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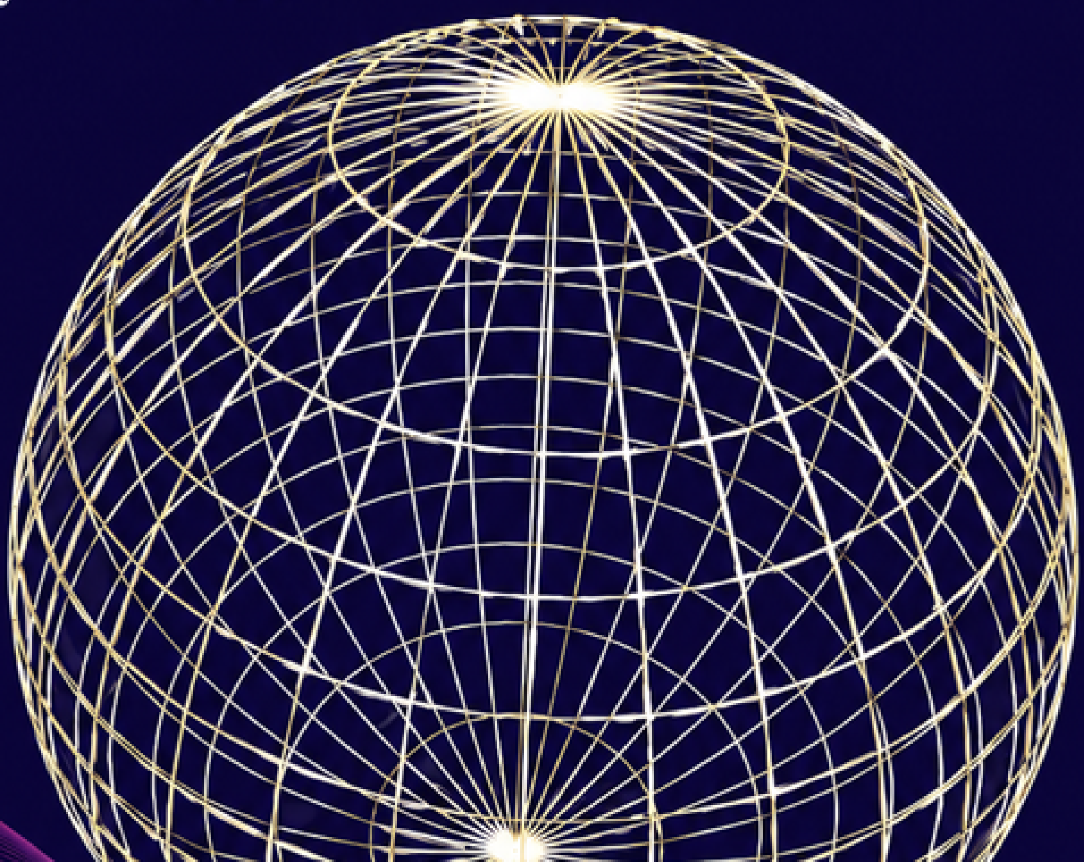
Some Boundary Value Problems for the Polyharmonic Equation in a Ball

Valeriy Karachik



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Abstruct

This monograph is devoted to the study of some boundary value problems for the polyharmonic equation in the unit ball $S \subset \mathbb{R}^n$, having high-order normal derivatives or Laplacians in the boundary conditions. First, the formulation of a general boundary value problem for the polyharmonic equation in S with high-order normal derivatives on the boundary is presented and conditions for its solvability are obtained. For further research, based on the method of normalized systems of functions, a complete system of orthogonal harmonic polynomials $\{H_k^{(i)}(x) : i = 1, \ldots, h_k\}$ is constructed and its properties are investigated.

Further in the book, specific problems of Dirichlet, Neumann (the formulation of the problem is proposed by A.V. Bitsadze), problems of the Neumann type, the Navier problem and the Riquier-Neumann problem are considered. In addition, in the last chapter boundary value problems for a nonlocal biharmonic equation with involution are studied and eigenfunctions of the nonlocal Laplace operator with double involution are also investigated.

To find the solvability conditions for the homogeneous boundary value problems of Neumann type, various number triangles are introduced and studied. For example, the Neumann triangle $\mathbb{P}$, triangles B and H are presented as well as related integral identities on the unit sphere for normal derivatives of polyharmonic functions are obtained. These identities are important for the solvability of Neumann-type problems. Here, using rows of matrices of a special form $\mathbb{H}_m$, the values at the origin of the Laplacians of m-harmonic in S functions are found.

To find solutions to the inhomogeneous polyharmonic equation, the concept of an elementary solution to the polyharmonic equation is introduced, as well as a special form of the fundamental solution $\mathcal{E}_{2m}(x, \xi)$, with the help of which an explicit representation of the Green's function of the Dirichlet problem $G_{2m}(x, \xi)$ for the polyharmonic equation in the unit ball is obtained. In addition, an explicit representation of the solution to the Dirichlet problem for the homogeneous polyharmonic equation in the unit ball is found, without using the Green's function of the problem. This solution to the Dirichlet problem is represented as an operator polynomial acting on the solutions of various Dirichlet problems for the Laplace equation related to the boundary functions of the original Dirichlet problem. Solutions to a class of Neumann type problems $\mathcal{N}_k$ are presented through the solution of the auxiliary Dirichlet problem.

In addition, the solvability of two types of problems for the biharmonic equation are studied: the generalized Dirichlet-Riquier problem and one Neumann type problem not included in $\mathcal{N}_k$.

For the Navier and Riquier-Neumann boundary value problems for the m-harmonic equation in a ball, an integral representation of functions from $C^{2m}(D) \cap C^{2m-1}(\bar{D})$ is found and Green's functions are constructed. Using Green's functions, an explicit form of solution to the problems under consideration is obtained. In a particular case, the Green's function of the Robin problem for the Poisson equation in the unit ball S is presented.

Much attention in the work is paid to the construction of polynomial solutions of the problems under study with polynomial boundary data and the right-hand side of the equation. This includes the construction of polynomial solutions to Dirichlet, Neumann, Navier, Riquier-Neumann and Neumann type problems.

MSC: 31A30, 31B25, 31B30, 31B35, 35J05, 35J08, 35J40, 35J60, 35P10, 35R10

Contents

Preface

This monograph contains a presentation of some author's results obtained in the field of boundary value problems for polyharmonic equation in the unit ball, having high-order normal derivatives or Laplacians in the boundary conditions. The results obtained over the past 8 years are mainly presented. The Dirichlet and Neumann problems, Neumann type problems, the Navier problem and the Riquier-Neumann problem are considered. Both conditions for the solvability of boundary value problems and methods for constructing solutions to these problems in explicit form are presented. For this purpose, the method of finding the Green's functions of the problem is used, as well as another method based on the operator representation of solutions. For boundary value problems with polynomial boundary data and the right side of the equation, formulas are given for finding polynomial solutions.

In addition, the last chapter of the monograph presents two results on the study of nonlocal boundary value problems. Four problems for a nonlocal biharmonic equation with involution and the problem of finding and studying the eigenfunctions of the nonlocal Laplace operator with double involution are considered.

