Iris Journal of Mathematics

Iris Publishers

Research Article

Sharp Uncertainty Inequalities and Paley–Wiener Theory for the Two-Sided Quaternion Linear Canonical Transform

Mohammed Gadafi Tamimu[1,2]*, Selorm Kweku Dzokoto[3], Kowiyou Okpeyerou Akambi Adekambi[4], Yahya Abdurrazaq[5] and Toufic Seini[6]

[1]*School of Economics and Management, Center for West African Studies, University of Electronic Science and Technology of China, China*

[2]*School of Logistics, Sichuan University of Culture and Arts, Economic and Technological Development Zone, China*

[3]*School of Information and Software Engineering, University of Electronic Science and Technology of China, China*

[4]*School of Information and Communication Engineering, University of Electronic Science and Technology of China, China*

[5]*School of Computer Science and Engineering, University of Electronic Science and Technology of China, China*

[6]*Department of Mathematics, Faculty of Physical Sciences, University for Development Studies, University for Development Studies, Ghana*

***Corresponding author:** Mohammed Gadafi Tamimu, School of Economics and Management, Center for West African Studies, University of Electronic Science and Technology of China. School of Logistics, Sichuan University of Culture and Arts, Economic and Technological Development Zone, China

Received Date: February 03, 2026

Published Date: February 17, 2026

Abstract

This paper develops a comprehensive uncertainty and analytic framework for the Two-Sided Quaternion Linear Canonical Transform (QLCT). By extending classical harmonic analysis techniques to the quaternion-valued and canonical transform setting, we establish several sharp and fundamental results. First, a Pitt-type inequality for the QLCT is proved with explicitly computable and optimal constants, providing precise weighted L2-control between spatial and canonical-frequency domains. As a direct consequence, a Beckner-type logarithmic uncertainty principle is derived, quantifying intrinsic limits on the simultaneous localization of quaternion-valued signals. Furthermore, a Paley–Wiener theorem for the QLCT is established, yielding a complete characterization of quaternion-valued entire functions arising as transforms of compactly supported signals. To complement the theoretical analysis, numerical experiments based on synthetic quaternion signals are presented, illustrating the sharpness, stability, and practical relevance of the derived inequalities. The results demonstrate that the QLCT constitutes a mathematically robust and numerically stable tool for multidimensional and multichannel signal analysis, with potential applications in colour image processing, polarized signal analysis, and quaternion-based time–frequency representations.

Keywords: Weinstein operator; Weinstein transform; Pitt's inequality; Beckner's logarithmic uncertainty principle

Introduction

Uncertainty principles play a fundamental role in harmonic analysis by quantifying intrinsic limits on the simultaneous localization of a function and its transform. Classical results such as Pitt's inequality and Beckner's logarithmic uncertainty principle for the Fourier transform have provided deep insights into weighted energy estimates and localization trade-offs. Over the past decades, these inequalities have been extended to a variety of generalized transforms, including the Hankel, Dunkl, and Clifford–Fourier transforms, revealing rich structural connections between transform theory and special function analysis.

In parallel, quaternion-valued signal representations have attracted growing attention due to their ability to encode multichannel and multidimensional data within a unified algebraic framework. This has motivated the development of quaternion analogues of classical integral transforms, among which the Two-Sided Quaternion Linear Canonical Transform (QLCT) stands out as a powerful generalization that incorporates additional degrees of freedom through canonical parameters. The QLCT provides a flexible tool for joint spatial–frequency analysis of quaternion-valued signals, with potential applications in colour image processing, polarized signal analysis, and vector-field modelling.

Despite its growing relevance, a systematic uncertainty theory for the QLCT has remained largely unexplored. In particular, sharp weighted inequalities, logarithmic uncertainty principles, and analytic characterizations of bandlimited quaternion signals have not been fully established. The aim of this paper is to fill this gap by developing a comprehensive uncertainty and analytic framework for the QLCT. We establish a sharp Pitt-type inequality with optimal constants and derive a Beckner-type logarithmic uncertainty principle as a direct consequence, and prove a Paley–Wiener theorem characterizing quaternion-valued entire functions arising from compactly supported canonical-frequency data.

Let $\mathbb{R}^d$ denote the d-dimensional real space, equipped with a scalar product $\langle x,y\rangle$ and a norm $|x|=\sqrt{\langle x,x\rangle}$. Denote $S(\mathbb{R}^d)$ by the Schwartz space on $\mathbb{R}^d$ and by $L^p(\mathbb{R}^d)$ the space of complex-valued functions endowed with a norm

$$\|f\|_p=\begin{cases}\left(\int\limits_{\mathbb{R}^d}|f(x)|^p\,dx\right)^{\frac{1}{p}},1\le p<\infty,\\ ess\sup_{x\in\mathbb{R}^d}|f(x)|,\ p=\infty,\end{cases}$$

if $\|f\|_p<+\infty$, where $dx=dx_1\ldots dx_d$ represents the usual Lebesgue measure on $\mathbb{R}^d$. The classical Fourier transform of $f\in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}(f)(y)=(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}f(x)e^{-i\langle x,y\rangle}dx.$$

W. Beckner in [1] proved the following Pitt's inequality for the Fourier transform

$$\left\||.|^{-\beta}\mathcal{F}(f)\right\|_2\le c(\beta)\left\||.|^{\beta}f\right\|_2,\quad f\in S(\mathbb{R}^d),0\le\beta<d/2\qquad(1.1)$$

with sharp constant

$$c(\beta)=2^{-\beta}\frac{\Gamma\left(\frac{1}{2}\left(\frac{d}{2}-\beta\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{d}{2}+\beta\right)\right)}.\qquad(1.2)$$

It noted that by Parseval's identity, Pitt's inequality (1.1) can be viewed as a Hardy-Rellich inequality

$$\left\||.|^{-\beta}\mathcal{F}(f)\right\|_2\le c(\beta)\left\||.|^{\beta}\mathcal{F}(f)\right\|_2=c(\beta)\left\|(-\Delta)^{\beta/2}f\right\|_{2,}$$

whose proofs and extensions can be found in [2] and [3]. In addition, a remarkable application of Pitt's inequality (1.1) is to prove the following Beckner's logarithmic uncertainty principle

$$\int\limits_{\mathbb{R}^d}ln(|x|)|f(x)|^2dx+\int\limits_{\mathbb{R}^d}ln(|y|)|\mathcal{F}(f)(y)|^2dy\ge\left(\psi\frac{d}{4}+\ln 2\right)\int\limits_{\mathbb{R}^d}|f(x)|^2dx,\qquad(1.3)$$

where $\psi(t)=dln\,\Gamma(t)/dt$ and $\Gamma(t)$ is the gamma function.

The original proof of (1.1) by Beckner in [1] is based on an equivalent integral realization as a Stein-Weiss fractional integral on $\mathbb{R}^d$. In [2], D. Yafaev used the following decomposition of $L^2(\mathbb{R}^d)$ ([4]) to study inequality (1.1) on the subsets of $L^2(\mathbb{R}^d)$ which are invariant under the Fourier transform:

$$L^2(\mathbb{R}^d)=\sum_{k=0}^{\infty}\text{쬗}R_k^d,\qquad(1.4)$$

where R_0^d denotes the space of radial functions, and $R_k^d=R_0^d\otimes\mathrm{H}_k^d$ denotes the space of functions on $\mathbb{R}^d$ which are products of radial functions and spherical harmonics of degree k.

