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An Understanding The Application Of Helly Theorem For A Topology Of R^2

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Abstract

In this article, the *topological Helly theorem* is proven by using *Molnar's theorem* and the main theorem from a *Helly-type theorem for simple polygons*. First, the description of relevant theorem used is introduced. Then, we describe similarities and differences of theorems and theorems are related to prove the goal that the intersection is a simple polygon. The below described is the topological Helly's theorem.

Introduction

Let $X_1,....,X_n \subset \mathbb{R}^2$ to be simple polygons in the plane. If all double and triple intersections of X_i are also simple polygons and non empty, then the intersection of all X_i is also a simple polygon [3] (Figure 1).

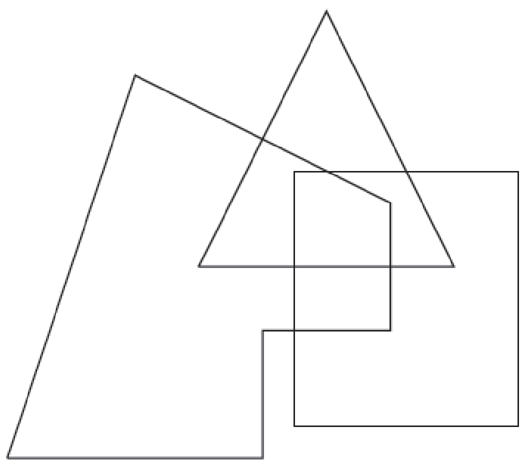

Figure 1: An illustration of intersecting simple polygons.

Definition

A *simple polygon* is defined to be connected without holes, simply connected finite union of convex polygons.

A Description of Theorem

The definition and description are based on "*A Helly-type Theorem for simple polygons*" and "*A Helly-type theorem for Intersections of compact connected sets in the plane*" by Marilyn Breen [1,2].

Definition

A planar set S is simply connected if for every simple closed curve λ in *S* and every point q in the bounded region determined by the λ, *q* belongs to *S* (Figure 2).

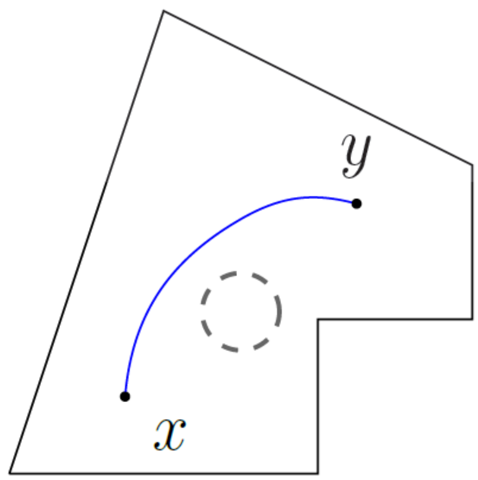

Figure 2: An illustration of a simple polygon and path connected.

While the convex set allows only straight lines, the path connected in the simple polygon can be any curves for two distinct points. As simple polygons are not necessarily convex sets, the classic Helly theorem is not applicable. However, we proceed to use compact sets and theorems to show for the intersection of simple polygons.

Theorem: (Molnar's theorem) If $C = \{C_1 ..., C_n\}$ is a family of *simply connected compact sets* in the plane such that every two (not necessarily distinct) sets of C have a *nonempty intersection*, then $\bigcap_{i=1}^{n} C_i \neq \varnothing$ and *simply connected*.

Theorem: (98') ⇔ (the main theorem)

- Every two (not necessarily distinct) sets of *C* have a connected union and every three (not necessarily distinct) sets of *C* have a *simply connected union*.
- *C* is a family of *simply connected sets* such that every two (not necessarily distinct) sets of *C* have a *connected intersection* and every three (not necessarily distinct) sets of *C* have a *nonempty intersection*.

Part (1) implies (2) using *Lemma 1* and *Lemma 2*. Part (2) implies (1) using Proposition 1 applied by the *Lemma 3*.

a. **Lemma 1:** Let C_1 and C_2 to be compact connected sets in the plane. If $C_1 \cup C_2$ is simply connected.

b. **Lemma 2:** For C_1, C_2, C_3 nonempty compact connected sets in the plane, if every two of these sets have a connected union and all three sets have a simply connected union, then $C_1 \cap C_2 \cap C_3 \neq \varnothing$.

c. **Lemma 3:** Let λ be a simple closed curve in the plane, with point q in the bounded region determined by λ and point r in the unbounded region determined by λ.