In order not to refer the reader to the author's earlier results used in the monograph, these results are also presented in the text with proof.

Statements of specific problems, the state of research and literature on the problems considered, as well as an overview of the results obtained in the monograph are described at the beginning of the corresponding chapters and sections.

Chapter 1

Boundary value problems with high order normal derivatives in boundary conditions

The first chapter is in the nature of an introduction to the results obtained by the author on the study of a class of boundary value problems for the polyharmonic equation, mainly in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$. First, in Section 1.1 a class of boundary value problems for the polyharmonic equation in the unit ball, which can contain in the boundary conditions normal derivatives of a higher order than the order of the equation is defined. We call such problems as problems of type $\mathcal{H}$. In Section 1.1.1 the necessary conditions for the solvability of the problem $\mathcal{H}_1$ for the harmonic equation are obtained. Example 1.1 is given. Then, in Section 1.1.2, basing on Lemmas 1.1-1.5, in Theorem 1.2 the necessary and sufficient conditions for the existence and uniqueness of a solution to the problem $\mathcal{H}_m$ for homogeneous m-harmonic equation are presented. The class of considered problems is quite wide, since the well-known Dirichlet and Neumann problems are included in it. The first results known to the author on such problems are the papers [19, 59, 163]. As the most general results on the study of the index of a boundary value problem for the polyharmonic equation on a plane containing normal derivatives and Laplacians in the boundary conditions, we note the papers [132, 164].

Section 1.2 gives a brief description of the method of normalized systems of functions [89] and presents the properties of such systems. Theorem 1.3 demonstrates a normalized system of functions with respect to the Laplace operator Δ. Example 1.2 is given. Based on the obtained system, Theorem 1.4 investigates the solvability of the Navier problem for the polyharmonic equation. It is worth mentioning the paper [189], where solutions to the super-Poisson and super-polyharmonic equations are obtained using normalized systems of functions.

In Section 1.3, based on the method of normalized systems of functions, some special polynomials and functions are constructed that are used in the study of boundary value problems for the polyharmonic equation. Here, a complete system of homogeneous harmonic polynomials in n variables $\{G_{(\nu)}(x)\}$ is constructed (Theorem 1.5), which is orthogonal in $L_2(\partial S)$ (Theorem 1.6) and in $\mathcal{P}$ (Theorem 1.7). In Section 1.3.2 the concept of a G-function is introduced. Connections between G-functions and the Legendre polynomials (Theorem 1.11), and G-functions and the Chebyshev polynomials (Theorem 1.12) are established. Rodrigue's formula is also given (Theorem 1.13). In Section 1.3.3 the orthogonality of G-functions $G_k^{s,n}(t)$ on the interval $[-1, 1]$ with weight $\rho(t) = (1 - t^2)^{(n-3)/2}$ is proved (Theorem 1.14). Illustrative examples 1.3-1.5 are given. The results of Chapter 1 are given in more detail in papers [60–62, 64–69, 72].

1.1 Problem $\mathcal{H}_m$ for the m-harmonic equation in the unit ball

In order to clarify the solvability conditions for the problem $\mathcal{H}_m$ for the m-harmonic equation, let us first consider a simpler problem for the Laplace equation containing high-order normal derivatives in boundary conditions. It is the problem $\mathcal{H}_1$.

1.1.1 Problem $\mathcal{H}_1$ for the Laplace equation

Problem $\mathcal{H}_1$. *Find a function $u \in C^2(S)$ such that $|x|^k \partial^k u/\partial \nu^k \in C(\bar{S})$ for $0 \le k \le m$, satisfying in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$ the conditions*

$$\Delta u(x) = 0,\ x \in S; \quad P_m\left(\frac{\partial}{\partial \nu}\right) u_{|\partial S} = f(s),\ s \in \partial S, \tag{1.1.1}$$

where $f \in C(\partial S)$, $\partial u/\partial \nu$ is the derivative with respect to the direction of the outer normal to the sphere of radius $|x|$, and $P_m(t)$ is an arbitrary polynomial of degree m over $\mathbb{R}$.

To study this problem, we introduce the following differential operator

$$\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

which will also be used extensively in all the chapters below. It has the property: if $u(x)$ is a harmonic function, then $\Lambda u(x)$ is also a harmonic function. This immediately follows from the equality

$$\Delta \Lambda u(x) = \sum_{i=1}^{n} \Delta(x_i u_{x_i}) = \sum_{i=1}^{n} x_i \Delta u_{x_i} + 2 \sum_{i=1}^{n} u_{x_i x_i} = (\Lambda + 2)\Delta u(x).$$

Definition 1.1. Let the polynomial $P_m(t) = \sum_{k=0}^{m} p_k t^k$ be given, then the polynomial

$$P_{[m]}(t) = \sum_{k=0}^{m} p_k t^{[k]},$$

which is obtained from the polynomial $P_m(t)$ by replacing the monomials of the kth order – t^k with the factorial powers of these monomials – $t^{[k]} \equiv t(t-1)\cdots(t-k+1)$, we call the factorial polynomial corresponding to $P_m(t)$.

For brevity, we also sometimes use the notation $t^{k,!} = t^k/k!$.

Stirling numbers of the first kind $s(n,k)$ are associated with the monomials $t^{[n]}$. They are found from equality

$$t^{[n]} = \sum_{k=0}^{n} s(n,k)t^k.$$

For Stirling numbers of the first kind $s(n,k)$ the recurrence relation holds

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad 0 < k < n,$$

provided $s(0,0) = 1$, $s(n,0) = 0$. Stirling numbers of the first kind $s(n,k)$ can be arranged in the form of a number triangle

$$\begin{array}{c}
1 \\
0 \quad 1 \\
0 \quad -1 \quad 1 \\
0 \quad 2 \quad -3 \quad 1 \\
0 \quad -6 \quad 11 \quad -6 \quad 1 \\
0 \quad 24 \quad -50 \quad 35 \quad -10 \quad 1 \\
s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)
\end{array} \tag{1.1.2}$$

Along with Stirling numbers of the first kind, there are also Stirling numbers of the second kind $S(n,k)$, which are found from the equality

$$t^n = \sum_{k=0}^{n} S(n,k)t^{[k]}.$$

For Stirling numbers of the second kind $S(n,k)$ the recurrence relation holds

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad 0 < k \le n$$

provided $S(0,0) = 1$, $S(n,0) = 0$. Stirling numbers of the second kind $S(n,k)$ can be arranged in the form of a number triangle

$$\begin{array}{c} 1 \\ 0 \quad 1 \\ 0 \quad 1 \quad 1 \\ 0 \quad 1 \quad 3 \quad 1 \\ 0 \quad 1 \quad 7 \quad 6 \quad 1 \\ 0 \quad 1 \quad 15 \quad 25 \quad 10 \quad 1 \\ S(n,k) = S(n-1,k-1) + kS(n-1,k) \end{array} \tag{1.1.3}$$

We denote the Poisson kernel of the Dirichlet problem in S by $D(x,\xi)$, and its derivatives with respect to variable x through $D_x^{(\alpha)}(x,\xi)$.