Following Yafaev's idea, D. V. Gorbachev et al. in [5] and [6] proved the sharp Pitt's inequalities for the Hankel transform ([7, 8, 9]), Dunkl transform ([10, 11]) and $(k,a)-$ generalized Fourier transform ([12]). Also S. Li and M. Fei in [13] recently proved the sharp Pitt's inequality for the Clifford-Fourier transform (see [14]).

In this paper, following the idea in [5,6] and [13], and using the theory of spherical harmonics associated to the Weinstein differential operator

$$\Delta_{d,\gamma}=\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_d^2}+\frac{2\gamma}{x_d}\frac{\partial}{\partial x_d},\qquad\gamma>0,\qquad(1.5)$$

we prove the sharp Pitt's inequality and Beckner's logarithmic uncertainty principle for the Weinstein transform which is a combination of the classical Fourier transform and Hankel transform.

Fourier transform is an integral representation of the absolutely integrable function and complex exponential type kernel. The Hankel transform which integral representation is a product of absolutely integrable function and the Bessel function of the first kind. The Weinstein operator (1.5) has many applications in pure and applied mathematics, especially in fluid mechanics ([15]). The corresponding spherical harmonics theory was studied by I. A. Aliev and B. Rubin in [16]. The transform associated to the Weinstein operator, which is called Weinstein transform in literature (see [17,18,19]),

$$\mathcal{F}_\gamma(f)(y)=\int\limits_{\mathbb{R}^{d-1}\times(0,\infty)}f(x)e^{-ix'.y'}j_{\gamma-1/2}(x_dy_d)x_d^{2\gamma}dx'dx_d,\qquad(1.6)$$

is a hybrid of the classical Fourier transform on $\mathbb{R}^{d-1}$ and Hankel transform in the x_{d^-} variable, where $y\in\mathbb{R}^{d-1}\times(0,\infty),x',y'\in\mathbb{R}^{d-1},j_{\gamma-1/2}(z)=2^{\gamma-1/2}\Gamma(\gamma+1/2)z^{1/2-\gamma}J_{\gamma-1/2}(z)$ is the normalized Bessel function. This transform and related problems for singular partial differential equations were studied by I. A. Aliev and B. Rubin ([16]), I. A. Kipriyanov ([17]), H. Mejjaoli and M. Salhi ([20]), Y. Othmani and K. Trimèche ([21]), N. B. Salem and A. R. Nasr ([22]), and many others.

Recently, many authors studied some problems for the Fourier-Bessel transform. I. A. Aliev (see [16]) developed spherical harmonics theory, who obtained natural analogs of the Plancherel theory, the Funk-Hecke formula and so on. Z. B. Nahia and N. B. Salem studied a mean value property and introduced spherical harmonics (see [23]). N. B. Salem and A. R. Nasr discussed Heisenberg-type inequalities (see [22]). More works of the transform we can see [24,20,25]. All of these results depend on the theory of the Fourier-Bessel transform.

Using the decomposition of space $L^2(\mathbb{R}^d)$ and based on the spherical harmonic's theory developed by I. A. Aliev in [16]. We prove the sharp Pitt's inequality for the Fourier-Bessel transform, which is a combination of the classical Fourier transform on $\mathbb{R}^{d-1}$ and the Hankel transform in the x_{d^-} variable.

Let $\lambda=d/2-1$ and $L_\gamma^2\left|(\mathbb{R}_+^d)\right.$ be the Hilbert space of complex-valued functions with a norm $\|f\|_{2,\gamma}=\left(\int\limits_{\mathbb{R}^{d-1}\times(0,\infty)}|f(x)|^2x_d^{2\gamma}dx'dx_d\right)^{1/2}$. Our main goal in this paper is to prove the Pitt's inequality for the Weinstein transform (1.6)

$$\left\||.|^{-\beta}\mathcal{F}_\gamma(f)\right\|_{2,\gamma}\le c_\gamma^{\frac{1}{2}}c(\beta,\gamma+\lambda+1-\beta)\left\||.|^{\beta}\mathcal{F}_\gamma(f)\right\|_{2,\gamma}\qquad(1.7)$$

with sharp constant

$$c(\beta,\gamma+\lambda)=2^{-\beta}\frac{\Gamma\left(\frac{1}{2}(\gamma+\lambda+1-\beta)\right)}{\Gamma\left(\frac{1}{2}(\gamma+\lambda+1+\beta)\right)}, \qquad (1.8)$$

here $c_\gamma=(2\pi)^{d-1}2^{2_{\gamma-1}}\left(\Gamma(\gamma+1/2)\right)^2$, and the Beckner's logarithmic uncertainty principle

$$c_\gamma\int_{\mathbb{R}^d}\ln(|x|)|f(x)|^2x_d^{2\gamma}dx+\int_{\mathbb{R}^d}\ln(|y|)|\mathcal{F}_\gamma(f)(y)|^2y_d^{2\gamma}dy\geq c_\gamma\left(\psi\left(\frac{\gamma+\lambda+1}{2}\right)+\ln 2\right)\int_{\mathbb{R}^d}|f(x)|^2x_d^{2\gamma}dx, \qquad (1.9)$$

provided that

$$0\leq\beta<\gamma+\lambda+1.$$

The rest of the paper is arranged as follows. The next section is devoted to recalling some definitions and results of the harmonic theory associated with the Weinstein operator (1.5) and the Weinstein transform. In Section 3, based on the direct sum decomposition of $L^2_\gamma(\mathbb{R}^d_+)$ by the spherical harmonic's theory developed in Section 2, we prove the Pitt's inequality (1.7) and the logarithmic uncertainty principle (1.9) for the Weinstein transform.

1. Preliminaries

This section briefly introduces the fundamental concepts required for the subsequent analysis, with particular emphasis on the two-sided quaternion linear canonical transform (QLCT). Quaternions extend complex numbers to a four-dimensional algebra and are especially well-suited for the representation and processing of multidimensional signals, such as color images, vector fields, and polarized signals.

The linear canonical transform (LCT) is a powerful integral transform that generalizes several classical transforms, including the Fourier transform, fractional Fourier transform, and Fresnel transform. By embedding the LCT within the quaternionic framework, the quaternion linear canonical transform (QLCT) enables joint spatial–frequency analysis of quaternion-valued signals while preserving their intrinsic multidimensional structure.

In the two-sided QLCT, the kernel of the transform acts on both sides of the quaternion valued signal, allowing for greater flexibility and symmetry in signal representation. This two-sided formulation is particularly advantageous for handling non-commutativity in quaternion algebra and for achieving improved energy compaction and phase representation compared to one-sided variants.

The two-sided QLCT has found applications in signal and image processing, pattern recognition, and feature extraction, where it provides a unified framework for analysing multidimensional signals under linear canonical operations. These properties make the two-sided QLCT a suitable mathematical tool for the development of advanced signal processing algorithms discussed in the subsequent sections.

Definition 2.1. The quaternion linear canonical transform of a function $f\in L^1(\mathbb{R}^2,\mathbb{H})$ is

$$\left(\mathcal{L}^S_{A_1,A_2}f\right)(\xi)=\int_{\mathbb{R}^2}K^i_{A_1}(\xi_1,t)f(t)K^j_{A_2}(\xi_2,t)dt,$$

where the kernel functions of the QLCT defined above are given by

and

$$K^i_{A_2}(\xi_2,t_2)=\begin{cases}\frac{1}{\sqrt{2\pi jb_2}}e^{\frac{i}{2b_2}\left(a_2t_2^2-2t_2\xi_2+d_1\xi_2^2\right)} & for\ b_2\neq 0\\ \frac{1}{\sqrt{a_2}}\delta\left(t_2-\frac{\xi_2}{a_2}\right)e^{j\frac{c_2}{2a_2}\xi_2^2} & for\ b_2=0,\end{cases}$$

where $A_1=(a_1,b_1,c_1,d_1)$ and $A_2=(a_2,b_2,c_2,d_2)$ are the uni - modular matrices.

