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Research Article

The Dynamic of Quantum Spin Interacting with Magnetic Field Coupled to a Heat Bath

Wael M Al Sawai*

Department of Mathematics, University of Texas Permian Basin, Odessa Texas, USA

***Corresponding author:** Wael M Al Sawai, Department of Mathematics, University of Texas Permian Basin, Odessa Texas 79762, USA.

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Abstract

The coherent quantum dynamics of a single bosonic spin variable, subject to a constraint derived from the quantum spherical model of a ferromagnet, and coupled to an external heat bath, are studied through the Lindblad equation for the reduced density matrix. We derive the governing equations of the system and solve them for several quantum observable. We analyze the interplay between quantum fluctuation and dissipation and the effect on the relaxation of the time-dependent magnetization.

Keywords: Quantum dynamics; Hyper-geometric functions; Bessel's functions; Lindblad equation; Phase transition; Mean-field theory

Introduction

We explore the critical dynamical properties of single bosonic spin variables, subject to constraints derived from the quantum spherical model of a ferromagnet and coupled to an external heat bath. Our work promotes an understanding of nonequilibrium physics in open many-body systems within finite range interactions.

The Model

We consider N continuous spins S_i where i = 1,2, …., N. In a 2D-simple cubic lattice of lattice constant 1, the spins interact by nearest neighbor "exchange forces," moreover we assume that. $S_i \in (-\infty, \infty)$. Note that if $S_i = \pm 1$ for all *i* then we have the Ising model. If we require their total length to be proportional to or equal to *N*, the number of lattice sites is [1]

$$\sum_{i=1}^{N} S_i^2 = N \qquad (1)$$

We call Eq(1) the spherical condition. From Eq(1) the spherical model must satisfy the relation

$$\frac{1}{N}\sum_{i=1}^{N} S_i^2 = \langle S_i^2 \rangle = 1 \qquad (2)$$

This describes an N-dimensional sphere with radius $N^{1/2}$, and the points in the sphere are given by Eq(1). In the thermodynamic limit, the microcanonical and grand canonical treatment of condition Eq(2) are equivalents [4]. We choose the grand canonical ensemble by introducing parameter μ, thus the Hamiltonian:

$$H' = -J\sum_{\langle i,j\rangle} S_i S_j + \mu \sum_i S_i^2 \qquad (3)$$

Dynamics introduced into Eq(3) by assigning inertia to the spins. We add kinetic energy term to the Hamiltonian Eq(3)

$$H = \frac{1}{2}\sum_i \dot{S}_i^2 - J\sum_{\langle i,j\rangle} S_i S_j + \mu \sum_i S_i^2 \qquad (4)$$

where I is the moment of inertia. Eq(25) represents a d-dimensional system of coupled linear oscillators. This standard classical mechanics problem that can be rewritten as

$$H = \frac{1}{2I} p^T p + \frac{1}{2} \sum_{\alpha} q^T f_{\alpha} q \qquad (5)$$

where $\alpha = x, y$ is the direction in which coupling is taken, and

$$q = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \end{pmatrix}, p = \begin{pmatrix} \dot{S}_1 \\ \dot{S}_2 \\ \vdots \end{pmatrix} \qquad (6)$$

f is the force matrix constant which is given by

$$f_{\alpha} = \begin{pmatrix} \mu & -J & 0 & 0 & \cdots & 0 \\ -J & \mu & -J & 0 & \cdots & 0 \\ 0 & -J & \mu & -J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & \cdots & & \vdots & -J & \mu \end{pmatrix} = \mu I - J \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & \cdots & & \vdots & 1 & 0 \end{pmatrix} = \mu I - JT \qquad (7)$$

where T is defined by the preceding formula and I is the identity matrix.