Let μ_1, μ_2, μ_3 be simple closed curves from q to r, with $\lambda \cap \mu_i$ finite for $1 \le i \le 3$ and $\lambda \cap \mu_i \cap \mu_j \neq \varnothing$, for $1 \le i \le j \le 3$. Then, for some point $b_1 \in \mu_1 \cap \lambda$ is in a bounded region determined by $\mu_2 \cup \mu_3$.

In summary, for three different closed curve connecting points $q \in \lambda$ to r we are able to find a point b_1 that is the intersection of μ_1 and λ bounded inside the connected region of $\lambda_i \cup \lambda_j$. Note that the point b_1 from $\lambda \cap \mu_i$ does not belong to $\lambda \cap \mu_i \cap \mu_j \neq \varnothing$. This means there are three cases to find the point b_1 from $\mu_1 \cap \lambda$ defined in $\mu_2 \cup \mu_3$ or $\mu_2 \cap \lambda$ defined in $\mu_1 \cup \mu_3$ or $\mu_3 \cap \lambda$ defined in $\mu_1 \cup \mu_2$.

d. Proposition 1: For family of set C of $n \geq 2$ in the plane, if every two sets of *C* have a connected, simply connected union and $\bigcap_{i=1}^{n} C_i \neq \emptyset$, then $\bigcup_{i=1}^{n} C_i$ is simply connected.

The goal of *Proposition 1* with *Lemma 3* is to show for three sets with Part (2) to be simply connected union of Part (1).

Theorem: (An intersection theorem)

Let *P* be the family of simply polygons $P_1, \ldots, P_n$ in the plane. If every three (not necessarily distinct) polygons of P have a *simply connected union* and every two polygons of *P* have a *nonempty intersection*, the $\bigcap_{i=1}^{n} P_i \neq \emptyset$.

The intersection theorem could not directly apply to simple polygons to derive a common intersection as every three polygons are simply connected union for a given condition.

A Relationship of Theorem

Difference

While the Molnar's theorem applicable for compact connected sets and an intersection theorem is applicable for simple polygons. Generally, simple polygons in the plane are identified to be compact connected sets.

While the Molnar's theorem is conditioned for *triple intersection of sets to be nonempty*, the intersection theorem is conditioned for *triple sets to be simply connected union*.

In the Molnar's theorem, $\bigcap_{i=1}^{n} C_i$ is *simply connected*, the intersection theorem does not support $\bigcap_{i=1}^{n} P_i$ to be simply connected.

Similarity

Using the induction, both theorems resulted in the common intersection of finitely many sets to be non-empty. Note that *the intersection theorem* is a strong version of *theorem 1* with the same conclusion for *Molnar's theorem* of common intersection for finitely many sets.

With loss of generality, the intersection theorem where all triples polygons have simply connected union is stronger than Part (2) with a condition that *all triple sets of intersection is nonempty*.

Corollary

Based on the Molnar's theorem and theorem 1 of part (1), $\bigcap_{i=1}^{n} C_i$ is nonempty, connected and simply connected and $\bigcup_{i=1}^{n} C_i$ is simply connected.

Basis

The conclusion that $\bigcap_{i=1}^{n} C_i \neq \emptyset$, is supported by the *Molnar's theorem* as $\bigcap_{i=1}^{n} C_i$ is nonempty, simply connected, which is based on the induction. However, $\bigcup_{i=1}^{n} C_i$ is connected, simply connected is supported by the *theorem 1* of part (1) for induction with *proposition 1* to be applied with.

Consequence

This *corollary proves* that $\bigcap_{i=1}^{n} X_i \neq \emptyset$, is a finite union of convex sets for simple polygons of $X_1, \ldots, X_n$. This fact that $\bigcap_{i=1}^{n} X_i$ is a simple polygon is supported by the *corollary* only for the basis of *main theorem* and *Molnar's theorem* where a simple polygon consists of finite union of convex polygons.

Acknowledgement

None.

Conflict of Interest

No conflict of interest.

References

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