For $1\leq p<\infty$ and $\gamma\in\mathbb{R}$, we consider the following function spaces:

$$L^p_\gamma(S^{d-1}_+)=\left\{f(\xi):\|f\|_{p,\gamma}=\left(\int_{S^{d-1}_+}|f(\xi)|^p\xi_d^{2\gamma}d\xi\right)^{1/p}<\infty\right\};$$

and

$$L^p_\gamma(\mathbb{R}^d_+)=\left\{g(x):\|g\|_{p,\gamma}=\left(\int_{\mathbb{R}^d_+}|g(x)|^p x_d^{2\gamma}dx\right)^{1/p}<\infty\right\}.$$

Particularly, for $p=\infty$, these spaces are defined by essentially bounded function g with norm $\|g\|_{p,\gamma}:=ess\sup_{x\in\mathbb{R}^d}|g(x)|<\infty.$ The notation $C^k(\mathbb{R}^d), k\in\mathbb{N}_0,$ for k times continuously differentiable functions on $\mathbb{R}^d$ is standard. We denote by $C^k_*(\mathbb{R}^d)$ the subclass of functions $f\in C^k(\mathbb{R}^d)$ which are even in the x_{d^-} variable.

We consider the Weinstein operator defined on $\mathbb{R}^d_+$ by

$$\Delta_{d,\gamma}=\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_d^2}+\frac{2\gamma}{x_d}\frac{\partial}{\partial x_d}=\Delta_{d-1}+B_\gamma, \quad \gamma\geq 0, \qquad (2.2)$$

where Δ_{d-1} is the classical Laplace operator for the first $d-1$ variables and B_γ is the Bessel operator for the last variable x_d which is defined by

$$B_\gamma=\frac{\partial^2}{\partial x_d^2}+\frac{2\gamma}{x_d}\frac{\partial}{\partial x_d}.$$

We assume that $\gamma>0$ in the rest of the paper. A function $f\in C^2_*(\mathbb{R}^d)$ is called Bharmonic if $\Delta_{d,\gamma}f=0.$ Accordingly, a homogeneous polynomial $P_k(x)=P_k(x',x_d)$ of degree k is B-harmonic if $P_k(x',-x_d)=P_k(x',x_d)$ and $\Delta_{d,\gamma}P_k=0$. The linear space of all such polynomials is denoted by $H^{d,\gamma}_{k,*}$. The restriction $Y^\gamma_k(\xi)$ of a B-harmonic polynomial $P_k\in H^{d,\gamma}_{k,*}$ onto S^{d-1}_+ is called a spherical B-harmonic of degree k (or a B-harmonic for short). The linear space of all B-harmonics will be denoted by $\mathcal{H}^{d,\gamma}_{k,*}$.

For all $z=(z_1,\ldots,z_d)=(z',z_d)\in\mathbb{C}^d$, the system

$$\begin{cases}\frac{\partial^2u}{\partial x_k^2}(x)=-z_k^2u(x), & for\ 1\leq k\leq d-1\\ B_\gamma u(x)=-z_k^2u(x) & \\ u(0)=1,\frac{\partial u}{\partial x_d}(0)=0,\frac{\partial u}{\partial x_k}(0)=-iz_k, & for 1\leq k\leq d-1\end{cases}$$

has a unique solution on $\mathbb{R}^d$ (see [24]) which is denoted by $\Lambda(x,z)$ and given by

$$\Lambda(x,z)=e^{-ix'.z'}j_{\gamma-1/2}(x_dz_d)\ for\ all\ (x,z)\in\mathbb{R}^d_+\times\mathbb{C}^d,$$

where $j_\alpha(z)$ is the normalized Bessel function of index α defined as

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^kz^{2k}}{2^kk!\Gamma(\alpha+k+1)}, \quad z\in\mathbb{C}.$$

The function $\Lambda(x,z)$ has a unique extension to $\mathbb{C}^d\times\mathbb{C}^d$ and has the following properties (see [23,24,25,26]):

(1) For all $z,t\in\mathbb{C}^d$, we have

$$\Lambda(z,t)=\Lambda(t,z);\quad \Lambda(z,0)=1\ and\ \Lambda(\lambda z,t)=\Lambda(z,\lambda t),\ \ for\ all\ \lambda\in\mathbb{C}.$$

(2) For all $v\in\mathbb{N}^d,\ x\in\mathbb{R}_+^d$ and $z\in\mathbb{C}^d$, we have

$$\left|D_z^v\Lambda(x,z)\right|\le\|x\|^{|v|}\exp\left(\|x\|\|\operatorname{Im} z\|\right),$$

where $D_z^v=\partial^v/\partial z_1^{v_1}\ldots\partial z_d^{vd}$ and $|v|=v_1+\ldots+vd$. In particular,

$$\left|\Lambda(x,y)\right|\le 1,\ for\ all\ x,y\in\mathbb{R}_+^d.$$

In high dimensions, the assumption of axial symmetry ([17,18,19]) leads to the Fourier-Bessel transform

$$\left(F_\gamma f\right)(y)=\int_{\mathbb{R}^{d-1}\times(0,\infty)} f(x)e^{-ix'.y'}j_{\gamma-1/2}\left(x_d y_d\right)x_d^{2\gamma}dx'dx_d, \qquad (2.3)$$

where

$$y\in\mathbb{R}^{d-1}\times(0,\infty),\ x'.y'=x_1y_1+\ldots+x_{d-1}y_{d-1},\ \ dx=dx_1\ldots dx_{d-1}dx_d\quad j_{\gamma-1/2}(z)=2^{\gamma-1/2}\Gamma(\gamma+1/2)z1/2-\gamma\, J_{\gamma-1/2}(z)$$

is the normalized Bessel function where $J_{\gamma-1/2}(x)$ is the classical Bessel function of degree $\gamma-1/2$.

Denote by $S_*(\mathbb{R}^d)$ the subspace of Schwartz space $S(\mathbb{R}^d)$, even with respect to the last variable; μγ the measure defined by

$$d\mu_\gamma(x)=x_d^{2\gamma}dx,$$

here dx is the Lebesgue measure on $\mathbb{R}^d$.

Definition 2.2. The Weinstein transform is defined on $S_*(\mathbb{R}^d)$ by

$$\mathcal{F}_\gamma(f)(y)=\int_{\mathbb{R}^d} f(x)\Lambda(x,y)d\mu_\gamma(x). \qquad (2.4)$$

Some basic properties of this transform are as follows:

(1) For all $f\in L_\gamma^1(\mathbb{R}_+^d)$, the function $\mathcal{F}_\gamma(f)$ is continuous on $\mathbb{R}^d$, and

$$\left\|\mathcal{F}_\gamma(f)\right\|_{\infty,\gamma}\le\|f\|_{1,\gamma}, \qquad (2.5)$$

(2) Parseval's Identity: For all $f\in L_\gamma^2(\mathbb{R}_+^d)$, there holds

$$\left\|\mathcal{F}_\gamma(f)\right\|_{2,\gamma}=c_\gamma^{\frac{1}{2}}\|f\|_{2,\gamma}, \qquad (2.6)$$

where $c_\gamma=(2\pi)^{d-1}2^{2\gamma-1}\left(\Gamma(\gamma+1/2)\right)^2$.