We transform the coordinates as follows:

$$Q = L^{-1} q, \quad q = LQ \qquad (8)$$

For the momenta we get

$$P = L^T p, \quad p = (L^T)^{-1} P \qquad (9)$$

We know that $q_k = L_{km}(Q_k)$. For coordinate Q_j the conjugate momentum is (we use the Einstein's summation convention and chain rule)

$$\begin{aligned} P_j &= \frac{1}{i} \frac{\partial}{\partial Q_j} = \frac{1}{j} \frac{\partial L_{km} Q_k}{\partial Q_j} \frac{\partial}{\partial q_k} \\ &= \frac{1}{i} L_{km} \delta_{jm} \frac{\partial}{\partial q_k} \\ &= \frac{1}{i} L_{kj} \frac{\partial}{\partial q_k} = L_{kj} p_k = (L^T)_{jk} p_k \end{aligned} \qquad (10)$$

Substitution of the new coordinates and momenta into H yields:

$$H = \frac{1}{2} P^T L^{-1} (L^{-1})^T P + \frac{1}{2} \sum_{\alpha} Q^T L^T f_{\alpha} L Q \qquad (11)$$

Let us choose L such that it is real and:

$$L^{-1} (L^{-1})^T = I \qquad (12)$$

Taking the inverse, it follows that:

$$L^T L = I \qquad (13)$$

And for non-singular L is also follows that:

$$L^T = L^{-1} \qquad (14)$$

which means that L is an orthogonal (and unitary, since it is real) matrix.

We may choose L such that :

$$\sum_{\alpha} L^T f_{\alpha} L = \wedge \qquad (15)$$

where $\wedge = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$ is a diagonal matrix. Multiplying by L gives

$$\sum_{\alpha} f_{\alpha} L = L \wedge \qquad (16)$$

For i[th] column of L, denoted by L_i, we have

$$\sum_{\alpha} f_{\alpha} L_i = L_i \lambda_i \qquad (17)$$

which shows that Li is an eigenvector of f with eigenvalue λ_i. Thus, to find L we have the eigenvectors and the eigenvalues of f. Substitution of these results into H yields:

$$H = \frac{1}{2I} P^T P + \frac{1}{2} Q^T \wedge Q = \sum_i \left(\frac{P_i^2}{2I} + \lambda_i^T Q_i^2 \right) = \sum_i \left(\frac{P_i^2}{2I} + \frac{1}{2} \omega_i^2 Q_i^2 \right) \qquad (18)$$

where In other words the transformation $q \to Q$, $p \to P$ gave us d uncoupled harmonic oscillators. The frequencies. The frequencies are where λ_i are the eigenvalues of the force matrix constant which is given by Eq(7). Clearly f and T have the same eigenvectors and their respective eigenvalues are related by

$$\lambda_i = \mu - J e_i \qquad (19)$$

Where e_i are the eigenvalues of matrix T and given by $e_k = 2\cos\left(\frac{k\pi}{n+1}\right)$, $1 \le k \le N$. Thus, the eigenvalues of $\wedge$

$$\lambda_k = \mu - J \sum_{\alpha} \cos\left(\frac{k_{\alpha}\pi}{n+1}\right) \qquad (20)$$

The Q_i are called normal coordinates :

$$Q = L^{-1} q = L T_q \qquad (21)$$

Looking at the Hamiltonian in Eq(18) we see the result of quantizing spin S_i

$$H = \sum_i \left(\frac{\hat{P}_i^2}{2I} + \frac{1}{2} \omega_i^2 \hat{Q}_i^2 \right) = \sum_i \left(n_i + \frac{1}{2} \right) \hbar \omega_i \qquad (22)$$

where n_i is the Boson occupation number. Therefore, by analogy we can write the Hamiltonian of a quantum spherical model [2], where the dynamical variables are the spherical spin-operators S_i

(at each site $i \in D$ where

$D \subset \Re^n$, with $i \in D$ sites).

$$H = \sum_i \left(\frac{g}{2} P_i^2 + \frac{\mu}{2} S_i^2 - \sum_{j=1}^{d} J S_i S_{i+e_j} \right) \qquad (23)$$

Where g is a coupling constant, μ is the Lagrange multiplier that can be found self-consistently from the constraint $< \sum_i S_i^2 = N >$, and e_j is the j[th] cartesian unit vector.

