Iris Journal of Mathematics

Iris Publishers

Research Article

Weighted Boundedness of Toeplitz Operator Related to Singular Integral Operator

Lanzhe Liu* and Xiaoqi Zhou

School of General Education, Hunan University of Information Technology, China

***Corresponding author:** Lanzhe Liu, School of General Education, Hunan University of Information Technology, China

Received Date: April 14, 2026

Published Date: April 29, 2026

Abstract

In this paper, the boundedness of the Toeplitz type operator associated to the singular integral operator with variable Calder´on-Zygmund kernel on weighted Morrey spaces is obtained. To do this, some weighted sharp maximal function inequalities for the operator are proved.

Keywords: Toeplitz type operator; Singular integral operator; Sharp maximal function; Morrey space; Weighted BMO; Weighted Lipschitz function

MR Subject Classification: 42B20, 42B25.

Introduction

Let b be a locally integrable function on R^n and T be an integral operator. For a suitable function f, the commutator generated by b and T is defined by $[b,T]f = bT(f) - T(bf)$. The investigation of the commutator begins with Coifman-Rochberg-Weiss pioneering study and classical result (see [3]). There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutator can produces some characterizations of function spaces (see [3,9,23]). The other one is that the theory of commutator plays an important role in the study of the regularity of solutions to elliptic and parabolic partial differential equations (PDEs) of the second order (see [5,22]). The well-posedness problem of solutions to many PDEs can be attributed to the corresponding boundedness of commutators of integral operators. Now, with the development of singular integral operators (see [7,27]), their commutators have been well studied. In [3,25,26], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ *for* $1 < p < \infty$. In [9,23], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ and Triebel-Lizorkin spaces are obtained. In [1,8], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [11,12,17,18], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In [2], Calder´on and Zygmund introduce certain singular integral operator with variable kernel and discuss its boundedness. In [14-16,28], the authors obtain the boundedness for the commutator generated by the singular integral operator with variable kernel and BMO function. In [19], the authors prove the boundedness for the multilinear oscillatory singular integral operator generated by the operator and BMO function.

On the other hand, the classical Morrey space was introduced by Morrey in [21] to investigate the local behaviour of solutions to second order elliptic partial differential equations (also see [22]). As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of operator on the Morrey spaces. The boundedness of the maximal operator, the singular integral operator, the fractional integral operator and their commutators on Morrey spaces have been studied by many authors (see [4,5,13,20]). In [10], Komori and Shirai studied the boundedness of these operators on weighted Morrey spaces.

Motivated by these, in this paper, we will study the Toeplitz type operator generated by the singular integral operator with variable kernel and the weighted Lipschitz and BMO functions on weighted Morrey spaces.

Preliminaries

First, let us introduce some notations. Throughout this paper, $Q=Q(x,r)$ will denote a cube of R^n with sides parallel to the axes and centered at x and edge length is r, and 2^kQ denote a cube with same center as Q and edge length is 2^kr for $k\ge 0$. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$ and $f(Q)=\int_Q f(x)dx$. It is well-known that (see [7,27])

$$M^{\#}(f)(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta^{\#}(f)(x)=M^{\#}\left(|f|^\eta\right)^{1/\eta}(x)$ and $M_\eta(f)(x)=M\left(|f|^\eta\right)^{1/\eta}(x)$.

For $0<\eta<n,\ 1\le p<\infty$ and the non-negative weight function w, set

$$M_{\eta,p,w}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{w(Q)^{1-p\eta/n}}\int_Q|f(y)|^p w(y)dy\right)^{1/p}.$$

We write $M_{\eta,p,w}(f)=M_{\eta,w}(f)$ if $\eta=0$ and $M_w(f)=M_{p,w}(f)$ if $p=1$.

The Ap weight is defined by (see [7]), for $1<p<\infty$,

$$A_p=\left\{0<w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},$$

$$A_1=\left\{0<w\in L^1_{loc}(R^n):M(w)(x)\le Cw(x),a.e\right\}$$

and

$$A_\infty=\bigcup_{p\ge1}A_p.$$

For $0<\beta<1$ and the non-negative weight function w, the weighted Lipschitz space $Lip\beta(w)$ is the space of functions b such that

$$\|b\|_{Lip\beta(w)}=\sup_Q\frac{1}{w(Q)^{\beta/n}}\left(\frac{1}{w(Q)}\int_Q|b(y)-b_Q|^p w(x)^{1-p}dy\right)^{1/p}<\infty,$$

and the weighted *BMO* space *BMO(w)* is the space of functions b such that

$$\|b\|_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q \left| b(y) - b_Q \right|^p w(x)^{1-p} \, dy \right)^{1/p} < \infty.$$