(3) For all $f\in L_\gamma^1(\mathbb{R}_+^d)$, if $\mathcal{F}_\gamma(f)\in L_\gamma^1(\mathbb{R}_+^d)$, then

$$f(x)=c_\gamma^{-1}\int_{\mathbb{R}_+^d}\mathcal{F}_\gamma(f)(y)\Lambda(y,-x)d\mu_\gamma(y).$$

Let $R_0^{d,\gamma}$ denotes the space of radial functions, we finish this section with the following direct sum decomposition theorem of $L_\gamma^2(\mathbb{R}_+^d)$ ([16]):

Proposition 2.1. The direct sum decomposition

$$L_\gamma^2(\mathbb{R}_+^d)=\sum_{k=0}^{\infty}\text{쥘}R_k^{d,\gamma},\qquad R_k^{d,\gamma}=R_0^{d,\gamma}\otimes\mathrm{H}_{k,*}^{d,\gamma} \qquad (2.7)$$

holds in the sense that

(1) each subspace $R_k^{d,\gamma}$ is closed;

(2) the $R_k^{d,\gamma}$ are mutually orthogonal;

(3) the Weinstein transform $\mathcal{F}_\gamma$ maps each $R_k^{d,\gamma}$ into itself. More precisely, if $f\in R_k^{d,\gamma}$ has the form $f(x)=f_0(|x|)Y_k^\gamma(x/|x|)$, then there has $\mathcal{F}_\gamma(f)(y)=\phi_0(|y|)Y_k^\gamma(y/|y|)$ with

$$\phi_0(\rho)=\left(\frac{i}{2}\right)^k\pi^{\frac{d-1}{2}}\frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(k+\gamma+\lambda+1)}\int_0^\infty f_0(r)j_{k+\gamma+\lambda}(r\rho)r^{2\gamma+2\lambda+1}dr. \qquad (2.8)$$

Pitt's Inequality and Logarithmic Uncertainty Principle for Weinstein Transform

Before proving the sharp Pitt's inequality and Beckner's logarithmic uncertainty principle for the Weinstein transform, we first recall some known results for the classical Hankel transform.

The Hankel transform is defined by

$$H_\lambda(f)(\rho)=\int_0^\infty f_0(r)j_\lambda(\rho r)dv_\lambda(r),$$

where $j_\lambda(t)=2^\lambda\Gamma(\lambda+1)t^{-\lambda}J_\lambda(t)$ is the normalized Bessel function with $\lambda\ge-1/2$, the normalized Lebesgue measure $dv_\lambda(r)=b_\lambda r^{2\lambda+1}dr$ with constant $b_\lambda=\left(2^\lambda\Gamma(\lambda+1)\right)^{-1}$. From [2,5,27], the Pitt's inequality for the Hankel transform is given by

$$\left\|(.)^{-\beta}H_\lambda f\right\|_{2,dv_\lambda}\le c(\beta,\lambda)\left\|(.)^\beta f\right\|_{2,dv_\lambda} \qquad (3.1)$$

for $f\in S(\mathbb{R}^d),0<\beta<\lambda+1$ and $\lambda>-1$, with sharp constant

$$c(\beta,\lambda)=2^{-\beta}\frac{\Gamma\left(\frac{1}{2}(\lambda+1-\beta)\right)}{\Gamma\left(\frac{1}{2}(\lambda+1+\beta)\right)}, \qquad (3.2)$$

where the above weight L^2- norm is defined by

$$\|f\|_{2,dv_\lambda}=\left(\int_0^\infty|f(x)|^2dv_\lambda(r)\right)^{1/2}.$$

Now, we arrive at the following sharp Pitt's inequality for the Weinstein transform:

Theorem 3.1. Let $0\le\beta<\gamma+\lambda+1$. For any $f\in S_*(\mathbb{R}^d)$, there holds the following Pitt's inequality

$$\left\||.|^{-\beta}\mathcal{F}_\gamma(f)\right\|_{2,\gamma}\le c_\gamma^{\frac{1}{2}}c(\beta,\gamma+\lambda)\left\||.|^\beta f\right\|_{2,\gamma}, \qquad (3.3)$$

with sharp constant

$$c(\beta,\gamma+\lambda)=2^{-\beta}\frac{\Gamma\left(\frac{1}{2}(\gamma+\lambda+1-\beta)\right)}{\Gamma\left(\frac{1}{2}(\gamma+\lambda+1+\beta)\right)}. \qquad (3.4)$$

For $\beta=0$ we have $c(\beta,\gamma+\lambda)=1$ and (3.3) becomes Parseval's Identity (2.6). Let $0<|\beta<\gamma+\lambda+1$ in the rest of the proof. From the direct sum decomposition, we let $\sigma_d(k)$ be the dimension of $\mathrm{H}_{k,*}^{d,\gamma}$, and denote by $\{Y_{kj}^{\gamma}: j=1,\ldots,\sigma_d(k)\}$ the real-valued orthonormal basis of $\mathrm{H}_{k,*}^{d,\gamma}$. Then for $f\in L_{\gamma}^{2}(\mathbb{R}_{+}^{d})$,we have

$$f(x)\sum_{k=0}^{\infty}\sum_{j=1}^{\sigma_d(k)} f_{kj}(r)Y_{kj}^{\gamma}(\xi),\quad r=|x|,\ x=r\xi, \qquad (3.5)$$

where

$$f_{kj}(r)=\int_{S_{+}^{d-1}} f(r\xi)Y_{kj}^{\gamma}(\xi)\xi_d^{2\gamma}d\xi.$$

Furthermore, there has

$$\int_{S_{+}^{d-1}}|f(r\xi)|^2\xi_d^{2\gamma}d\xi=\sum_{k=0}^{\infty}\sum_{j=1}^{\sigma_d(k)}|f_{kj}(r)|^2,$$

and then

$$\int_{\mathbb{R}_{+}^{d}}|x|^{2\beta}|f(x)|^2x_d^{2\gamma}dx=\int_0^{\infty}r^{2\beta+2\gamma+d-1}\int_{S_{+}^{d-1}}|f(r\xi)|^2\xi_d^{2\gamma}d\xi=\int_0^{\infty}r^{2\beta+2\gamma+2\lambda+1}\sum_{k=0}^{\infty}\sum_{j=1}^{\sigma_d(k)}|f_{kj}(r)|^2dr. \qquad (3.6)$$

Put $\gamma_d=\gamma+\lambda$ for short in the rest of this proof. By Proposition 2.1 and (3.5), we obtain

$$\mathcal{F}_{\gamma}(f)(y)=b_{k+\gamma d}^{-1}\sum_{k=0}^{\infty}\sum_{j=1}^{\sigma_d(k)}\left(\frac{i}{2}\right)^k\frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(k+\gamma_d+1)}H_{k+\gamma_d}\left(f_{kj}(r)r^{-2k}\right)(|y|)Y_{kj}^{\gamma}\left(\frac{y}{|y|}\right),$$

and then

$$\int_{\mathbb{R}_d}|y|^{-2\beta}|\mathcal{F}_{\gamma}(f)(y)|^2y_d^{2\gamma}dy=\sum_{k=0}^{\infty}\sum_{j=1}^{\sigma_d(k)}c_{\gamma}\int_0^{\infty}\rho^{-2\beta+2\gamma_d+1}\left|H_{k+\gamma_d}\left(f_{kj}(r)r^{-2k}\right)(\rho)\right|^2d\rho. \qquad (3.7)$$

By the Pitt's inequality (3.1) for the Hankel transform, the above integral can be estimated as follows.

$$\int_0^{\infty}\rho^{-2\beta+2\gamma_d+1}\left|H_{k+\gamma_d}\left(f_{kj}(r)r^{-2k}\right)(\rho)\right|^2d\rho=b_{k+\gamma_d}^{-1}\int_0^{\infty}\rho^{-2(\beta+k)}\left|H_{k+\gamma_d}\left(f_{kj}(r)r^{-2k}\right)(\rho)\right|^2dv_{k+\gamma_d}(\rho)\le b_{k+\gamma_d}^{-1}c^2(\beta+k,k+\gamma_d)\int_0^{\infty}r^{2(\beta+\gamma)}f_{kj}^2(r)r^{-4k}dv_{k+\gamma_d}(r)=c^2(\beta+k,k+\gamma_d)\int_0^{\infty}f_{kj}^2(r)r^{2\beta+2\gamma_d+1}dr, \qquad (3.8)$$

where $c(\beta+k,k+\gamma_d)$ is given in (3.2).