Let $\lambda_1,\dots,\lambda_l$ be the roots of the polynomial $P_{[m]}(t)$, and $k_1,\dots,k_l$ be their multiplicities and $\lambda = \max_i\{[\operatorname{Re}\lambda_i]\}$, where $[.]$ is an integer part of a number. Let $H_k(x)$ denote homogeneous harmonic polynomials of kth degree. Let us introduce the function

$$R_\lambda(x,\xi) = \int_0^1 \sum_{i=1}^{l}\sum_{j=1}^{k_i}(-1)^{j-1}C_{ij}\ln^{j-1,!}\tau \\ \times\left(D(\tau x,\xi) - \sum_{|\alpha|=0}^{\lambda}(\tau x)^{\alpha,!}D_x^{(\alpha)}(0,\xi)\right)\frac{d\tau}{\tau^{\lambda_i+1}},$$

in which the constants C_{ij} satisfy the equality

$$\frac{1}{P_{[m]}(t)} = \sum_{i=1}^{l}\sum_{j=1}^{k_i}C_{ij}(t-\lambda_i)^{-j}$$

and a polynomial

$$u_\lambda(x) = \sum_{|\alpha|=0}^{\lambda}\frac{a_\alpha}{P_{[m]}(|\alpha|)}x^{\alpha,!},$$

such that

$$a_\alpha = \frac{1}{\omega_n}\int_{\partial S}D_x^{(\alpha)}(0,\xi)f(\xi)\,d\xi$$

for $|\alpha| \notin \{\lambda_1,\dots,\lambda_l\}$ and $a_\alpha = 0$ otherwise. Here ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Let us introduce the set $I = \{\lambda_1, \dots, \lambda_l\} \cap \mathbb{N}_0$ consisting of non-negative integer zeros of the polynomial $P_{[m]}(t)$.

Theorem 1.1. *A solution to problem* (1.1.1) *exists if and only if the following orthogonality condition is satisfied*

$$\lambda \in I \Rightarrow \forall H_\lambda(x), \int_{\partial S} H_\lambda(\xi) f(\xi)\, d\xi = 0.$$

If a solution to problem $\mathcal{H}_1$ exists, then it can be represented in the form

$$u(x) = u_\lambda(x) + \frac{1}{\omega_n} \int_{\partial S} R_\lambda(x, \xi) f(\xi)\, d\xi$$

and this solution is unique up to homogeneous harmonic polynomials, the degrees of which belong to the set I.

Proof. The proof of the theorem repeats the proof of the more general Theorem 1.2 for a more general problem and therefore we do not present it completely. To clarify the essence of the conditions for the existence of a solution to the problem $\mathcal{H}_1$, we present only a proof of the necessity of these conditions.

So, let a solution to problem $\mathcal{H}_1$ exist. Since the conditions $|x|^k \partial^k u/\partial \nu^k \in C(\bar{S})$, $0 \le k \le m$ are satisfied, then using the following equality for the normal ν to a sphere of radius $|x|$ (see equality (2.1.10) from Lemma 2.1 on page 72)

$$\frac{\partial^k}{\partial \nu^k} = |x|^{-k} \Lambda^{[k]},$$

from which it immediately follows that (see Corollary 2.1 on page 73)

$$\left(P_m \left(\frac{\partial}{\partial \nu} \right) u(x) - P_{[m]}(\Lambda) u(x) \right) \Big|_{\partial S} = 0,$$

we can easily find that the function $v(x) = P_{[m]}(\Lambda) u(x)$ is harmonic in S, continuous in $\bar{S}$ and $v|_{\partial S} = f$. Let us use the system of harmonic polynomials $\{H_k^{(i)}(x) : i = 1, \dots, h_k\}$ from Definition 1.5. These polynomials are normalized so that $\int_{\partial S} (H_k^{(i)}(x))^2\, ds_x = \omega_n$.

The function $v(x)$ in some neighborhood of zero can be represented as a series

$$v(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{h_k} h_k^{(i)} H_k^{(i)}(x), \quad h_k^{(i)} = \frac{1}{\omega_n} \int_{\partial S} H_k^{(i)}(\xi) f(\xi)\, ds_\xi,\ |x| \le a < 1$$

(see Theorem 1.8 on page 50). Substituting into the equality $P_{[m]}(\Lambda)u(x) = v(x)$ the function $u(x)$, also taken as a series

$$u(x) = \frac{1}{\omega_n}\sum_{k=0}^{\infty}\sum_{i=1}^{h_k} U_k^{(i)} H_k^{(i)}(x), \quad |x| \le a < 1,$$

and then multiplying the resulting equality by $H_l^{(j)}(x)$ and integrating over a sphere of radius $a < 1$, taking advantage of the uniform convergence of the series and the orthogonality of the polynomials $H_k^{(i)}(x)$ on the sphere, we get

$$U_l^{(j)} P_{[m]}(l) = h_k^{(j)} = \int_{\partial S} H_l^{(j)}(\xi) f(\xi)\, d\xi.$$

In fact, the polynomials $H_k^{(i)}(x)$ are differently designated harmonic polynomials $G_{(\nu)}(x)$ from Theorem 1.5 on page 44. An equation is obtained for unknown coefficients $U_l^{(j)}$. It is solvable only if the condition is met

$$P_{[m]}(l) = 0 \Rightarrow \forall j = \overline{1, h_l}, \int_{\partial S} H_l^{(j)}(\xi) f(\xi)\, d\xi = 0,$$

which coincides with the conditions of the theorem. Therefore, the solution $u(x)$ can be written in the form

$$u(x) = \frac{1}{\omega_n}\sum_{k=0}^{\infty}\sum_{i=1}^{h_k} \frac{H_k^{(i)}(x)}{P_{[m]}(k)} \int_{\partial S} H_k^{(i)}(\xi) f(\xi)\, d\xi, \quad |x| \le a < 1$$

and transformed to the form given in the theorem (see Theorem 1.2 below). The necessity of the conditions of the theorem is proved. □

Remark 1.1. The problem $\mathcal{H}_1$ for the Poisson equation in a ball is considered in [62], in the half-space is given in [66], and for the Helmholtz equation contains in [61].

Example 1.1. Let us give an example of using Theorem 1.1 for a problem of the form

$$\Delta u(x) = 0,\ x \in S; \quad \left.\frac{\partial^m u}{\partial \nu^m}\right|_{\partial S} = f(s),\ s \in \partial S,$$

which can be called the generalized Neumann problem [30, 66, 163].