Remark. (1) It has been known that (see [6]), for $b \in Lip\beta(w),\ w \in A_1\ and\ x \in Q,$

$$\left| b_Q - b_{2^k Q} \right| \le Ck \|b\|_{Lip\beta(w)} w(x) w(2^k Q)^{\beta/n}.$$

(2) It has been known that (see [6]), for $b \in BMO(w),\ w \in A_1\ and\ x \in Q,$

$$\left| b_Q - b_{2^k Q} \right| \le Ck \|b\|_{BMO(w)} w(x).$$

(3) Let $b \in Lip\beta(w)\ or\ b \in BMO(w)\ and\ w \in A_1$. By [6,7], we know that spaces $Lip\beta(w)$ or $BMO(w)$ coincide and the norms $\|b\|_{Lip\beta(w)}$ or $\|b\|_{BMO(w)}$ are equivalent with respect to different values $1 \le p < \infty$.

Definition 1. Let $1 \le p < \infty,\ 0 < k < 1,\ u\ and\ v$ be two non-negative weight functions on R^n and f be a locally integrable function on R^n. Set

$$\|f\|_{L^{p,k}(u,v)} = \sup_Q \left(\frac{1}{v(Q)^k} \int_Q \left| f(y) \right|^p u(y) \, dy \right)^{1/p}.$$

The generalized weighted Morrey space $L^{p,k}(u,v)$ is defined by

$$L^{p,k}(u,v) = \left\{ f \in L^1_{loc}(R^n) : \|f\|_{L^{p,k}(u,v)} < \infty \right\}.$$

Remark. (4) We write $L^{p,k}(u,v) = L^{p,k}(u)\ if\ u = v.$

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3,24]). In this paper, we will study certain singular integral operator with variable Calder´on- Zygmund kernel as following (see [2]).

Definition 2. Denote a multi-indices by $\alpha = (\alpha_1, \ldots \alpha_n)$ with α_j is a nonnegative integer for $1 \le j \le n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. Let $K(x) = \Omega(x)/|x|^n : R^n/\{0\} \to R$. K is said to be a Calder´on-Zygmund kernel if

a) $\Omega \in C^\infty(R^n/\{0\})$;

b) Ω is homogeneous of degree zero;

c) $\int_\Sigma \Omega(x) x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition3. Let $K(x,y) = \Omega(x,y)/|y|^n : R^n \times (R^n/\{0\}) \to R.$ K is said to be a variable Calder´on-Zygmund kernel if

(d) $K(x,.)$ is a Calder´on-Zygmund kernel for a.e. $x \in R^n$;

$$\max_{|\gamma| \le 2n} \left\| \frac{\partial^\gamma}{\partial^\gamma y} \Omega(x,y) \right\|_{L^\infty(R^n \times \Sigma)} = L < \infty.$$

(e)

Moreover, let *b* be a locally integrable function on R^n and *T* be the singular integral operator with variable Calder´on-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x-y) f(y) dy,$$

where $K(x, x-y) = \dfrac{\Omega(x, x-y)}{|x-y|^n}$ and that $\Omega(x,y)/|y|^n$ is a variable Calder´on-Zygmund kernel.

The Toeplitz type operator associated to *T* is defined by

$$T^b = \sum_{j-1}^m T^{j,1} M_b T^{j,2},$$

where $T^{j,1}$ are the singular integral operator with variable Calder´on-Zygmund kernel *T* or ±*I* (the identity operator), $T^{j,2}$ are the linear operators, $j = 1, \ldots, m,\ M_b(f) = bf$.

Remark. (5) Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the

Toeplitz type operator Tb. The Toeplitz type operator Tb is the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [25,26]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator *Tb*. As the application, we obtain the weighted boundedness on Morrey space for the Toeplitz type operator *Tb*.

Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. Let T be the singular integral operator as **Definition 3**, $w\in A_1$, $0<\eta<1$, $1<s<\infty$, $0<\beta<1$ and $b\in Lip\beta(w)$. If $T^1(g)=0$ for any $g\in L^r(R^n)(1<r<\infty)$, then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,

$$M_\eta^{\#}\left(T^b(f)\right)(\tilde{x})\le C\|b\|_{Lip\beta(w)}\,w(\tilde{x})\sum_{j=1}^{m}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x}).$$

Theorem 2. Let T be the singular integral operator as **Definition 3**, $w\in A_1$, $0<\eta<1$, $1<s<\infty$ and $b\in BMO(w)$. If $T^1(g)=0$ for any $g\in L^r(R^n)(1<r<\infty)$, then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,

$$M_\eta^{\#}\left(T^b(f)\right)(\tilde{x})\le C\|b\|_{BMO(w)}\,w(\tilde{x})\sum_{j=1}^{m}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x}).$$

Theorem 3. Let T be the singular integral operator as **Definition 3**, $w\in A_1$, $0<\beta<1$, $b\in Lip\beta(w)$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $0<k<p/q$. If $T^1(g)=0$ for any $g\in L^r(R^n)(1<r<\infty)$ and $T^{j,2}$ are bounded on $L^{p,k}(w)$ for $1\le j\le m$, then T^b is bounded from $L^{p,k}(w)$ to $L^{q,kq/p}(w^{1-p},w)$.

Theorem 4. Let T be the singular integral operator as **Definition 3**, $w\in A_1$, $1<p<\infty$, $b\in BMO(w)$ and $0<k<1$. If $T^1(g)=0$ for any $g\in L^r(R^n)(1<r<\infty)$ and $T^{j,2}$ are bounded on $L^{p,k}(w)$ for $1\le j\le m$, then T^b is bounded from $L^{p,k}(w)$ to $L^{p,k}(w^{1-p},w)$.

Corollary. Let $[b,T](f)=bT(f)-T(bf)$ be the commutator generated by the singular integral operator T as **Definition 3** and b. Then Theorems 1-4 hold for [*b, T*].

To prove the theorems, we need the following lemmas.

Lemma 1. (see [7]) Let $0<p<q<\infty$ and for any function $f\ge 0$. We define that, for $1/r=1/p-1/q$,

$$\|f\|WL^q=\sup_{\lambda>0}\lambda\left|\left\{x\in R^n:f(x)>\lambda\right\}\right|^{1/q},\qquad N_{p,q}(f)=\sup_Q\|fXQ\|L^p/\|XQ\|L^r,$$

where the sup is taken for all measurable sets Q with $0<|Q|<\infty$. Then

$$\|f\|WL^q\le N_{p,q}(f)\le\left(q/(q-p)\right)^{1/p}\|f\|WL^q.$$

Lemma 2. (see [2,19]) Let T be the singular integral operator as **Definition 3** and $1<p<\infty$. Then T is bounded on $L^p(R^n,w)$ for $w\in A_p$ with $1<p<\infty$, and weak (L^1,L^2) bounded.

Lemma 3. Let $1<p<\infty$, $0<\eta<\infty$, $0<k<1$ and $u,v\in A_\infty$. Then, for any smooth function f for which the left-hand side is finite,

$$\|M_\eta(f)\|L^{p,k}(u,v)\le C\|M_\eta^{\#}(f)\|L^{p,k}(u,v).$$

In particular,

$$\|M_\eta(f)\|L^{p,k}(u)\le C\|M_\eta^{\#}(f)\|L^{p,k}(u).$$

Proof. By [7,27], we have the following weighted version of the local good inequality, for all cube Q and $\lambda,\delta>0$,

$$w\left(\left\{x\in Q: M_\eta\left(f\right)\left(x\right)>\lambda, M_\eta^{\#}\left(f\right)\left(x\right)\le\delta\lambda\right\}\right)<C\delta w\left(\left\{x\in Q: M_\eta\left(f\right)\left(x\right)>\lambda\right\}\right).$$

Thus, the conclusion is obtained by using the standard argument of Whitney decomposition theorem (see [7,27]). This finishes the proof.

Lemma 4. Let $0<k<1,\ 1\le s<p<\infty$ and $w\in A_\infty$. Then

$$\left\|M_{s,w}\left(f\right)\right\|L^{p,k}\left(w\right)\le C\left\|f\right\|L^{p,k}\left(w\right).$$

Proof. By [10], we know

$$\left\|M_w\left(f\right)\right\|L^{p,k}\left(w\right)\le C\left\|f\right\|L^{p,k}\left(w\right).$$

Notice that $M_{s,w}\left(f\right)=\left(M_w\left(\left|f\right|^s\right)\right)^{1/s}$, we get

$$\left\|M_{s,w}\left(f\right)\right\|L^{p,k}\left(w\right)=\left\|M_w\left(\left|f\right|^s\right)\right\|_{L^{p/s,k}(w)}^{1/s}\le C\left\|\left|f\right|^s\right\|_{L^{p/s,k}(w)}^{1/s}\le C\left\|f\right\|_{L^{p,k}(w)}.$$

This completes the proof.