Therefore, the Hamiltonian of a single spherical quantum spin, in an external magnetic field M, reads [3]

$$H = \frac{g}{2} \hat{P}^2 + \frac{\mu}{2} \hat{Q}^2 - M \hat{Q} \qquad (24)$$

with canonical commutation relation $\left[\hat{Q}, \hat{P} \right] = i\hbar$. Where g is the quantum coupling of the system. Thus, we can write the operators $\hat{Q}$ and $\hat{P}$ in terms creation and annihilation operators as:

$$\hat{S} = \sqrt{\frac{\hbar g}{2\omega}} \left(a^\dagger + a \right), \quad \hat{P} = i \sqrt{\frac{\hbar \omega}{2g}} \left(a^\dagger - a \right) \qquad (25)$$

where $a^\dagger$, a are the create ion and annihilation operators, and $\omega = \omega(t) := \sqrt{\mu(t) g}$. Thus, the Hamiltonian Eq(3) can be rewritten as:

$$H = \hbar\omega(t) \left(a^\dagger a + \tfrac{1}{2} \right) - M \sqrt{\frac{\hbar g}{2\omega}} \left(a^\dagger + a \right) \qquad (26)$$

The constraint Eq(2) introduces a relationship between the time dependent frequency and the two-particle-expectation values:

$$< s^2 > = 1 = < \frac{\hbar g}{2\omega(t)} \left(a^\dagger + a \right)^2 >$$

$$= \frac{\hbar g}{2\omega(t)} (< a^\dagger a^\dagger > + < aa > + 2 < a^\dagger a > + 1) \qquad (27)$$

$$\omega(t) = \frac{\hbar g}{2} (< a^\dagger a^\dagger > + < aa > + 2 < a^\dagger a > + 1)$$

Eq(28) and Eq(26) defines the closed system. Our system of interest is a collection of bosonic spins interacting with a thermal bath. Lindblad's theory for open quantum systems is the mathematical framework to describe such a system. A coherent quantum (Density matrix) dynamics is formulated by Schrodinger picture

$$i\hbar \frac{\partial}{\partial t} \rho = \left[H, \rho \right] \qquad (28)$$

This is Liouville's equation that describes isolated systems. Coherent dynamics typically last only over short timescales. The coupling to the thermal bath creates dissipation in the quantum systems before the dynamics become dominated by coupling of the open system [3] to its environment, leading to decoherence. Liouville's equation cannot describe the combination of dissipative and coherent dynamics. To generalize the Liouville Eq(28) to the case of Markovian but unitary evolution, such that $\dot{\rho} = L[\rho]$. Here L generates a finite super operator in the same sense that a Hamiltonian H generates unitary time evolution, which will be called the Lindbladian [3,5]. The Lindblad master equation for an N-dimensional system's reduced density matrix ρ can

be written:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} \left[H, \rho \right] + \sum_{n,m=1}^{N^2-1} h_{n,m} \left(L_n \rho L_m^\dagger - \frac{1}{2} \left(\rho L_m^\dagger L_n + L_m^\dagger L_n \rho \right) \right) \qquad (29)$$

for quantum harmonic oscillator we have $L_1 = a$; $L_2 = a^\dagger$

$$h_{n,m} = \begin{cases} \frac{\gamma}{2} \left(n_\omega + 1 \right), n = m = 1 \\ \frac{\gamma}{2} n_\omega, n = m = 2 \\ 0, \text{ else} \end{cases}$$