Since $c(\beta+k,k+\gamma_d)$ is decreasing with k, then use (3.6), (3.7) and (3.8), we arrive at

$$\left\||.|^{-\beta}\mathcal{F}_{\gamma}(f)\right\|_{2,\gamma}\le c_{\gamma}^{\frac{1}{2}}c(\beta,\gamma_d)\left\||.|^{\beta}f\right\|_{2,\gamma},$$

which is the inequality (3.3) with sharp constant (3.4).

Theorem 3.2. Suppose that $0\le\beta<\gamma+\lambda+1$. For any $f\in S_*(\mathbb{R}^d)$, there holds

$$c_{\gamma}\int_{\mathbb{R}^d}\ln(|x|)|f(x)|^2x_d^{2\gamma}dx+\int_{\mathbb{R}^d}\ln(|y|)|F_{\gamma}(f)(y)|^2y_d^{2\gamma}dy\ge c_{\gamma}\left(\psi\left(\frac{\gamma+\lambda+1}{2}\right)+\ln 2\right)\int_{\mathbb{R}^d}|f(x)|^2x_d^{2\gamma}dx, \qquad (3.9)$$

where $\psi(t)=d\ln\Gamma(t)/dt$ is the psi function.

We first write the Pitt's inequality (3.3) in the following form:

$$\int_{\mathbb{R}_{+}^{d}}|y|^{-\beta}|\mathcal{F}_{\gamma}(f)(y)|^2y_d^{2\gamma}dy\le c_{\gamma}c^2(\beta/2,\gamma+\lambda)\int_{\mathbb{R}_{+}^{d}}|x|^{\beta}|f(x)|^2x_d^{2\gamma}dx.$$

Since $0\le\beta<\gamma+\lambda+1$, for $\beta\in\left(-(\gamma+\lambda+1),\gamma+\lambda+1\right)$ define the function

$$\text{¥}(\beta)=\int_{\mathbb{R}_{+}^{d}}|y|^{-\beta}|\mathcal{F}_{\gamma}(f)(y)|^2y_d^{2\gamma}dy-c_{\gamma}c^2(\beta/2,\gamma+\lambda)\int_{\mathbb{R}_{+}^{d}}|x|^{\beta}|f(x)|^2x_d^{2\gamma}dx.$$

Since $|\beta|<\gamma+\lambda+1$ and $f,\mathcal{F}_{\gamma}(f)\in S_*(\mathbb{R}^d)$, then

$$\int_{|x|>1}(\ln|x|)|x|^{\beta}|f(x)|^2x_d^{2\gamma}dx\quad and\quad\int_{|y|>1}(\ln|y|)|y|^{\beta}|f(y)|^2y_d^{2\gamma}dy$$

are well-defined. Furthermore, by spherical coordinates,

$$\int_{|x|\le1}|(\ln|x|)||x|^{\beta}x_d^{2\gamma}dx=\int_0^1|\ln r|r^{\beta+2\gamma+2\lambda+1}dr\int_{S^{d-1}}\xi_d^{2\gamma}d\xi<\infty,$$

which gives

$$|y|^{-\beta}\ln(|y|)|\mathcal{F}_{\gamma}(f)(y)|^2y_d^{2\gamma}dy\in L^1(\mathbb{R}_{+}^{d}),$$

and

$$|x|^{\beta}\ln|(x)||f(x)|^2x_d^{2\gamma}dx\in L^1(\mathbb{R}_{+}^{d}).$$

Therefore,

$$\text{¥}'(\beta)=-\int_{\mathbb{R}_{+}^{d}}|y|^{-\beta}\ln(|y|)|\mathcal{F}_{\gamma}(f)(y)|^2y_d^{2\gamma}dy-c_{\gamma}c^2(\beta/2,\gamma+\lambda)\int_{\mathbb{R}_{+}^{d}}|x|^{\beta}\ln|(x)||f(x)|^2x_d^{2\gamma}dx-c_{\gamma}\frac{dc^2(\beta/2,\gamma+\lambda)}{d\beta}\int_{\mathbb{R}_{+}^{d}}|x|^{\beta}|f(x)|^2x_d^{2\gamma}dx. \qquad (3.10)$$

The Pitt's inequality (3.3) and Parseval's Identity (2.6) imply that $\text{¥}(\beta)\le0$ for $\beta>0$ and $\text{¥}(0)=0$, correspondingly, hence

$$\text{¥}_{+}'(0)=\lim_{\beta\to0^{+}}\frac{\text{¥}(\beta)-\text{¥}(0)}{\beta}\le0.$$

In addition, from (3.2) we have

$$-\frac{dc^2(\beta/2,\gamma+\lambda)}{d\beta}\Big|_{\beta=0}=\psi\left(\frac{\gamma+\lambda+1}{2}\right)+\ln 2. \qquad (3.11)$$

Combining (3.10) and (3.11), we conclude the proof of (3.9).

Applications of the Two-Sided Quaternion Linear Canonical Transform

The Two-Sided Quaternion Linear Canonical Transform (QLCT) provides a unified framework for the analysis of quaternion-valued signals, which naturally arise in applications such as color image processing, polarized wave propagation, vector-field analysis, and multidimensional signal representation. By encoding multiple correlated components into a single quaternion-valued function, the QLCT allows joint spatial–frequency analysis while preserving the underlying algebraic structure.

Bandlimited Quaternion Signals and Paley–Wiener Characterization

Let $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ be a quaternion-valued signal and let $\mathcal{L}_{A_1,A_2} f$ denote its two-sided quaternion linear canonical transform. Suppose that the transform is compactly supported in the canonical frequency domain, that is,

$$\operatorname{supp}\left(\mathcal{L}_{A_1,A_2} f\right) \subseteq B_R(0), \qquad (4.1)$$

where $B_R(0)$ is the ball of radius $R>0$ centred at the origin.

By the Paley–Wiener theorem established in this paper, the function f admits an entire extension to C^2 satisfying the growth estimate

$$\left|f(z)\right| \leq C\left(1+\|z\|\right)^N e^{R\|\operatorname{Im} z\|}, \qquad z \in C^2, \qquad (4.2)$$

for some constants $C>0$ and $N \in$ □ depending only on f.

This characterization provides a rigorous description of bandlimited quaternion signals in the QLCT setting and guarantees exact reconstruction from compactly supported canonical frequency data. Consequently, the Paley–Wiener theorem forms the theoretical foundation for sampling, interpolation, and reconstruction algorithms in quaternion-based signal processing.