In this case $P_m(t) = t^m$, $m \in \mathbb{N}_0$. Since $I = \{0, 1, \dots, m-1\}$, from Theorem 1.1 it follows that a solution to such a problem $\mathcal{H}_1$ exists if and only if, for any polynomial $H_k(x)$ of degree $k < m$ the equality holds

$$\int_{\partial S} H_k(\xi) f(\xi)\, d\xi = 0.$$

The solution to the problem is unique up to polynomials of $(m-1)$th degree and presented in the form

$$u(x) = u_{m-1}(x) + \frac{1}{\omega_n} \int_{\partial S} R_{m-1}(x, \xi) f(\xi)\, d\xi.$$

The unknowns C_k included in the function $R_{m-1}(x, \xi)$ are determined from the condition

$$\lambda^{[m]} \sum_{k=1}^{m} \frac{C_k}{\lambda - k + 1} = 1$$

and look like

$$C_k = \frac{(-1)^{m-k}}{(k-1)!(m-k)!}.$$

Therefore we have

$$R_{m-1}(x, \xi) = \int_0^1 (1-\tau)^{m-1,!} \left(D(\tau x, \xi) - \sum_{|\alpha|=0}^{m-1} (\tau x)^{\alpha,!} D_x^{(\alpha)}(0, \xi) \right) \frac{d\tau}{\tau^m}.$$

In addition, according to the definition of the polynomial $u_\lambda(x)$, the equality $u_{m-1}(x) = 0$ is true. For $m = 1$ we come to the result obtained by A.V. Bitsadze in [30].

1.1.2 Problem $\mathcal{H}_m$ for polyharmonic equation

Consider the following non-classical problem for the polyharmonic equation, which naturally generalizes the $\mathcal{H}_1$ problem from the previous section and Sections 2.1, 3.3 and 3.4. Let now m denote the order of polyharmonicity of the equation.

Problem $\mathcal{H}_m$. Find a function $u \in C^{2m}(S)$, satisfying in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$ the conditions

$$\begin{gathered} \Delta^m u(x) = 0,\ x \in S; \\ P_{m_i}\left(\frac{\partial}{\partial \nu}\right) u|_{\partial S} = f_i(s),\ s \in \partial S,\ i = \overline{1, m}, \end{gathered} \tag{1.1.4}$$

where $f_i \in C^{s_i}(\partial S)$, $s_i \in \mathbb{N}_0$, $\partial u / \partial \nu$ is the derivative in the direction of the outer normal to the sphere of radius $|x|$, $P_{m_i}(t)$ are arbitrary polynomials of degrees m_i over $\mathbb{R}$, and such that $|x|^j \partial^j u / \partial \nu^j \in C(\bar{S})$ for $j \le \max_i\{m_i\}$. The numbers s_i will be specified below.

Let us investigate the existence and uniqueness of the posed problem $\mathcal{H}_m$, and also try to construct its solution. In the simplest case, when $P_{m_i}(t) = t^{m_i}$, $s_i = m - i$ and for $m_i = i - 1$ the problem $\mathcal{H}_m$ turns into the Dirichlet problem (see Chapter 2), when $m_i = i$ becomes a Neumann problem, and when $m_i = k + i$ ($k \in \mathbb{N}$) turns into a Neumann type problem (see Chapter 3).

Suppose that the system of polynomials $\{P_{m_i}(t) : i = \overline{1, m}\}$ is linearly independent. As for the Laplace equation, let us denote by $P_{[k]}(t)$ the polynomial, obtained from the polynomial $P_k(t)$ by replacing the monomials t^i at $i \in \mathbb{N}_0$ with $t^{[i]} \equiv t(t - 1) \cdots (t - i + 1)$.

Lemma 1.1. *Let $u(x)$ be a solution to problem $\mathcal{H}_m$, then*

$$u(x) = \sum_{i=1}^{m} \left(1 - |x|^2\right)^{i-1} u_i(x), \tag{1.1.5}$$

where the harmonic functions $u_i(x)$ in S are such that the function $U(x) = (u_1(x), \ldots, u_m(x))$ satisfies the matrix equation

$$P(tD_t)U(tx) = V(tx), \quad |t| \leq 1, \tag{1.1.6}$$

in which $x \in \bar{S}$, the coordinates of the vector $V(x)$ are solutions to the Dirichlet boundary value problems

$$\Delta v_i(x) = 0,\ x \in S; \quad v_i|_{\partial S} = f_i(s),\ s \in \partial S,$$

$i = 1, \ldots, m$, and the matrix

$$P(\lambda) = Q(\lambda) \times C \tag{1.1.7}$$

has coefficients

$$Q(\lambda) = (P_{[m_i]}(\lambda + 2j - 2))_{i,j=\overline{1,m}}, \quad C = \left((-1)^i \binom{j}{i}\right)_{i,j=\overline{0,m-1}},$$

where $\binom{j}{i} = 0$ when $j < i$.

Proof. In accordance with the Almansi formula [5], every k-harmonic function $u(x)$ in S can be written in the form (1.1.5) whence

$$u(x) = \sum_{j=1}^{m} (-1)^{j-1} \sum_{i=1}^{m} \binom{i-1}{j-1} |x|^{2j-2} u_i(x).$$

Let $v(x)$ be some function harmonic in S. Using the equality (see Lemma 2.1 on page 72 and proof of Theorem 3.1 on page 156)

$$\frac{\partial^k}{\partial\nu^k}|x|^l v(x) = |x|^{l-k}(\Lambda + l)^{[k]} v(x),$$

where, as before, $\Lambda \equiv x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$, ν is the outer normal to the sphere of radius $|x|$ and bearing in mind that the function $q_k(\Lambda + c)v(x)$ is harmonic in S for any polynomial $q_k(x)$ and $c \in \mathbb{C}$, we can verify that the function

$$\begin{aligned} V(x) &= P(\Lambda)U(x) \\ &= \Big(\sum_{k=1}^{m}(-1)^{k-1}\binom{j-1}{k-1}P_{[m_i]}(\Lambda + 2k - 2)\Big)_{i,j=\overline{1,m}} U(x), \end{aligned} \tag{1.1.8}$$

$x \in \bar{S}$ due to the conditions (1.1.4) must be a solution to the Dirichlet problem

$$\Delta V(x) = 0,\ x \in S; \quad V|_{\partial S} = f(s),\ s \in \partial S,$$

where $f = (f_1, \ldots, f_m)$. So, the function $U(x)$ is a solution to the equation (1.1.8). Further, it is easy to verify that if $U(x)$ is a solution to the equation (1.1.8), then $U(tx)$ satisfies the equation (1.1.6) for $|t| \le 1$. The lemma is proved. □

Remark 1.2. If the function $U(tx)$ satisfies the equation (1.1.6) with $|t| \le 1$, $x \in \bar{S}$, then $U(x)$ satisfies the equation (1.1.8), which means that $u(x)$ from (1.1.5) is a solution to problem $\mathcal{H}_m$.

Let $P'(t)$ denote a matrix having the property

$$P(t)P'(t) = \operatorname{diag}(\det P(t)),$$

and let $P^*(t)$ be the conjugate matrix with $P(t)$. We introduce the numbers

$$\omega_i = \max_k \max_j(\deg p_{jk}(t) + \deg p'_{ki}(t)), \tag{1.1.9}$$

where $p_{i,j}(t)$ and $p'_{i,j}(t)$ are elements of the matrices $P(t)$ and $P'(t)$, respectively, $t^{k!} = t^k$.