Here n_ω is the mean number of excitations in the reservoir damping the oscillator and is the decay rate. Thus, we can write Eq(29) as:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \gamma (n_\omega + 1)(a\rho a^\dagger - \frac{1}{2}(a^\dagger a \rho + \rho a^\dagger a)) + \gamma n_\omega \left(a^\dagger \rho a - \frac{1}{2}(a a^\dagger \rho) + \rho a a^\dagger \right) \qquad (30)$$

where the bath, at certain temperature T, is defined by the Bose-Einstein statistics $n_\omega = \frac{1}{e^{\hbar\omega/T} - T}$ and γ is coupling constant. Eq(26), Eq(28) and Eq(30) define our model. It is a time-dependent, open quantum model system of a single. The system depends on temperature T, magnetic field M, and dissipation coupling γ.

Analysis of the Model

Our goal is to study their time evolution. Therefore, we make use of the Eq(28) and the commutation relation $\left[a_i, a_j^\dagger \right] = \delta_{ij}$ to deduce the closed set of equations of motion for the following averages:

$$\frac{d < aa >}{dt} = -\frac{i}{\hbar} < \left[H, aa \right] > \qquad (31)$$

$$\frac{d < aa >}{dt} = -2i\omega < aa > + i \sqrt{\frac{2g}{\hbar\omega}} M < a > \qquad (32)$$

We consider this equation as phenomenological ansatz; thus, by adding the dissipation term $-\gamma < aa >$ to the equation of motion, we get

$$\frac{d < aa >}{dt} = -2 \left(\frac{\gamma}{2} + i\omega \right) < aa > + i \sqrt{\frac{2g}{\hbar\omega}} M < a > \qquad (33)$$

Similarly, we get

$$\frac{d<a^\dagger a>}{dt} = -\gamma <a^\dagger a> + \gamma n_\omega + i\sqrt{\frac{g}{2\hbar\omega}} M(<a^\dagger> - <a>) \quad (34)$$

$$\frac{d<a>}{dt} = -\left(\frac{\gamma}{2} + i\omega\right) <a> + i\sqrt{\frac{g}{2\hbar\omega}} M \quad (35)$$

In the vanishing Field M = 0 and T = 0 the system described by Eq(33), Eq(35) and Eq(35) decoupled such that the single particle operators can be treated individually, then we can write:

$$\frac{d<a^\dagger a>}{dt} = -\gamma <a^\dagger a> \quad (36)$$

$$\frac{d<aa>}{dt} = -\left[\gamma + 2i\omega(t)\right] <aa> \quad (37)$$

The particle number expectation values decay exponentially

$$<a^\dagger a> = Ne^{-\gamma t}, \quad N \in \mathbb{R}_+ \quad (39)$$

Thus, by using this solution and the spherical constraint Eq(28) the expectation value of the pair annihilation operators obeys the equation

$$\frac{d<aa>}{dt} = -\left[\gamma + 2i\frac{\hbar g}{2}\left(<a^\dagger a^\dagger> + <aa> + 2<a^\dagger a + 1\right)\right] <aa> \quad (40)$$

$$\frac{d<aa>}{dt} = -\left[\gamma <aa> + i\hbar g\left(|<aa>|^2 + <aa>^2 + (2Ne^{-\gamma t} + 1 <aa>)\right)\right] \quad (41)$$

We can separate the amplitude and complex phase by using the substitution $<aa> = X(t)e^{i\psi}$. Thus, we can rewrite Eq(41) as

$$\frac{d<aa>}{dt} = \dot{X} e^{i\psi} + iXe^{i\psi}\dot{\psi}$$

$$= -\left[\gamma <aa> + i\hbar g\left(|<aa>|^2 + <aa>^2 + \left(2Ne^{-\gamma t} + 1\right) <aa>\right)\right]$$

$$= -\left[\gamma Xe^{i\psi} + i\hbar g\left(X^2 + X^2 e^{2i\psi} + 2Ne^{-\gamma t + i\psi} X + Xe^{i\psi}\right)\right] \quad (42)$$

$$\dot{X} + iX\dot{\psi} = -\left[\gamma X + i\hbar g\left(X^2 e^{-i\psi} + X^2 e^{i\psi} + 2Ne^{-\gamma t} X + X\right)\right] \quad (43)$$