Lemma 5. Let $0<\eta<n,\ 1<r<p<n/\eta,\ 1/r=1/p-\eta/n,\ 0<k<p/q$ and $w\in A_\infty$.

Then

$$\left\|M_{\eta,r,w}\left(f\right)\right\|_{L^{q,kq/p}(w)}\le C\left\|f\right\|_{L^{p,k}(w)}.$$

Proof. By using a similar argument as in the proof of [10, Theorem 3.5], we get

$$\left\|M_{\eta,1,w}\left(f\right)\right\|_{L^{q,kq/p}(w)}\le C\left\|f\right\|_{L^{p,k}(w)}.$$

It is observe that $M_{\eta,r,w}\left(f\right)=\left(M_{r\eta,1,w}\left(\left|f\right|^r\right)\right)^{1/r}$, thus, for $r/q=r/p-r\eta/n$,

$$\left\|M_{\eta,r,w}\left(f\right)\right\|_{L^{q,kq/p}(w)}=\left\|M_{\eta r,1,w}\left(\left|f\right|^r\right)\right\|_{L^{q/r,kp/q}(w)}^{1/r}\le C\left\|\left|f\right|^r\right\|_{L^{p/r,k}(w)}^{1/r}\le C\left\|f\right\|_{L^{p,k}(w)}.$$

This completes the proof.

Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f\in C_0^\infty\left(R^n\right)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\left|T^b\left(f\right)\left(x\right)-C_0\right|^\eta dx\right)^{1/\eta}\le C\left\|b\right\|_{Lip\beta(w)}w(\tilde{x})\sum_{j=1}^m M_{\beta,s,w}\left(T^{j,2}\left(f\right)\right)(\tilde{x}).$$

Without loss of generality, we may assume $T^{j,1}$ are $T\left(j=1,\ldots,m\right)$. Let $\tilde{x}\in Q$. Fix a cube $Q=Q\left(x_0,d\right)$ and $\tilde{x}\in Q$. Write

$$T^b\left(f\right)\left(x\right)=T^{b-b_{2Q}}\left(f\right)\left(x\right)=T^{\left(b-b_{2Q}\right)X2Q}\left(f\right)\left(x\right)+T^{\left(b-b_{2Q}\right)X(2Q)^c}\left(f\right)\left(x\right)=I_1\left(x\right)+I_2\left(x\right).$$

Then, for $C_0=I_2\left(x_0\right)$,

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x)-I_2(x_0)\right|^\eta dx\right)^{1/\eta} \le C\left(\frac{1}{|Q|}\int_Q \left|I_1(x)\right|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q \left|I_2(x)-I_2(x_0)\right|^\eta dx\right)^{1/\eta} = I_1 + I_2.$$

For I_1, by the weak $\left(L^1, L^2\right)$ boundedness of $T^{j,1}$ (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)(x)\right|^\eta dx\right)^{1/\eta}\\
&\le \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{\left\|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)XQ\right\|L^\eta}{\|XQ\|L^{\eta/(1-\eta)}}\\
&\le \frac{C}{|Q|}\left\|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)\right\|WL^1\\
&\le \frac{C}{|Q|}\int_{R^n}\left|M_{(b-b_{2Q})X2Q}T^{j,2}(f)(x)\right|dx\\
&\le \frac{C}{|Q|}\int_{2Q}\left|b(x)-b_{2Q}\right|w(x)^{-1/s}\left|T^{j,2}(f)(x)\right|w(x)^{1/s}dx\\
&\le \frac{C}{|Q|}\left(\int_{2Q}\left|b(x)-b_{2Q}\right|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\int_{2Q}\left|T^{j,2}(f)(x)\right|^s w(x)dx\right)^{1/s}\\
&\le \frac{C}{|Q|}\|b\|_{Lip\beta(w)}w(2Q)^{1/s'}+\beta/n\,w(2Q)^{1/s-\beta/n}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x})\\
&\le C\|b\|_{Lip\beta(w)}\frac{w(2Q)}{|2Q|}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x})\\
&\le C\|b\|_{Lip\beta(w)}w(\tilde{x})M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\le C\sum_{j=1}^m\left(\frac{1}{|Q|}\int_{2Q}\left|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)(x)\right|^\eta dx\right)^{1/\eta}\\
&\le C\|b\|_{Lip\beta(w)}w(\tilde{x})\sum_{j=1}^m M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x}).
\end{aligned}$$