Lemma 1.2. *Let* $\det P(\lambda) \not\equiv 0$ *and functions* $\varphi_i(t)$*, where* $i = \overline{1,m}$ *be such that* $\det P(k) = 0 \Rightarrow \forall i,\ \varphi_i^{(k)}(0) = 0$*. Let also equations*

$$\det P(tD_t)w_i(t) = \varphi_i(t), \quad i = \overline{1,m} \tag{1.1.10}$$

have analytical solutions in $(-1,1)$. *Then, any analytic solution in* $(-1,1)$ *of the matrix equation*

$$P(tD_t)U(t) = \Phi(t), \tag{1.1.11}$$

in which $\Phi = (\varphi_1, \dots, \varphi_m)$ *is denoted, can be written as*

$$U(t) = P'(tD_t)W(t). \tag{1.1.12}$$

Moreover, if $w_i(t) \in C^{\omega_i}(|t| \le 1)$*, then the solution* (1.1.12) *satisfies the equation* (1.1.11) *for* $t \to \pm 1$.

Proof. Let $U(t) = (u_1(t), \dots, u_m(t))$ be an analytical solution in $(-1,1)$ to the equation (1.1.11). Then, by virtue of equality

$$P(tD_t)\left(U_k t^k\right) = t^k P(k) U_k, \tag{1.1.13}$$

in which U_k is a constant vector and $k \in \mathbb{N}_0$, from the equation (1.1.11), taking into account the Taylor formula and the conditions of the lemma, for small t we obtain

$$U(t) = \sum_{k=0}^{\infty} P'(k)\, \Phi^{(k)}(0) \frac{t^{k,!}}{\det P(k)}.$$

Using again the equality (1.1.13) for the matrix $P'(t)$, we have

$$U(t) = P'(tD_t) \sum_{k=0}^{\infty} \Phi^{(k)}(0) \frac{t^{k,!}}{\det P(k)}. \tag{1.1.14}$$

From the equations (1.1.10), for small t we find

$$W(t) = \sum_{m=0}^{\infty} \Phi^{(k)}(0) \frac{t^{k,!}}{\det P(k)},$$

which means, taking into account (1.1.14), the equality (1.1.12) for small t is proved. Due to the fact that the function $U(t)$ obtained from (1.1.12) is a solution to the system (1.1.11) for all $t \in (-1,1)$, then the equality (1.1.12) can be considered valid for all $t \in (-1,1)$.

It is easy to see that for the equation (1.1.11) to be satisfied in $[-1,1]$ it is enough to require that $u_j \in C^{s_j}(|t| \le 1)$, where $s_j = \max_k \deg p_{k,j}(\lambda)$. From (1.1.12) we can conclude that the condition on $u_j(t)$ are satisfied if $p_{i,j}(tD_t)w_i(t) \in C^{s_j}(|t| \le 1)$ for any $i = \overline{1,m}$. The last condition obviously holds true if $w_i \in C^{\omega_i}(|t| \le 1)$ with the exponent ω_i found by the equality (1.1.9). The lemma is proved. □

Lemma 1.3. *Let $\lambda_1, \ldots, \lambda_l$ be different roots of the polynomial $Q_k(\lambda)$, and their multiplicities be $k_1, \ldots, k_l$, and let $q_k \neq 0$ be the leading coefficient of $Q_k(\lambda)$. Let us assume that there is at least one system of functions $v_{i,j} \in C^j(|t| \leq 1)$ that satisfies the condition*

$$(tD_t - \lambda_i)\, v_{i,j}(t) = v_{i,j-1}(t), \quad v_{i,0}(t) = \varphi(t), \tag{1.1.15}$$

where $1 \leq i \leq l$, $1 \leq j \leq k_i$. Then, any solution to the equation

$$Q_k(tD_t)W(t) = \varphi(t), \tag{1.1.16}$$

in which $|t| \leq 1$, $\varphi \in C(|t| \leq 1)$ can be represented as

$$W(t) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} C_{i,j} u_{i,j}(t), \tag{1.1.17}$$

where the constants $C_{i,j}$ are unique and satisfy the equality

$$Q_k(\lambda) \sum_{i=1}^{l} \sum_{j=1}^{k_i} C_{i,j} (\lambda - \lambda_i)^{-j} = 1, \tag{1.1.18}$$

and the functions $u_{i,j}(t)$ obey the conditions (1.1.15).

Proof. It is enough to prove that at least one particular solution of the equation (1.1.16) can be represented in the form (1.1.17).

1^0. Let us choose $u_{i,j}(t) = v_{i,j}(t)$. Taking into account the condition (1.1.15), we find

$$(tD_t - \lambda)\, u_{i,j}(t) = (\lambda_i - \lambda)\, u_{i,j}(t) + u_{i,j-1}(t). \tag{1.1.19}$$

Acting on the function $W(t)$ from (1.1.17) k_1 times with the operator $(tD_t - \lambda_1)$, then applying the operator $(tD - \lambda_2)$ to the result k_2 times, etc. ., using (1.1.15) and (1.1.19), we equate to zero all coefficients arising for the function $\varphi(t)$, except for the last one, which we equate to q_k^{-1}.

As a result of executing the described algorithm, a linear system of equations for $C_{i,j}$ arises. If we now choose $C_{i,j}$ to satisfy this system, then the function $W(t)$ from (1.1.17) will be k times continuously differentiable in $[-1, 1]$ and will satisfy the equation (1.1.16).

2^0. Multiplying the expression

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} C_{i,j} (\lambda - \lambda_i)^{-j}$$

k_1 times on $(\lambda-\lambda_1)$, then k_2 times on $(\lambda-\lambda_2)$, etc., using the equalities

$$(\lambda-\lambda_i)(\lambda-\lambda_j)^{-s}=(\lambda_j-\lambda_i)(\lambda-\lambda_j)^{-s}+(\lambda-\lambda_i)^{1-s}, \tag{1.1.20}$$

we equate to zero all terms that do not contain powers $(\lambda-\lambda_i)$, except for the last one, which we equate to q_k^{-1}.

Comparing the algorithms described in 1^0 and 2^0, we are convinced that the systems of equations obtained from them for $C_{i,j}$ are identical. It is known that $C_{i,j}$ satisfying the system obtained by algorithm from 2^0 exist and are unique for any polynomial $Q_k(\lambda)$. The lemma is proved. □

Let us denote $\varphi^{(k)}(t)=\partial^k\varphi/\partial^k t$, and as usual $\mathcal{O}(-1,1)$ is the set of analytic in $(-1,1)$ functions. Let $[l]$ be the integer part of the number l.