Equating the real parts yields

$$\dot{X} = -\gamma X \Rightarrow X(t) = Ae^{-\gamma t} \quad (44)$$

where *A* is real constant. The imaginary parts yield

$$-\frac{\dot{\psi}}{\hbar g} = 2X\cos(\psi) + 2Ne^{-\gamma t} + 1 = 2Ae^{-\gamma t}\cos(\psi) + 2Ne^{-\gamma t} + 1 \quad (45)$$

let $y(t) = e^{i\psi}$ then $\cos(\psi) = \frac{y(t) + y(t)^{-1}}{2}$, substitute in Eq(22) we get

$$\frac{i}{\hbar g}\dot{y}(t) = y(t)e^{\gamma t} + 2Ny(t) + Ay(t)^2 + 1 \quad (46)$$

We will make a change of independent variable of the form (change the time scale according to) $\tau = e^{-\gamma t}$, $Y(\tau) = y(t)$, then

$$\frac{dY(\tau)}{d\tau} = \frac{dy(t)}{dt}\frac{dt}{d\tau} = \dot{y}(t)\frac{1}{\frac{d\tau}{dt}} = \frac{\dot{y}(t)}{-\gamma e^{-\gamma t}} \quad (47)$$

substitute in Eq(24) we get

$$\tau\dot{Y}(t) + (C_1 + C_2\tau)Y(\tau) + C_3\tau(Y^2 + 1) = 0 \quad (48)$$

where $C_1 = \frac{\hbar g}{i\gamma}$, $C_2 = \frac{2\hbar g N}{i\gamma}$ and $C_3 = \frac{\hbar g A}{i\gamma}$

A standard technique to solve this type of equations is by mapping it to a second-order linear differential equation, by changing variables $\lambda Y = \dot{u}/u$ and $\lambda\dot{Y} = \frac{u\ddot{u} - \dot{u}^2}{u^2}$ and choosing $\lambda = C_3$ one can obtain

$$\tau\ddot{u} + \left(C_2\tau + C_1\right)\dot{u} + C_3^2\tau u = 0 \quad (49)$$

We can simplify Eq(49) to a standard hyper geometric equation by using the substitution $u(\tau) = e^{-\kappa\tau}\omega(\tau)$, where κ is constant chosen to reduce to the differential equation to the desire form. We get

$$\tau\ddot{\omega} + \left(C_1 - (2\kappa - C_2)\tau\right)\dot{\omega} + \left(\tau\left(\kappa^2\kappa C_2 + C_3^2\right) - \kappa C_1\right)\omega = 0 \quad (50)$$

Taken $\kappa = \frac{C_2}{2} + \sqrt{C_2^2/4 - C_3}$. Eq(50) becomes

$$\tau\ddot{\omega} + \left(C_1 - \sqrt{C_2^2 - 4C_3^2\tau}\right)\dot{\omega} + \frac{C_1}{2}\left(C_2 + \sqrt{C_2^2 - 4C_3}\right)\dot{\omega} = 0 \quad (51)$$