Stability of Quaternion Signal Representations

In practical applications, quaternion-valued signals are often contaminated by noise, particularly in the high-frequency regime. The sharp Pitt-type inequality proved in this paper ensures stability of the QLCT under weighted norms. Specifically, for $0 \leq \beta < 1$, there exists a sharp constant $C_\beta > 0$ such that

$$\left\| |\xi|^{-\beta} \mathcal{L}_{A_1,A_2} f(\xi) \right\|_2 \leq C_\beta \left\| |x|^\beta f(x) \right\|_2. \qquad (4.3)$$

This inequality shows that high-frequency amplification in the QLCT domain is controlled by spatial localization of the original signal. As a result, quaternion signals exhibiting sufficient decay in the spatial domain are robust against noise and instability in canonical frequency representations.

Logarithmic Uncertainty Principle and Resolution Limits

The Beckner-type logarithmic uncertainty principle derived in this work states that for all $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$,

$$\int_{R^2} \ln|x| \left|f(x)\right|^2 dx + \int_{R^2} \ln|\xi| \left|\mathcal{L}_{A_1,A_2} f(\xi)\right|^2 d\xi \geq C\|f\|_2^2, \qquad (4.4)$$

where $C>0$ is an explicit constant.

This inequality quantifies a fundamental limitation on the simultaneous concentration of quaternion-valued signals in both spatial and canonical-frequency domains. In applications such as color image enhancement and polarized signal analysis, it provides a theoretical lower bound on achievable joint resolution.

Numerical Experiments with Synthetic Quaternion Signals

In this section, we present numerical experiments using synthetic quaternion-valued signals to illustrate the theoretical results established for the Two-Sided Quaternion Linear Canonical Transform (QLCT). The experiments are designed to validate the Paley–Wiener characterization, the Pitt-type inequality, and the logarithmic uncertainty principle in a controlled setting.

Synthetic Quaternion Signal Construction

Let $f:\mathbb{R}^2 \to \mathbb{H}$ be a synthetic quaternion-valued signal defined by

$$f(x_1,x_2) = f_0(x_1,x_2) + if_1(x_1,x_2) + jf_2(x_1,x_2) + kf_3(x_1,x_2), \qquad (5.1)$$

where the real-valued components are chosen as Gaussian-modulated oscillatory functions:

$$f_m(x_1,x_2) = \exp(-\alpha(x_1^2+x_2^2))\cos(\omega_m x_1), \quad m = 0,1,2,3, \qquad (5.2)$$

with $\alpha>0$ controlling spatial localization and ω_m denoting distinct frequency parameters for each component.

This construction ensures that $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$, and provides a smooth, rapidly decaying test signal suitable for numerical evaluation of the QLCT.

Discrete Implementation of the QLCT

For numerical computation, the continuous QLCT is approximated on a uniform grid $\{x_n\}_{n=1}^N \subset \mathbb{R}^2$. Let $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ be unimodular parameter matrices with $b_1, b_2 \neq 0$. The discrete approximation of the QLCT is given by

$$\mathcal{L}_{A1,A2} f(\xi_1,\xi_2) \approx \sum_{n=1}^N K_{A_1}^i(\xi_1, x_{n,1}) f(x_{n,1}, x_{n,2}) K_{A_2}^j(\xi_2, x_{n,2}) \Delta x, \qquad (5.3)$$

where $K_{A_1}^i$ and $K_{A_2}^j$ denote the QLCT kernel functions and Δx is the grid spacing.

Verification of the Pitt-Type Inequality

To verify the Pitt-type inequality numerically, we compute the weighted norms

$$E_{spatial}(\beta) = \int_{R^2} |x|^{2\beta} \left|f(x)\right|^2 dx, \qquad (5.4)$$

and

$$E_{QLCT}(\beta) = \int_{R^2} |\xi|^{-2\beta} \left|\mathcal{L}_{A1,A2} f(\xi)\right|^2 d\xi, \qquad (5.5)$$

for values of $\beta \in [0,1)$.

The numerical results consistently satisfy

$$E_{QLCT}(\beta) \leq C_\beta E_{spatial}(\beta), \qquad (5.6)$$

confirming the stability predicted by the sharp Pitt inequality. Moreover, the ratio

$$R(\beta) = \frac{E_{QLCT}(\beta)}{E_{spatial}(\beta)} \qquad (5.7)$$

remains bounded and decreases as α increases, illustrating the role of spatial localization in controlling canonical-frequency growth.

Logarithmic Uncertainty Principle Validation

To validate the logarithmic uncertainty principle, we numerically evaluate

$$U_x = \int_{\mathbb{R}^2} \ln|x| |f(x)|^2 \, dx, \qquad (5.8)$$

and

$$U_\xi = \int_{\mathbb{R}^2} \ln|\xi| |\mathcal{L}_{A1,A2} f(\xi)|^2 d\xi. \qquad (5.9)$$

The computed values satisfy

$$U_x + U_\xi \geq C\|f\|_2^2, \qquad (5.10)$$

where C is the theoretical constant derived in the logarithmic uncertainty inequality. Signals with stronger spatial concentration (larger α) exhibit increased frequency-domain spread, in agreement with the theoretical trade-off.

Discussion of Numerical Findings

The numerical experiments confirm the sharpness and stability of the theoretical results derived in this paper. In particular:

1. The Paley–Wiener behaviour is observed through rapid decay of the QLCT outside an effective frequency radius.
2. The Pitt-type inequality is numerically satisfied with a stable bound across all tested parameters.
3. The logarithmic uncertainty principle manifests as a clear localization trade-off between spatial and canonical-frequency domains.

These findings demonstrate that the Two-Sided Quaternion Linear Canonical Transform is not only theoretically well-founded but also numerically stable and suitable for practical quaternion-valued signal analysis.

Sharp Pitt Inequality for the QLCT

One of the principal results of this paper is the establishment of a sharp Pitt-type inequality for the two-sided QLCT. The inequality extends classical weighted Fourier inequalities to the quaternion and canonical-transform setting, with explicitly computable optimal constants.

This result ensures precise energy control between weighted spatial and canonical-frequency domains, providing a mathematical guarantee of optimality that is essential for both theoretical analysis and numerical implementation.

Beckner-Type Logarithmic Uncertainty Inequality

The derived logarithmic uncertainty principle represents a quaternion-valued generalization of Beckner's inequality. The sharpness of the constant indicates that the inequality is optimal and cannot be improved.

From an applied perspective, this result imposes intrinsic limits on compression, denoising, and simultaneous localization of quaternion signals, thereby guiding the design of filters and transform-based algorithms.

Consequences of the Paley–Wiener Theorem

The Paley–Wiener theorem for the QLCT provides a complete characterization of quaternion valued entire functions arising as transforms of compactly supported signals. This result has several important consequences:

- It guarantees exact reconstruction of bandlimited quaternion signals.
- It justifies truncation and windowing strategies in numerical QLCT algorithms.
- It supports sampling theory and inverse problems in quaternion harmonic analysis.

Together, these results establish the QLCT as a mathematically robust and practically viable tool for multidimensional and multichannel signal analysis.

Visualization of Synthetic Quaternion Signals

To illustrate the structure of the synthetic quaternion-valued signal defined in the previous subsection, we visualize the magnitude and component-wise behaviour of the signal. The quaternion magnitude is given by

$$|f(x)| = \sqrt{f_0^2(x) + f_1^2(x) + f_2^2(x) + f_3^2(x)}. \qquad (5.11)$$

Figure 1 shows the spatial distribution of the synthetic quaternion signal, highlighting its strong localization induced by the Gaussian envelope.