Lemma 1.4. *The equation*

$$(tD_t-\lambda)^k\, w(t)=\varphi(t),\quad |t|\le 1 \tag{1.1.21}$$

for $w,\varphi\in\mathcal{O}(-1,1)$ *is solvable if and only if*

$$\lambda\in\mathbb{N}_0\Rightarrow\varphi^{(\lambda)}(0)=0.$$

The only solution to the equation (1.1.21)*, for* $\lambda\in\mathbb{N}_0$ *up to monomials of powers* $\lambda\in\mathbb{N}_0$*, can be represented in the form*

$$w(t)=\sum_{i=0}^{[\operatorname{Re}\lambda]}\frac{\varphi^{(i)}(0)}{(i-\lambda)^k}t^{i,!}$$
$$+\,t^\lambda\int_0^t\frac{\ln^{k-1,!}(t/\tau)}{\tau^{\lambda+1}}\Big(\varphi(\tau)-\sum_{i=0}^{[\operatorname{Re}\lambda]}\varphi^{(i)}(0)\tau^{i,!}\Big)d\tau. \tag{1.1.22}$$

Proof. Let t be small enough. If $\lambda\notin\mathbb{N}_0$, then, using the equality $(tD_t-\lambda)t^s=(s-\lambda)t^s$, we easily arrive at the statement

$$\varphi(t)=\sum_{i=0}^{\infty}\varphi_i t^i\Rightarrow w(t)=\sum_{i=0}^{\infty}\frac{\varphi_i}{(i-\lambda)^k}t^i. \tag{1.1.23}$$

If $\lambda\in\mathbb{N}_0$, then the necessary condition for the existence of a solution to the equation (1.1.21) is immediately obtained in the form $\varphi_\lambda=0$, and the solution can be written as

$$w(t)=\sum_{i=0,\,i\ne\lambda\in\mathbb{N}_0}^{\infty}\frac{\varphi_i}{(i-\lambda)^k}t^i. \tag{1.1.24}$$

It is easy to verify that for $i > \operatorname{Re}\lambda$ the equality

$$t^\lambda \int_0^t \tau^i \ln^{k-1,!}(t/\tau)\frac{d\tau}{\tau^{\lambda+1}} = \frac{t^i}{(i-\lambda)^k},$$

holds true and therefore for small t, in accordance with (1.1.23) and (1.1.24), the solution of (1.1.21) can be written in the form (1.1.22).

Let now $t \in [-1, 1]$. The function on the right side of (1.1.22) is defined if

$$\lambda \in \mathbb{N}_0 \Rightarrow \varphi_\lambda \equiv \varphi^{(\lambda)}(0)/\lambda! = 0.$$

Due to the analyticity of $w(t)$ in a neighborhood of zero (1.1.24), the analyticity of $w(t)$ in $(-1, 1)$ follows from the analyticity in $(-1, 0) \cup (0, 1)$ the function

$$t^\lambda \int_0^t \psi(\tau)\ln^j(t/\tau)\frac{d\tau}{\tau^\sigma},$$

for which $\operatorname{Re}\sigma < 1$, $j \in \mathbb{N}_0$, and $\psi(t)$ is analytic in $(-1, 0) \cup (0, 1)$.

To prove the uniqueness of the solution (1.1.22) it is enough to note that a solution of the form $w(t) + t^\lambda C\ln^{j,!} t$ (component of the general solution) is analytical in $(-1, 1)$ solution of the equation (1.1.21) if and only if $\lambda \in \mathbb{N}_0$ and $j = 0$. The lemma is proved. □

Corollary 1.1. If $\varphi \in C^l(|t| \le 1) \cap \mathcal{O}(-1, 1)$ and there exists an analytical solution to the equation (1.1.16) in $(-1, 1)$, then $w \in C^{k+l}(|t| \le 1)$.

Proof. From Lemma 1.4, using the form of solution to the equation (1.1.21) for $k = 1, \varphi = \psi$, we deduce that

$$\psi \in C^l(|t| \le 1) \cap \mathcal{O}(-1, 1) \Rightarrow w \in C^{l+1}(|t| \le 1) \cap \mathcal{O}(-1, 1). \tag{1.1.25}$$

Let us represent the equation (1.1.16) in the form

$$(tD_t - \lambda_1)\, Q'_{m-1}(tD_t)w(t) = \varphi(t).$$

Denoting

$$Q'_{m-1}(tD_t)w(t) = w_1(t),$$

by (1.1.25) and the conditions of the corollary, we obtain $w_1 \in C^{l+1}(|t| \le 1) \cap \mathcal{O}(-1, 1)$. By doing the same with the polynomial $Q'_{m-1}(\lambda)$, and then continuing this process further, we arrive at the statement of the corollary. □

Lemma 1.5. *The system of polynomials $p_1(\lambda), \dots, p_k(\lambda)$ is linearly independent if and only if the determinant of the matrix*

$$\Delta_k(\lambda) = \big(p_i(\lambda + (j-1)\lambda_0)\big)|_{i,j=\overline{1,k}},$$

with some $\lambda_0 \neq 0$ is identically zero.

Proof. The necessity of the lemma's conditions is obvious, since the matrix $\Delta_k(\lambda)$, in the case of the linearly dependent system of polynomials $p_1(\lambda), \dots, p_k(\lambda)$, have linearly dependent rows.

Let us prove the sufficiency. Let $k = 2$. Let us assume that

$$\det \Delta_2(\lambda) = \det \begin{pmatrix} p_1(\lambda) & p_1(\lambda + \lambda_0) \\ p_2(\lambda) & p_2(\lambda + \lambda_0) \end{pmatrix} \equiv 0. \tag{1.1.26}$$

We introduce a strict order on $\mathbb{C}$

$$\lambda_1 < \lambda_2 \Leftrightarrow (\operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_2) \vee \big((\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2) \wedge (\operatorname{Im}\lambda_1 < \operatorname{Im}\lambda_2)\big).$$

Let us denote the roots of the polynomial $p_1(\lambda)$ by $\mu_1 < \cdots < \mu_s$, where s is obviously not greater than $\deg p_1(\lambda)$. From a simple statement

$$\lambda_0 \neq 0 \Rightarrow \big(\forall \lambda \in \mathbb{C},\ \lambda < \lambda + \lambda_0\big) \vee \big(\forall \lambda \in \mathbb{C},\ \lambda > \lambda + \lambda_0\big)$$

it follows that for the validity of (1.1.26) it is necessary either $p_2(\mu_s) = 0$ when $\lambda_0 > 0$, or $p_2(\mu_1) = 0$ otherwise. Designating

$$p_i'(\lambda) = p_i(\lambda)/(\lambda - \mu_s),$$

if $\lambda_0 > 0$, and replacing μ_s in this notation with μ_1, if $\lambda_0 < 0$, we find that the identity (1.1.26) is equivalent to the identity

$$\det \big(p_i'\,(\lambda + (j-1)\lambda_0)\,\big)_{|i,j=\overline{1,2}} \equiv 0.$$

Continuing this process, we conclude that the equality (1.1.26) is also equivalent to the following identity

$$\det \begin{pmatrix} 1 & 1 \\ r(\lambda) & r(\lambda + \lambda_0) \end{pmatrix} \equiv 0, \tag{1.1.27}$$

where the notation $r(\lambda) = p_2(\lambda)/p_1(\lambda)$ is introduced. The identity (1.1.27) is valid only when $r(\lambda) = \text{const}$. This means that the identity (1.1.26) is satisfied only in the case when $p_1(\lambda) = Cp_2(\lambda)$, i.e. for linear dependence of the polynomials $p_1(\lambda)$ and $p_2(\lambda)$.