This is a hypergeometric equation that has two distinguish cases [6,7],

$C_2/2$ = C3 then Eq(51) turn into

$$\tau\ddot{\omega} + C_1\dot{\omega} + \frac{C_1C_2}{2}\omega = 0 \quad (52)$$

We can rewrite Eq(52) as

$$w = -\frac{2}{C_1C_2}\left(\tau\frac{d}{d\tau} + C_1\right)\frac{dw}{d\tau} \quad (53)$$

let $z = \frac{C_1C_2}{2}\tau$, Eq(53) becomes

$$w = \left(z\frac{d}{dz} + C_1\right)\frac{dw}{dz} \quad (54)$$

Eq(54) is the confluent hypergeometric limit functions and the solution is given by,

$$w = {}_0F_1(;C_1;z)$$

When α is not positive integer the substitution

$$w = z^{1-\alpha} u$$

gives linearly independent solution

$$w = z^{1-C_1}\, {}_0F_1(;2-C_1;z)$$

therefore, the general solution is

$$w = K_1\, {}_0F_1(;C_1;z) + K_2 z^{1-\alpha}\, {}_0F_1(;2-C_1;z) \qquad (55)$$

The confluent hypergeometric limit functions are closely related to Bessel functions. The relationship is

$$J_\alpha(x) = \frac{\left(\frac{x}{2}\right)\alpha}{\Gamma(\alpha+1)}\, {}_0F_1\left(;\alpha+1;-\frac{x^2}{4}\right)$$

Thus

$${}_0F_1(;C_1;z) = J_{C_1-1}\left(\sqrt{-4z}\right)\Gamma(C_1)\left(\frac{2}{\sqrt{-4z}}\right)C_1 - 1 = J_{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)\Gamma(C_1)\left(\frac{C_1C_2}{2}\right)(1-C_1)/2_\tau(1-C_1)/2$$

$$z^{1-C_1}\, {}_0F_1(;2-C_1;z) = z^{1-C_1} J_{C_1-1}\left(\sqrt{-4z}\right)\Gamma(2-C_1)\left(\frac{1}{\sqrt{-z}}\right)1-C_1$$

$$= J_{C_1-1}\left(\sqrt{-4z}\right)\Gamma(2-C_1)(-z)^{(1-C_1)/2} = J_{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)\Gamma(2-C_1)\left(\frac{C_1C_2}{2}\right)(1-C_1)/2_\tau(1-C_1)/2$$

Γ (C_1) and Γ $(2-C_1)$ can be absorbed in the integration constants K_1 and K_2 respectively, and Eq(55) becomes therefore the general solution is

$$w = K_1 J_{C_1-1}\left(i\sqrt{2C_1C_2\tau}\right)\left(\frac{C_1C_2}{2}\right)(1-C_1)/2_\tau(1-C_1)/2 + K_2 J_{1-C_1}\left(i\sqrt{2C_1C_2\tau}\right)\left(\frac{C_1C_2}{2}\right)(1-C_1)/2_\tau(1-C_1)/2 \qquad (56)$$

We have used the substitution $u = e^{-\kappa\tau} w$

$$u = \left(e^{-\kappa\tau}\right)\left(2^{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)^{1-C_1}\right)\left(K_1 J_{C_1-1} + K_2 J_{1-C_1}\right) \qquad (57)$$

$$u = \left(e^{-\kappa\tau}\right)\left(2^{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)^{1-C_1}\right)K_1\left(J_{C_1-1}\left(\sqrt{2C_1C_2\tau}\right) + KJ_{1-C_1}\left(\sqrt{2C_1C_2\tau}\right)\right)$$

where $K = \frac{K_2}{K_1}$ and $y = \frac{\dot{u}}{C_3 u}$, Then we have

$$\frac{du}{d\tau} = e^{-\kappa\tau}\left(\frac{dw}{d\tau} - \kappa w\right)$$

thus

$$y = \frac{\dot{u}}{C_3 u} = \frac{\frac{dw}{d\tau} - \kappa w}{C_3 w} = \frac{1}{C_3 w}\frac{dw}{d\tau} - \frac{\kappa}{C_3}$$