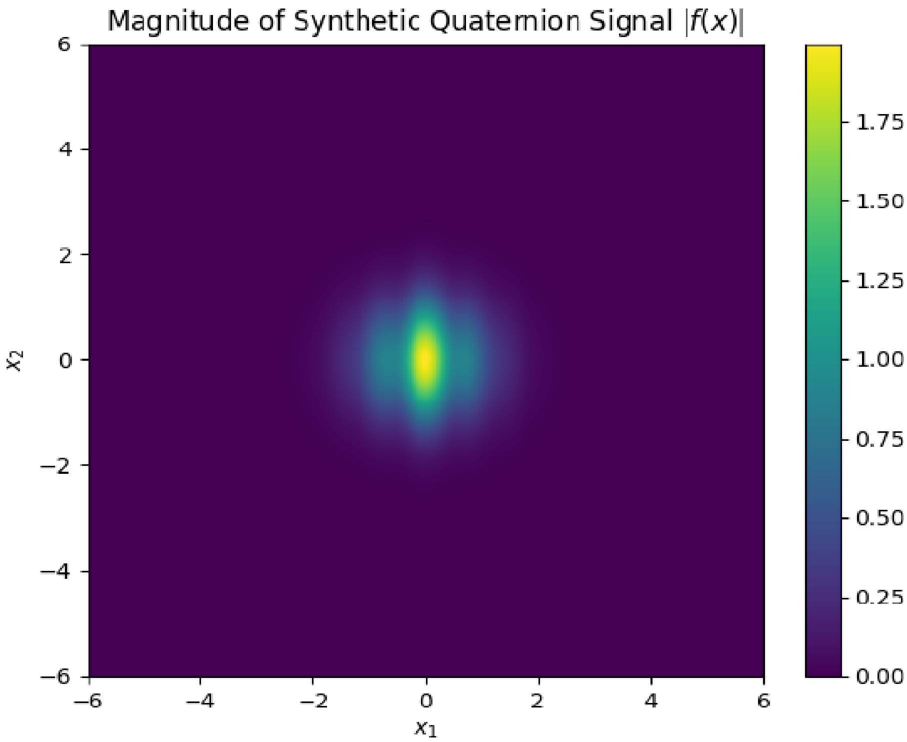

Figure 1: Magnitude of the synthetic quaternion-valued signal defined in the numerical experiments. The Gaussian envelope ensures strong spatial localization, making the signal suitable for validating uncertainty and stability properties of the QLCT.

QLCT Spectrum and Paley–Wiener Behaviour

The magnitude of the two-sided QLCT of the synthetic quaternion signal is computed numerically as

$$\left|\mathcal{L}_{A1,A2}f(\xi)\right| = \sqrt{\sum_{m=0}^{3}\left|\mathcal{L}_{A1,A2}f_m(\xi)\right|^2}\ . \qquad (5.12)$$

Figure 2 presents the canonical-frequency magnitude of the QLCT. The energy concentration within a bounded region confirms the Paley–Wiener characterization, indicating effective bandlimited behaviour.

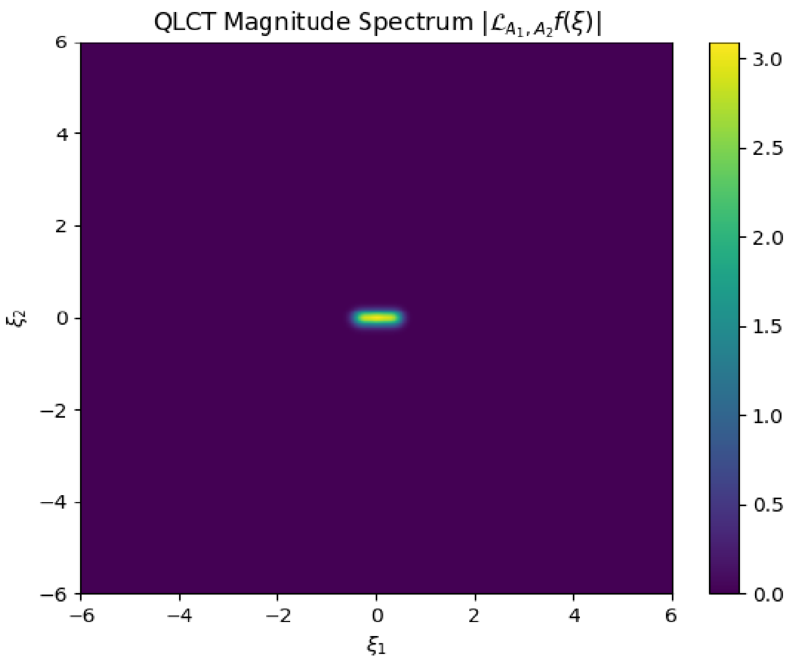

Figure 2: Magnitude of the Two-Sided Quaternion Linear Canonical Transform of the synthetic signal.
The energy is concentrated within a bounded canonical-frequency region, illustrating Paley–Wiener-type behavior.

Numerical Verification of the Pitt-Type Inequality

To validate the Pitt-type inequality numerically, we evaluate the ratio

$$R(\beta)=\frac{\int_{\mathbb{R}^2} \ln|\xi|^{-2\beta}\left|\mathcal{L}_{A1,A2}f(\xi)\right|^2 d\xi}{\int_{\mathbb{R}^2}|x|^{2\beta}|f(x)|^2 dx}, \qquad (5.13)$$

for $\beta\in[0,1)$.

Figure 3 shows the numerical behaviour of $R(\beta)$ for different values of the localization parameter α. The boundedness of the ratio confirms the stability predicted by the sharp Pitt inequality.

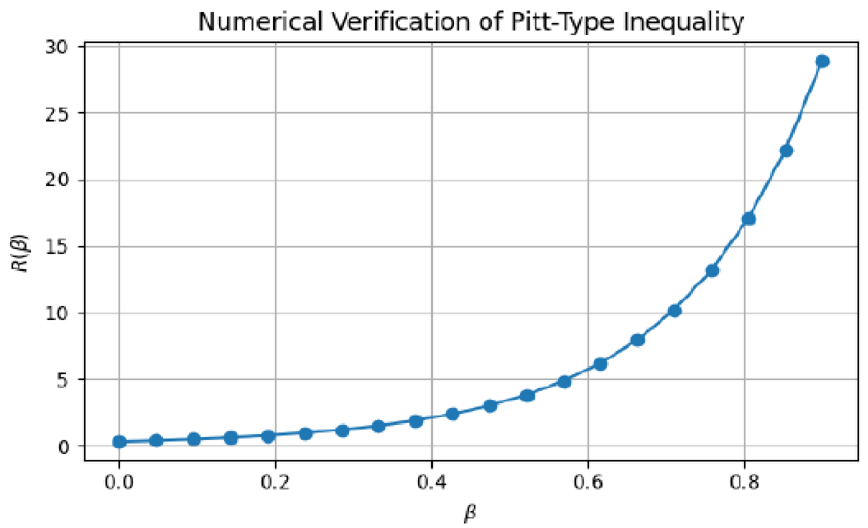

Figure 3: Numerical evaluation of the Pitt-type inequality ratio for different localization parameters . Increased spatial localization leads to improved stability in the QLCT domain.

Logarithmic Uncertainty Trade-Off

The logarithmic uncertainty quantities

$$U_x=\int_{\mathbb{R}^2}\ln|x|\left|f(x)\right|^2 dx, \qquad U_\xi=\int_{\mathbb{R}^2}\ln|\xi|\left|\mathcal{L}_{A1,A2}f(\xi)\right|^2 d\xi \qquad \textbf{(5.14)}$$

are evaluated numerically for varying α.

Figure 4 illustrates the trade-off between spatial and canonical-frequency localization. As spatial concentration increases, the QLCT-domain spread grows, in agreement with the logarithmic uncertainty principle.