Let the conditions of the lemma be sufficient for $k = s$. Let us show their sufficiency for $k = s + 1$. The equality takes place

$$\prod_{j=1}^{s-1} p_1(\lambda + j\lambda_0) \det \Delta_{s+1}(\lambda) =$$
$$= \det \left(\det \begin{pmatrix} p_1(\lambda + (j-1)\lambda_0) & p_1(\lambda + j\lambda_0) \\ p_i(\lambda + (j-1)\lambda_0) & p_i(\lambda + j\lambda_0) \end{pmatrix} \right)_{\substack{i=\overline{2,s+1} \\ j=\overline{1,s}}} \tag{1.1.28}$$

Let us prove it. We denote the right-hand side of (1.1.28) by I_s. It is obvious that

$$I_s = \det \left(r_i(\lambda + (j-1)\lambda_0) \right)_{\substack{i=\overline{2,s+1} \\ j=\overline{1,s}}},$$

where

$$r_i(\lambda) = p_1(\lambda)p_i(\lambda + \lambda_0) - p_i(\lambda)p_1(\lambda + \lambda_0).$$

Let us write I_s as a sum of determinants of order s, using the linearity property of the determinant with respect to the columns. Let us associate each term in the resulting sum with an integer vector $\alpha = (\alpha_1, \dots, \alpha_s)$ in the following way: if in this determinant in place of the column with number j there is a column of the form

$$\left(p_1(\lambda + (j-1)\lambda_0) p_i(\lambda + j\lambda_0) \right)^T_{i=\overline{2,s+1}},$$

then $\alpha_j = 1$, but $\alpha_j = 2$ otherwise. We denote each determinant of the sum under consideration by Λ_α. Then we get

$$I_s = \sum_\alpha \Lambda_\alpha. \tag{1.1.29}$$

It is easy to see that $\det \Lambda_\alpha = 0$ for those α for which $\alpha_l < \alpha_{l+1}$ for some number l, $(1 < l \le s)$, since the determinant Λ_α contains linearly dependent columns with numbers l and $l + 1$. Vectors α that do not satisfy the specified condition have the form

$$\alpha_{(0)} = (1, \dots, 1), \quad \alpha_{(1)} = (2, 1, \dots, 1),$$
$$\alpha_{(2)} = (2, 2, 1, \dots, 1), \dots, \alpha_{(s)} = (2, \dots, 2).$$

The vector $\alpha_{(l)}$ corresponds to a determinant whose first l columns can be represented in the form

$$p_1(\lambda + j\lambda_0)\big(p_i(\lambda + (j-1)\lambda_0)\big)^T_{i=\overline{2,s+1}}, \quad j = 1, \dots, l,$$

and the rest in the form

$$p_1(\lambda + (j-1)\lambda_0)\big(p_i(\lambda + j\lambda_0)\big)^T_{i=\overline{2,s+1}}, \quad j = l+1, \dots, s.$$

If now, from each term of the sum (1.1.29) we take out the common factor $\prod_{j=1}^{s-1} p_1(\lambda + j\lambda_0)$, then the remaining sum is represent the expansion of the determinant $\Delta_{s+1}(\lambda)$ by elements of the first row. The equality (1.1.28) is proved.

Let us continue the induction. Let $\det \Delta_{s+1}(\lambda) \equiv 0$. Then, using (1.1.28) and the induction hypothesis, we conclude that there are $c_i \in \mathbb{C}$, not all zero and such that $\sum_{i=2}^{s+1} c_i r_i(\lambda) = 0$. Introducing the designation

$$q(\lambda) = \sum_{i=2}^{s+1} c_i p_i(\lambda),$$

convert the resulting sum to the form

$$\det \begin{pmatrix} p_1(\lambda) & p_1(\lambda + \lambda_0) \\ q(\lambda) & q(\lambda + \lambda_0) \end{pmatrix} \equiv 0.$$

Hence, due to the validity of the lemma for $k = 2$, there is a $c_1 \neq 0$ such that $q(\lambda) + c_1 p_1(\lambda) = 0$. Taking into account the form of $q(\lambda)$, we conclude that the system of polynomials $p_1(\lambda), \dots, p_{s+1}(\lambda)$ is linearly dependent. Since s is arbitrary, the sufficiency of the conditions of the lemma is established. The lemma is proved. □

Let us denote the Poisson kernel of the Dirichlet problem in S and its partial derivatives of order α by $D_x^{(\alpha)}(x; y)$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is multiindex. Let $\lambda_1, \dots, \lambda_l$ be the roots of the polynomial $\det P(t)$, and $k_1, \dots, k_l$ be their multiplicities,

$$\nu = \max_i [\operatorname{Re} \lambda_i], \quad \mu = \deg\big(\det P(\lambda)\big), \quad I = \mathbb{N}_0 \cap \{\lambda_1, \dots, \lambda_l\}.$$

Let us denote harmonic polynomials of kth degree by $H_k(x)$ and set $f(x) = (f_1(x), \dots, f_m(x))$. Let $s_i = \omega_i - \mu$, where ω_i is taken from the equality (1.1.9). Let us introduce the matrix

$$R_\nu(x;\xi) = P'(tD_t)\Big(\frac{1}{t}\int_0^t \sum_{i=1}^{l}\sum_{j=1}^{k_i} C_{ij}\ln^{j-1,!}(t/\tau)\Big(D(\tau x;\xi)$$
$$- \sum_{|\alpha|=0}^{\nu}(\tau x)^{\alpha,!}D_x^{(\alpha)}(0;\xi)\Big)\frac{d\tau}{(\tau/t)^{\lambda_i+1}}\Big)\Big|_{t=1}, \quad (1.1.30)$$

in which C_{ij} satisfy the equality (1.1.18) with $Q_k(\lambda) = \det P(\lambda)$ and the vector

$$U_\nu(x) = \sum_{|\alpha|=0}^{\nu} A_\alpha x^{\alpha,!},$$

defined by the vectors A_α, which are some solutions of the equation

$$P(|a|)A_\alpha = \frac{1}{\omega_n}\int_{\partial S} D_x^{(\alpha)}(0;\xi)f(\xi)\,d\xi.$$

The main result of this section is formulated as follows.

Theorem 1.2. *Let $f_i \in C^{\omega_i-\mu+\varepsilon}$ for $\varepsilon > 0$, where the numbers ω_i are taken from* (1.1.9)*, $\mu = \deg\big(\det P(\lambda)\big)$ and the matrix $P(\lambda)$ is defined in* (1.1.7)*. A solution to problem $\mathcal{H}_m$ exists if and only if the following condition is satisfied*

$$\forall\lambda \in I,\ \forall b\ (P^*(\lambda)b = 0),\ \forall H_\lambda(x)\ \int_{\partial S}\sum_{i=1}^{m}\bar{b}_i f_i(\xi)H_\lambda(\xi)\,d\xi = 0. \quad (1.1.31)$$

If a solution to the problem $\mathcal{H}_m$ exists, then it can be represented in the form (1.1.5) *with the function $U(x) \equiv (u_1(x),\ldots,u_m(x))$ given by the equality*

$$U(x) = U_\nu(x) + \frac{1}{\omega_n}\int_{\partial S} R_\nu(x,\xi)f(\xi)\,d\xi, \quad (1.1.32)$$

where $R_\nu(x,\xi)$ is taken from (1.1.30)*. This solution is unique up to polynomials of the form*

$$v_\lambda(x) = \sum_{i=1}^{k} a_i\left(1-|x|^2\right)^{i-1}H_\lambda(x),$$

for which $\lambda \in I$, and the vector a is such that $P(\lambda)a = 0$.

