ 1, the following holds

$$L(\omega_k) = 2\pi\omega_k \Leftrightarrow Ai(-\omega_k) = 0$$

and

$$L'(\omega_k) = 2\pi \int_0^\infty Ai^2(x - \omega_k) dx,$$

where $\{-\omega_k\}_{k\geq 1}$ *denotes the zeros of the Airy function in decreasing order.*

Proof. Given that A+ is analytic with values in $\mathbb{C}$ and never vanishes on the real line, there exist unique analytic functions $\rho(\omega) > 0$ and $\theta(\omega) \in \mathbb{R}$ such that $A_+(\omega) = \rho(\omega) e^{i\theta(\omega)}$. As a consequence, $A_-(\omega) = \rho(\omega) e^{-i\theta(\omega)}$, and according to its definition $L(\omega) = \pi + 2\theta(\omega)$ is real on the real axis. At ω = 0, we find $A_\pm = e^{\mp i\pi/3} Ai(0)$ using (3.1). This results in

$$\frac{A_-(0)}{A_+(0)} = e^{-2i\theta(0)} = e^{2\pi i/3},$$

so $L(0) = \pi + 2\theta(0) = \frac{\pi}{3}$. As $Ai(-\omega) = \sum_{\pm} A_{\pm}(\omega) = 2\rho(\omega)\cos(\theta(\omega))$ and asymptotic expansions for θ *and* ρ *as* $\omega \to +\infty$,

$$\omega^{1/2}(2\rho(\omega))^2 \sim_{\frac{1}{\omega}} \frac{1}{\pi}\sum_{k=0}^{+\infty} \alpha_k \omega^{-3k}, \quad \alpha_0 = 1,$$

$$\theta(\omega) + \frac{\pi}{4} \sim_{\frac{1}{\omega}} \frac{2}{3}\omega^{3/2}\sum_{k=0}^{+\infty} \beta_k \omega^{-3k}, \quad \beta_0 = 1, \quad \beta_1 = -5/32$$

produce

$$L(\omega) = 2\theta(\omega) = \frac{4}{3}\omega^{3/2} + \frac{\pi}{2} - B(\omega^{3/2}),$$

where we set

$$B(\omega^{3/2}) \sim_{\frac{1}{\omega}} -\frac{4}{3}\omega^{3/2}\sum_{k=1}^{+\infty} \beta_k \omega^{-3k}.$$

Setting $b_{2k-1} := -\frac{4}{3}\beta_k$ and $b_{2k} := 0$ produces (3.2) with $b_1 = \frac{5}{24} > 0$. Additionally, we have $(2\rho(\omega))^2\theta'(\omega) = \frac{1}{\pi}$, which results in

$$L'(\omega) = 2\theta'(\omega) = \frac{1}{2\pi\rho^2(\omega)} > 0.$$

As a result, L is strictly increasing. Now, let $F(\omega) = Ai(-\omega)$. With $A_+(\omega) = \rho(\omega)e^{i\theta(\omega)}$, we get $F(\omega) = 2\rho(\omega)\cos(\theta(\omega))$. Therefore, the equation $F(\omega) = 0$ is equivalent to $\theta(\omega) = \pi/2 + l\pi, l \in \mathbb{Z},$, which is equivalent to $L(\omega) = 2\pi(1 + l)$. Since L is a diffeomorphism from onto $]0,\infty[$, one has for all integer $k \geq 1$, $Ai(-\omega_k) = 0$ if and only if $L(\omega_k) = 2\pi k.$.

Finally, using the Airy equation $F''(y) + yF(y) = 0$ and integration by part, we get

$$\begin{aligned}\int_{\infty}^{\omega} F^2(y)dy &= \omega F^2(\omega) - \int_{-\infty}^{\omega} 2yF(y)F'(y)dy \\ &= \omega F^2(\omega) + \int_{-\infty}^{\omega} 2yF''(y)F'(y)dy \\ &= \omega F^2(\omega) + F'^2(\omega).\end{aligned}$$

Since

$$F'(\omega_k) = 2\rho'(\omega_k)\cos(\theta(\omega_k)) + 2\rho(\omega_k)\theta'(\omega_k)\sin(\theta(\omega_k)),$$

we get

$$\int_0^{\infty} Ai^2(x-\omega_k)dx = F'^2(\omega) = 4\rho^2(\omega_k)\theta'^2(\omega_k) = \rho^2(\omega_k)L'^2(\omega_k) = c_0 L'(\omega_k),$$

From $2\pi Ai(0) = 3^{-1/6}\Gamma(1/3)$, $2\pi Ai'(0) = -3^{1/6}(2/3)$ and the Euler reflection formula for the Γ function, $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$, we get $2\pi c_0 = 1$, consequently

$$\int_0^{\infty} Ai^2(x-\omega_k)dx = \frac{L'(\omega_k)}{2\pi}.$$

This completes the proof of the lemma.