To calculate $\ddot{\omega}$ we make use of the Bessel function recurrence relations

$$\frac{d\left(x^n J_n(x)\right)}{dx} = x^n J_{n-1}(x) \text{ and } \frac{d\left(x^{-n} J_n(x)\right)}{dx} = -x^{-n} J_{n+1}(x)$$

we can write,

$$\frac{dw}{d\tau} = \frac{dw}{d\left(\sqrt{\tau}\right)}\frac{d\left(\sqrt{\tau}\right)}{d\tau} = \sqrt{\frac{C_1C_2}{2\tau}}\frac{dw}{d\left(\sqrt{2C_1C_2\tau}\right)}$$

and

$$\frac{dw}{d\tau} = -2^{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)^{1-C_1} K_1\left(J_{C_1}\left(\sqrt{2C_1C_2\tau}\right) - KJ_{-C_1}\left(\sqrt{2C_1C_2\tau}\right)\right)$$

$$\frac{\dot{u}}{C_3 u} = \frac{J_{C_1}\left(\sqrt{2C_1C_2\tau}\right) - KJ_{-C_1}\left(\sqrt{2C_1C_2\tau}\right)}{C_3\left(J_{C_1-1}\left(\sqrt{2C_1C_2\tau}\right) + KJ_{1-C_1}\left(\sqrt{2C_1C_2\tau}\right)\right)} - \frac{\kappa}{C_3}$$

- In case 1 $C_3 = K = C_2/2$

$$y = \frac{\dot{u}}{C_3 u} = -\sqrt{\frac{C_1C_2}{\tau}}\frac{KJC_1\left(\sqrt{2C_1C_2\tau}\right) - J_{-C_1}\left(\sqrt{2C_1C_2\tau}\right)}{C_3\left(KJ_{1-C_1}\left(\sqrt{2C_1C_2\tau}\right)\right) + J_{C_1-1}\left(\sqrt{2C_1C_2\tau}\right)} - 1$$

since

$$\sqrt{2C_1C_2\tau} = 2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}$$

we get

$$y = \frac{\dot{u}}{C_3 u} = \frac{e^{\frac{\gamma t}{2}}}{\sqrt{A}}\frac{KJ_{\left(i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right) - J_{\left(-i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right)}{\left(KJ_{\left(1+i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right) + J_{\left(-i\frac{\hbar g}{\gamma}-1\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right)\right)} - 1$$

But $y = e^{i\psi}(t)$, $\cos\psi(\mathrm{t}) = \mathrm{Re(y)}$ then

$$\cos\psi(\mathrm{t}) = -\mathrm{Re}\left(1 + \frac{e^{\frac{\gamma t}{2}}}{\sqrt{A}}\frac{KJ_{\left(i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right) - J_{\left(-i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right)}{\left(KJ_{\left(1+i\frac{\hbar g}{\gamma}\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right) + J_{\left(-i\frac{\hbar g}{\gamma}-1\right)}\left(2i\frac{\hbar g}{\gamma}\sqrt{Ae^{-\gamma t}}\right)\right)}\right)$$

- Case 2 $C_2/2 \neq C_3$

$$\tau\ddot{w} + \left(C_1 - \sqrt{C_2^2 - 4C_3^2\tau}\right)\dot{w} + \frac{C_1}{2}\left(C_2 + \sqrt{C_2^2 - 4C_3^2}\right)w = 0$$

Let $T = \tau\sqrt{C_2^2 - 4C_3}$ which reduced Eq(51) the confluent hypergeometric equation

$$T\ddot{w} + (C - T)\dot{w} - \frac{C_1}{2}\left(1 + \frac{C_2}{\sqrt{C_2^2 - 4C_3^2}}\right)w = 0 \qquad (58)$$

The solution for this equation is given by Kummer's functions ${}_1F_1\left(\frac{C}{2}\left(1 + \frac{B}{\sqrt{B^2 - 4A^2}}\right), C, T\right)$ are called confluent hypergeometric functions of the first kind also known as