For I_2, by [2,19], we know that

$$T(f)(x)=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}a_{uv}\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^n}f(y)dy,$$

Where $g_u \le Cu^{n-2}$, $\left|a_{uv}(x_0)\right| \le Cu^{-2n}$, $\left|a_{uv}(x)-a_{uv}(x_0)\right| \le Cu^{-2n+1}|x-x_0|/|x_0-y|$, $\left|Y_{uv}(x-y)\right| \le Cu^{n/2-1}$ for $x\in Q$, $y\in(2Q)^c$ and for $|x-y|>2|x_0-x|>0.$ Thus, notice $w\in A_1\subset A_s$, we get, for $x\in Q$,

$$\begin{aligned}
&\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\\
&\le \left|Y_{uv}(x-y)-Y_{uv}(x_0-y)\right|\frac{1}{|x-y|^n}+\left|\frac{1}{|x-y|^n}-\frac{1}{|x_0-y|^n}\right|\left|Y_{uv}(x_0-y)\right|\\
&\le Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&\left|T^{j,1}M_{(b-b_{2Q})\chi(2Q)^c}T^{j,2}(f)(x)-T^{j,1}M_{(b-b_{2Q})\chi(2Q)^c}T^{j,2}(f)(x_0)\right|\\
&\le \int_{(2Q)^c}\left|by-b_{2Q}\right|\left|K(x,x-y)-K(x_0,x_0-y)\right|\left|T^{j,2}(f)(y)\right|dy\\
&=\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}\left|b(y)-b_{2Q}\right|\left|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}\frac{a_{uv}(x)Y_{uv}(x-y)}{|x-y|^n}-\frac{a_{uv}(x_0)Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\left|T^{j,2}(f)(y)\right|dy\\
&\le C\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}\left|b(y)-b_{2Q}\right|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}\left|a_{uv}(x)-a_{uv}(x_0)\right|\frac{\left|Y_{uv}(x-y)\right|}{|x-y|^n}\left|T^{j,2}(f)(y)\right|dy\\
&+C\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}\left|b(y)-b_{2Q}\right|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}\left|a_{uv}(x_0)\right|\left|\frac{\left|Y_{uv}(x-y)\right|}{|x-y|^n}-\frac{\left|Y_{uv}(x_0-y)\right|}{|x_0-y|^n}\right|\left|T^{j,2}(f)(y)\right|dy\\
&\le C\sum_{u=1}^{\infty}u^{-2n+1}u^{n/2-1}u^{n-2}\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}\left|b(y)-b_{2Q}\right|\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|T^{j,2}(f)(y)\right|dy\\
&+C\sum_{u=1}^{\infty}u^{-2n}u^{n/2}u^{n-2}\sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d}\left|b(y)-b_{2Q}\right|\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|T^{j,2}(f)(y)\right|dy\\
&\le C\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\int_{2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}+b_{2^{k+1}Q}-b_{2Q}\right|w(y)^{-1/s}\left|T^{j,2}(f)(y)\right|w(y)^{1/s}dy\\
&\le C\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\left(\int_{2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}\right|^{s'}w(y)^{1-s'}dy\right)^{1/s'}\left(\int_{2^{k+1}Q}\left|T^{j,2}(f)(y)\right|^{s}w(y)dy\right)^{1/s}\\
&+\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\left|b_{2^{k+1}Q}-b_{2Q}\right|\left(\int_{2^{k+1}Q}w(y)^{-1/(s-1)}dy\right)^{1/s'}\left(\int_{2^{k+1}Q}\left|T^{j,2}(f)(y)\right|^{s}w(y)dy\right)^{1/s}\\
&\le C\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\|b\|_{Lip\beta(w)}w\left(2^{k+1}Q\right)^{1/s'+\beta/n}w\left(2^{k+1}Q\right)^{1/s-\beta/n}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x})\\
&+C\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\|b\|_{Lip\beta(w)}w(\tilde{x})kw\left(2^{k+1}Q\right)^{\beta/n}w\left(2^{k+1}Q\right)^{1/s-\beta/n}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x})\\
&\times\frac{\left|2^{k+1}Q\right|}{w\left(2^{k+1}Q\right)^{1/s}}\left(\frac{1}{\left|2^{k+1}Q\right|}\int_{2^{k+1}Q}w(y)dy\right)^{1/s}\left(\frac{1}{\left|2^{k+1}Q\right|}\int_{2^{k+1}Q}w(y)^{-1/(s-1)}dy\right)^{(s-1)/s}\\
&\le C\|b\|_{Lip\beta(w)}w(\tilde{x})M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x})\sum_{k=1}^{\infty}k2^{-k}\\
&\le C\|b\|_{Lip\beta(w)}w(\tilde{x})M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_2 &\le \frac{1}{|Q|}\int_Q\sum_{j=1}^{m}\left|T^{j,1}M_{(b-b_{2Q})\chi(2Q)^c}T^{j,2}(f)(x)-C_0\right|dx\\
&\le C\|b\|_{Lip\beta(w)}w(\tilde{x})\sum_{j=1}^{m}M_{\beta,s,w}\left(T^{j,2}(f)\right)(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f\in C_0^{\infty}(R^n)$ and some constant C_0, the following inequality holds:

$$
\left(\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-C_0\right|^{\eta}dx\right)^{1/\eta}\le C\|b\|_{BMO(w)}w(\tilde{x})\sum_{j=1}^{m}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x}).
$$

Without loss of generality, we may assume $T^{j,1}$ are $T(j=1,\ldots,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$T^{b}(f)(x)=T^{b-b_{2Q}}(f)(x)=T^{(b-b_{2Q})X2Q}(f)(x)+T^{(b-b_{2Q})X(2Q)^{c}}(f)(x)=I_{3}(x)+I_{4}(x)$$

and

$$\left(\frac{1}{|Q|}\int_{Q}\left|T^{b}(f)(x)-I_{2}(x_{0})\right|^{\eta}dx\right)^{1/\eta}\leq C\left(\frac{1}{|Q|}\int_{Q}\left|I_{3}(x)\right|^{\eta}dx\right)^{1/\eta}+C\left(\frac{1}{|Q|}\int_{Q}\left|I_{4}(x)-C_{0}\right|^{\eta}dx\right)^{1/\eta}=I_{3}+I_{4}.$$

By using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
I_{3}&\leq C\sum_{j=1}^{m}\left(\frac{1}{|Q|}\int_{Q}\left|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)(x)\right|^{\eta}dx\right)^{1/\eta}\\
&\leq C\sum_{j=1}^{m}\frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{\left\|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)(x)\right\|L^{\eta}}{\|XQ\|L^{\eta/(1-\eta)}}\\
&\leq C\sum_{j=1}^{m}\frac{C}{|Q|}\left\|T^{j,1}M_{(b-b_{2Q})X2Q}T^{j,2}(f)\right\|WL^{1}\\
&\leq C\sum_{j=1}^{m}\frac{C}{|Q|}\int_{R^{n}}\left|M_{(b-b_{2Q})X2Q}T^{j,2}(f)\right|dx\\
&\leq C\sum_{j=1}^{m}\frac{C}{|Q|}\int_{2Q}\left|b(x)-b_{2Q}\right|w(x)^{-1/s}\left|T^{j,2}(f)(x)\right|w(x)^{1/s}dx\\
&\leq C\sum_{j=1}^{m}\frac{C}{|Q|}\left(\int_{2Q}\left|b(x)-b_{2Q}\right|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\int_{2Q}\left|T^{j,2}(f)(x)\right|^{s}w(x)dx\right)^{1/s}\\
&\leq C\sum_{j=1}^{m}\frac{w(2Q)}{|2Q|}\left(\frac{1}{w(2Q)}\int_{2Q}\left|b(x)-b_{2Q}\right|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\frac{1}{w(2Q)}\int_{2Q}\left|T^{j,2}(f)(x)\right|^{s}w(x)dx\right)^{1/s}\\
&\leq C\|b\|_{BMO(w)}w(\tilde{x})\sum_{j=1}^{m}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x}),
\end{aligned}$$

$$I_4 \le \sum_{j=1}^{m} \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|b(y)-b_{2Q}\right|\left|K(x,x-y)-K(x_0,x_0-y)\right|\left|T^{j,2}(f)(y)\right| dydx$$

$$\le \sum_{j=1}^{m} \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|b(y)-b_{2Q}\right|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}\left|a_{uv}(x)-a_{uv}(x_0)\right|\frac{\left|Y_{uv}(x-y)\right|}{|x-y|^n}\left|T^{j,2}(f)(y)\right| dydx$$