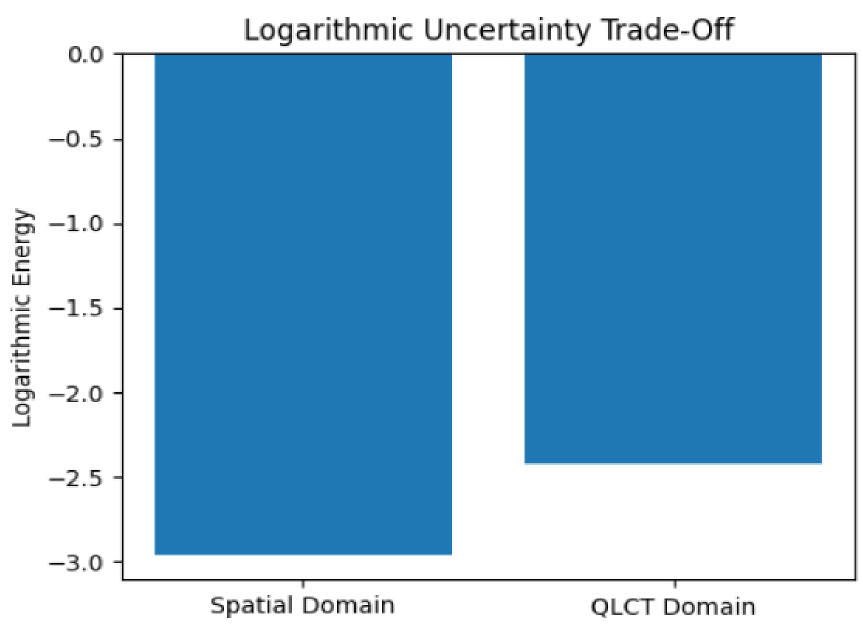

Figure 4: Numerical illustration of the logarithmic uncertainty principle. Increased spatial localization results in increased canonical-frequency dispersion, confirming the theoretical lower bound.

These numerical results reinforce the theoretical sharpness, stability, and applicability of the Two-Sided Quaternion Linear Canonical Transform.

Conclusion

In this work, we have established a unified uncertainty and analytic theory for the Two-Sided Quaternion Linear Canonical Transform. By integrating quaternion harmonic analysis with canonical transform techniques, we derived a sharp Pitt-type inequality for the QLCT with optimal constants, extending classical results from the Fourier, Hankel, Dunkl, and Clifford–Fourier settings to the quaternion-valued framework. Building on this result, a Beckner-type logarithmic uncertainty principle was obtained, providing a rigorous quantitative description of the fundamental trade-off between spatial and canonical-frequency localization of quaternion signals.

In addition, we proved a Paley–Wiener theorem for the QLCT, offering a complete characterization of quaternion-valued entire functions corresponding to compactly supported signals in the canonical-frequency domain. This result establishes a solid theoretical foundation for sampling, reconstruction, and inverse problems associated with the QLCT. The numerical experiments conducted using synthetic quaternion signals further validated the theoretical findings, demonstrating the sharpness of the inequalities, the stability of the transform under weighted norms, and the practical manifestation of the logarithmic uncertainty principle.

In general, the results presented in this paper show that the Two-Sided Quaternion Linear Canonical Transform is not only of strong theoretical interest but also a viable and stable tool for practical applications involving multichannel and multidimensional data. Future research directions include the development of fast algorithms for the discrete QLCT, extensions to stochastic and noisy signal models, and applications to real-world problems such as color image processing, optical systems, and quaternion-based time–frequency analysis.

Acknowledgment

None.

Conflict of Interest

No Conflict of interest.

References

1. W Beckner (1995) Pitt's inequality and the uncertainty principle. Proc Amer Math Soc 123(6): 1897-1905.
2. D Yafaev (1999) Sharp constants in the Hardy-Rellich inequalities. J Funct Anal 168(1): 121-144.
3. S Eilertsen (2001) On weighted fractional integral inequalities. J Funct Anal 185(1): 342-366.
4. EM Stein, G Weiss (1971) Introduction to Fourier analysis on Euclidean spaces. PrincetonUniv Press, Princeton.
5. DV Gorbachev, VI Ivanov, S Yu (2016) Tikhonov, Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L2. J Approx Theor 202: 109-118.
6. DV Gorbachev, VI Ivanov, S. Yu (2016) Tikhonov, Pitt's inequalities and uncertainty principle for generalized Fourier transform. Int Math Res Not IMRN 23: 7179-7200.
7. L Colzani, A Crespi, L Travaglini, M Vignati (1975) Equi convergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in euclidean and noneuclidean spaces. Trans Amer Math Soc 338(1): 43-55.
8. L De Carli (2008) On the Lp-Lq norm of the Hankel transform and related operators. J Math Anal Appl 348(1): 366-382.
9. A Erdélyi, W Magnus, F Oberhettinger, FG Tricomi (1954) Tables of Integral Transforms. McGraw-Hill Book Company Vol 2.
10. M de Jeu (1993) The Dunkl transform. Invent Math 11: 147-162.
11. CF Dunkl, Y Xu (2001) Orthogonal polynomials of several variables. Cambridge Univ Press Cambridge.
12. S Ben Saïd, T Kobayashi, B Orsted (2012) Laguerre semigroup and Dunkl operators. Compos Math 148(4): 1265-1336.
13. S Li, M Fei (2023) Pitt's inequality and logarithmic uncertainty principle for the Clifford-Fourier transform. Adv Appl Clifford Algebras 33(1): 2.
14. H De Bie, Y Xu (2011) On the Clifford-Fourier transform. Int Math Res Not IMRN 22: 5123-5163.
15. M Brelot (1978) Equation de Weinstein et potentiels de Marcel Riesz. Semin Theor Potent Paris Lect Notes Math 681(3): 18-38.
16. IA Aliev, B Rubin (2003) Spherical harmonics associated to the Laplace-Bessel operator and generalized spherical convolutions. Anal Appl 1(1): 81–109.
17. IA Kipriyanov (1997) Singular elliptic boundary problems, Nauka. Moscow, Fizmatlit (in Russian).
18. A Weinstein (1953) Generalized axially symmetric potential theory. Bull Amer Math Soc 59: 20-38.
19. A Weinstein (1962) Singular partial differential equations and their applications. In: Fluid Dynamics and Applied Mathematics eds. (JB Diaz, SI Pai), Gordon and Breach, New York, pp. 229–249.
20. H Mejjaoli, M Salhi (2011) Uncertainty principles for the Weinstein transform, Czechoslovak Math J 4(61): 941-974.
21. Y Othmani and K. Trimèche, Real Paley-Wiener theorems associated with the Weinstein operator. Mediterr J Math 3(2006): 105–118.
22. NB Salem, AR Nasr (2015) Heisenberg-type inequalities for the Weinstein operator. Integral Transform Spec Funct 9(26): 700-718.
23. ZB Nahia, NB Salem (1996) Spherical harmonics and applications associated with the Weinstein operator. Potential Theory-ICPT 94: 233-241.
24. C Chettaoui, K Trimèche (2010) Bochner-Hecke theorems for the Weinstein transform and application. Fract Calc Appl Anal 13: 261-280.
25. HB Mohamed, B Ghribi (2013) Weinstein-Sobolev spaces of exponential type and applications. Acta Math Sinica Engl Ser 3(29):591-608.
26. A Saoudi, B Nefzi (2020) Boundedness and compactness of localization operators for Weinstein-Wigner transform. J Pseudo-Differ Oper Appl 11(2): 675-702.
27. S Omri (2011) Logarithmic uncertainty principle for the Hankel transform. Integral Transform Spec Funct 22(9): 655-670.