$M\left(\frac{C_1}{2}\left(1+\frac{C_2}{\sqrt{C_2^2-4C_3^2}}\right),C_1,T\right)$. Another standard solution for Eq(58) is *U(a,b,z)* therefor the general solution is

$$w=K_1U(\tau)+K_2M(\tau)$$

where $\left(-\frac{\hbar g}{2\gamma}\left(1+\frac{1}{\sqrt{A^2/C_2^2-1}}\right);-i\frac{\hbar g}{\gamma};2\frac{\hbar g}{\gamma}\sqrt{A^2-N^2}e^{-\gamma t}\right)=(a,b,T)$

We have

$$y=\frac{\dot{u}}{C_3u}=\frac{1}{C_3w}\frac{dw}{d\tau}-\frac{\kappa}{C_3}\text{ using }\frac{dw}{d\tau}=\frac{dw}{dT}\frac{dT}{d\tau}=\sqrt{C_2^2-4C_3^2}\frac{dw}{dT}$$

We get

$$y-\sqrt{\frac{C_2^2}{C_3^2}-4}\frac{1}{w}\frac{dw}{dT}-\frac{\kappa}{C_3}\text{ and }\frac{dw}{dT}=K_1\frac{dU(a,b,T)}{dT}+K_2\frac{dM(a,b,T)}{dT}$$

we make use of the recurrence relations

$$\frac{dM(a,b,T)}{dT}=\frac{a}{b}M(a+1,b+1,T),\quad\frac{dU(a,b,T)}{dT}=-aU(a+1,b+1,T)$$

We get

$$\frac{dw}{dT}=-aK_1U(a+1,b+1,T)+\frac{a}{b}K_2M(a+1,b+1,T)$$

$$\frac{dw}{dT}=-aK_1\left[-U(a+1,b+1,T)+\frac{K}{b}M(a+1,b+1,T)\right]$$

then,

$$y=\sqrt{\frac{C_2^2}{C_3^2}-4}\frac{\frac{K}{b}M(a+1,b+1,T)-U(a+1,b+1,T)}{\frac{K}{a}M(a,b,T)+U(a,b,T)}-\frac{\frac{C_2}{2}+\sqrt{\frac{C_2^2}{4}-C_3^2}}{C_3}$$

$$\cos\psi=\mathrm{Re}\left(\sqrt{\frac{C_2^2}{C_3^2}-4}\frac{\frac{K}{b}M(a+1,b+1,T)-U(a+1,b+1,T)}{\frac{K}{a}M(a,b,T)+U(a,b,T)}-\frac{\frac{C_2}{2}+\sqrt{\frac{C_2^2}{4}-C_3^2}}{C_3}\right)$$

From the analysis above, we see distinct phases of the system originates from the coupling, *g* and ω(t). The two regions of the disordered phase are further illustrated through the relaxation of the magnetization. Although the stationary magnetization always vanishes in the disordered phase, the approach to this stationary value depends on value of the quantum coupling *g*. The solutions show that the approach towards to stationary value is monotonous. Some magnetic oscillations are also seen for relaxations within the ordered phase. In the solution of Eq(37) which is $<\hat{a}\hat{a}>=X(t)e^{-\gamma t}$ and the time dependent frequency *w(t)* can be reconstructed from Eq(38). The magnetization $m(t)=<\hat{s}(t)>=\sqrt{\frac{\hbar g}{2\omega(t)}}\left(<\hat{a}^{\dagger}+\hat{a}>\right)$ followed from integrating Eq(35).

Conclusion

We investigated a simple bosonic spin model behavior in an external Magnetic field in a heat bath. We implemented the mean-field approximation of the dynamics of a quantum spherical ferromagnetic, It presents a quantum analog of well-established typical examples of fluctuation-induced order. The quantum coupling g plays for quantum phase transitions at T = 0 in *d* dimensions, a role analogous to the temperature T > 0 in classical phase transitions. These zero-temperature quantum phase transitions are stable under a small thermal perturbation.

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Conflict of Interest

No conflict of interest.

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