$$+\sum_{j=1}^{m} \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|b(y)-b_{2Q}\right|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}\left|a_{uv}(x_0)\right|\times\left|\frac{\left|Y_{uv}(x-y)\right|}{|x-y|^n}-\frac{\left|Y_{uv}(x_0-y)\right|}{|x_0-y|^n}\right|\left|T^{j,2}(f)(y)\right| dydx$$

$$\le \sum_{j=1}^{m} \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|b(y)-b_{2Q}\right|\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|T^{j,2}(f)(y)\right| dydx$$

$$\le C\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\int_{2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}+b_{2^{k+1}Q}-b_{2Q}\right|w(y)^{-1/s}\left|T^{j,2}(f)(y)\right|w(y)^{1/s}\,dy$$

$$\le C\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\left(\int_{2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}\right|^{s'}w(y)^{1-s'}\,dy\right)^{1/s'}\left(\int_{2^{k+1}Q}\left|T^{j,2}(f)(y)\right|^{s}w(y)\,dy\right)^{1/s}$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\left|b_{2^{k+1}Q}-b_{2Q}\right|\left(\int_{2^{k+1}Q}w(y)^{-1/(s-1)}\,dy\right)^{1/s'}\left(\int_{2^{k+1}Q}\left|T^{j,2}(f)(y)\right|^{s}w(y)\,dy\right)^{1/s}$$

$$\le C\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\|b\|_{BMO(w)}\,w\left(2^{k+1}Q\right)M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x})$$

$$+C\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{d}{\left(2^{k+1}d\right)^{n+1}}\|b\|_{BMO(w)}\,w(\tilde{x})\,k\,w\left(2^{k+1}Q\right)^{1/s}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x})$$

$$\times\frac{\left|2^{k+1}Q\right|}{w\left(2^{k+1}Q\right)^{1/s}}\left(\frac{1}{\left|2^{k+1}Q\right|}\int_{2^{k+1}Q}w(y)\,dy\right)^{1/s}\left(\frac{1}{\left|2^{k+1}Q\right|}\int_{2^{k+1}Q}w(y)^{-1/(s-1)}\,dy\right)^{(s-1)/s}$$

$$\le C\|b\|_{BMO(w)}\,w(\tilde{x})\sum_{j=1}^{m}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x})\sum_{k=1}^{\infty}k2^{-k}$$

$$\le C\|b\|_{BMO(w)}\,w(\tilde{x})\sum_{j=1}^{m}M_{s,w}\left(T^{j,2}(f)\right)(\tilde{x}),$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Choose $1 < s < p$ in Theorem 1, we have, by Lemmas 3 and 5,

$$\left\|T^b(f)\right\|_{L^{q,kq/p}\left(w^{1-q},w\right)} \le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^{q,kq/p}\left(w^{1-q},w\right)} \le C\left\|M^{\#}_{\Phi,\eta}\left(T^b(f)\right)\right\|_{L^{q,kq/p}\left(w^{1-q},w\right)}$$

$$\le C\|b\|_{Lip\beta(w)}\sum_{j=1}^{m}\left\|wM_{\beta,s,w}\left(T^{j,2}(f)\right)\right\|_{L^{q,kq/p}\left(w^{1-q},w\right)}$$

$$= C\|b\|_{Lip\beta(w)}\sum_{j=1}^{m}\left\|M_{\beta,s,w}\left(T^{j,2}(f)\right)\right\|_{L^{q,kq/p}(w)}$$

$$\le C\|b\|_{Lip\beta(w)}\sum_{j=1}^{m}\left\|T^{j,2}(f)\right\|_{L^{p,k}(w)}$$

$$\le C\|b\|_{Lip\beta(w)}\|f\|_{L^{p,k}(w)}.$$

This completes the proof.

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 2, we have, by Lemmas 3 and 4,

$$\left\|T^b(f)\right\|_{L^{p,k}(w^{1-p},w)} \le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^{p,k}(w^{1-p},w)} \le C\left\|M^\#_{\Phi,\eta}\left(T^b(f)\right)\right\|_{L^{p,k}(w^{1-p},w)}$$

$$\le C\|b\|_{BMO(w)}\sum_{j=1}^m\left\|wM_{s,w}\left(T^{j,2}(f)\right)\right\|_{L^{p,k}(w^{1-p},w)}$$

$$\le C\|b\|_{BMO(w)}\sum_{j=1}^m\left\|M_{s,w}\left(T^{j,2}(f)\right)\right\|_{L^{p,k}(w)}$$

$$\le C\|b\|_{BMO(w)}\sum_{j=1}^m\left\|T^{j,2}(f)\right\|_{L^{p,k}(w)}$$

$$\le C\|b\|_{BMO(w)}\|f\|_{L^{p,k}(w)}.$$

This completes the proof.

Acknowledgement

The authors would like to express their deep gratitude to the referee for his/her valuable comments and suggestions.

References

1. S Bloom (1985) A commutator theorem and weighted BMO. Trans Amer Math Soc 29: 103-122.
2. AP Calderon, A Zygmund (1978) On singular integrals with variable kernels. Appl Anal 7: 221-238.
3. RR Coifman, R Rochberg, G Weiss (1976) Factorization theorems for Hardy spaces in several variables. Ann of Math 103: 611-635.
4. G Di FaZio, MA Ragusa (1991) Commutators and Morrey spaces. Boll Un Mat Ital 5-A (7): 323-332.
5. G Di Fazio, MA Ragusa (1993) Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J Func Anal 112: 241-256.
6. J Garcia-Cuerva (1979) Weighted Hp spaces. Dissert Math 162.
7. J Garcia-Cuerva, JL Rubio de Francia (1985) Weighted norm inequalities and related topics. North-Holland Math Amsterdam 16.
8. B Hu, J Gu (2008) Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces. J of Math Anal and Appl 340: 598-605.
9. S Janson (1978) Mean oscillation and commutators of singular integral operators. Ark Math 16: 263-270.
10. Y Komori, S Shirai (2009) Weighted Morrey spaces and a singular integral operator. Math Nachr 282: 219-231.
11. S Krantz, S Li (2001) Boundedness and compactness of integral operators on spaces of homogeneous type and applications. J of Math Anal Appl 258: 629-641.
12. Y Lin, SZ Lu (2006) Toeplitz type operators associated to strongly singular integral operator. Sci in China(ser.A) 36: 615-630.
13. LZ Liu (2005) Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators. Acta Math Scientia 25(B): 89-94.
14. LZ Liu (2005) The continuity for multilinear singular integral operators with variable Calder´on-Zygmund kernel on Hardy and Herz spaces. Siberia Electronic Math Reports 2: 156-166.
15. LZ Liu (2005) Good estimate for multilinear singular integral operators with variable Calder´on-Zygmund kernel. Kragujevac J of Math 27: 19-30.
16. LZ Liu (2006) Weighted estimates of multilinear singular integral operators with variable Calder´on-Zygmund kernel for the extreme cases. Vietnam J of Math 34(2006): 51-61.
17. LZ Liu (2013) Sharp maximal function inequalities and boundedness for Toeplitz type operator related to general fractional integral operators. Banach J of Math Analysis 7: 142-159.
18. SZ Lu, HX Mo (2009) Toeplitz type operators on Lebesgue spaces. Acta Math Scientia 29(B): 140-150.
19. SZ Lu, DC Yang, ZS Zhou (1999) Oscillatory singular integral operators with Calder´on-Zygmund kernels. Southeast Asian Bull of Math 23: 457-470.
20. T Mizuhara (1990) Boundedness of some classical operators on generalized Morrey spaces, in" Harmonic Analysis". Proceedings of a conference held in Sendai, Japan pp.183-189.
21. CB Morrey (1983) On the solutions of quasi-linear elliptic partial differential equations. Trans Amer Math Soc 43: 126-166.
22. DK Palagachev, LG Softova (2004) Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's. Potential anal 20: 237-263.
23. M Paluszynski (1995) Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. Indiana Univ Math J 44: 1-17.
24. J Peetre (1969) On the theory of Lp;-spaces. J Func Anal 4: 71-87.
25. C P´erez (1995) Endpoint estimate for commutators of singular integral operators. J Func Anal 128: 163-185.
26. CP´erez, R Trujillo-Gonzalez (2002) Sharp weighted estimates for multilinear commutators. J London Math Soc 65: 672-692.
27. EM Stein (1993) Harmonic analysis: real variable methods, orthogonality and oscillatory integrals. Princeton Univ Press, Princeton NJ.
28. H Xu, LZ Liu (2008) Weighted boundedness for multilinear singular integral operator with variable Calder´on-Zygmund kernel. African Diaspora J of Math 6: 1-12.