Proof. Using Lemma 1.1 and Remark 1.2 we reduce finding a solution of the problem $\mathcal{H}_m$ to solving the equation (1.1.11) with $\Phi(t) = V(tx)$, where $V(x)$ is the vector solution to the Dirichlet problems. Since the harmonic in S function $V(x)$, generates the function $V(tx)$ $(x \in S)$, analytic in $(-1,1)$, then the equation (1.1.11) must be solved in analytic functions, with the analytic right-hand side.

Let us show that $\det P(\lambda) \not\equiv 0$. Indeed, the mapping $P_k(\lambda) \to P_{[k]}(\lambda)$ is one-to-one (see Lemma 3.24), and hence the system of polynomials $\{P_{[m_i]}(\lambda) : i = \overline{1,m}\}$ is linearly independent. Therefore the matrix of the form (see Lemma 1.1)

$$Q(\lambda) = (q_{i,j}(\lambda))_{i,j=\overline{1,m}} = (P_{[m_i]}(\lambda + 2j - 2))_{i,j=\overline{1,m}}$$

by Lemma 1.5 at $\lambda_0 = 2$ must have a non-zero determinant. Since the matrix $C = ((-1)^i \binom{j}{i})_{i,j=\overline{0,m-1}}$ is non-singular, then $\det P(\lambda) = \det Q(\lambda) \det C \not\equiv 0$.

It we take $V(x)$ and denote

$$V_1(tx) = \sum_{i=1}^{\nu} D_t^i V(tx)|_{t=0} t^{i,!},$$

then there is a function $V_2(tx)$ such that $V(tx) = V_1(tx) + V_2(tx)$. Let $U_i(t,x)$ be an analytical solution of the equation (1.1.11) with $\Phi(t) = V_i(tx)$, and the conditions $D_t^k V_2(t,x)|_{t=0} = 0$ for $k \le \nu$. It is obvious that $V_2(tx)$ satisfies the requirements of Lemma 1.2, which means that to find the function $U_2(t,x)$ it is necessary to solve the equation (1.1.16) with $Q_k(\lambda) = \det P(\lambda)$, $\varphi = V_2(tx)$. The polynomial $U_1(t,x)$ exists if and only if

$$\lambda_i \in \mathbb{N}_0 \Rightarrow \forall x \in S, \quad D_t^{\lambda_i} V(tx)|_{t=0} \in P(\lambda_i)\mathbb{R}^m.$$

After transformations, this condition takes the form (1.1.31). The solution of equation (1.1.11) for $\Phi(t) = V(tx)$ obviously decomposes into the sum of functions $U_i(t,x)$, $i = 1,2$ and can be represented in the form

$$U(t,x) = \sum_{i=1}^{\nu} B_i(x) t^{i,!} + U_2(t,x), \tag{1.1.33}$$

where the coefficients $B_i(x)$ satisfy the equation

$$P(i)B_i(x) = D_t^i V(tx)|_{t=0},\ x \in S.$$

By simple reasoning it is easy to show that $U_1(1,x) = U_\nu(x)$ and the polynomial $U_1(1,x)$ is unique up to terms of the form $t^\lambda H_\lambda(x)a$, for $\lambda \in I$ and a vector a satisfying the condition $P(\lambda)a = 0$.

To find the function $U_2(t,x)$, we use Lemma 1.3 under the condition $Q_k(\lambda) = \det P(\lambda)$, $\varphi = V_2(tx)$, and then Lemma 1.4. The function $U_2(t,x)$ defined in this way always exists, is unique, and has the property $U_2(t,x) = U_2(tx)$. The polynomial $U_1(t,x)$ has the same property, and therefore the function $U(t,x)$ from (1.1.33). Therefore, by virtue of the remark 1.2, the function $u(x)$ obtained from $U(1,x)$ using the equality (1.1.5) is a solution to the problem $\mathcal{H}_m$. It can easily be converted to the form (1.1.32).

The uniqueness of the resulting solution is determined by the uniqueness of finding the functions $U_1(t,x)$ and $U_2(t,x)$.

In order for the equation (1.1.11) to be satisfied in $[-1,1]$, by Lemma 1.2 and Corollary 1.1, it is sufficient that $v_i(tx) \in C^{\omega_i-\mu}$, and for this it is sufficient that $f_i \in C^{\omega_i-\mu+\varepsilon}$ [4] for some $\varepsilon > 0$.

Concluding the proof of the theorem, we note that if the equation (1.1.11) for $\Phi(t) = V(tx)$ does not have an analytical solution in $(-1,1)$, then the problem $\mathcal{H}_m$ does not have a solution for any $s_i \in \mathbb{N}_0$. The theorem is proved. □

Remark 1.3. If we consider the problem $\mathcal{H}_1$, then from Theorem 1.2 it follows that the conditions of Theorem 1.1 are sufficient.

Dirichlet and Neumann boundary value problems. As an example of the application of Theorem 1.2, consider the Dirichlet boundary value problem, i.e. when $P_{m_i}(\lambda) = \lambda^{i-1}$ and the Neumann boundary value problem, i.e. when $P_{m_i}(\lambda) = \lambda^i$. In the case of the Dirichlet problem we denote the matrix $P(\lambda)$ by $D_m(\lambda)$, and in the case of the Neumann problem by $N_m(\lambda)$.

Let us show that for the problems under consideration $s_i \le m - i$. First, we make sure that the equalities

$$\det D_m(\lambda) = \text{const}, \quad \det N_m(\lambda) = \det D_m(\lambda - 1) \cdot \prod_{i=0}^{m-1}(\lambda + 2i), \tag{1.1.34}$$

hold true, which implies $\mu = 0$ for the Dirichlet problem and $\mu = m$ for the Neumann problem. The first equality is easily established if we notice that there is a non-singular matrix Q such that

$$D_m(\lambda) = Q \cdot \Big((\lambda + 2j - 2)^{i-1}\Big)_{i,j=\overline{1,m}} \cdot C,$$

where the matrix C is defined in Lemma 1.1, and differentiate the determinant of the right-hand side of the last equality with respect to λ. The validity of the second equality is easy to see if we take out the common factor $\lambda+2i-2$ from the ith column of the matrix $N_m(\lambda)$.

Next we establish that

$$\deg d_{is}(\lambda) = i-s, \quad \deg h_{is}(\lambda) = i-s+1,$$

where $d_{is}(\lambda)$ and $h_{is}(\lambda)$ are elements of the matrices $D_m(\lambda)$ and $N_m(\lambda)$, respectively. Indeed, the above equalities are a consequence of the following equality

$$\deg \sum_{s=0}^{i}(-1)^s\binom{i}{s}(\lambda+2s)^{[j]} = j-i,$$

which in turn is derived from the relations

$$j<i \Rightarrow \sum_{s=0}^{i}(-1)^s\binom{i}{s}s^j = 0, \quad \sum_{s=0}^{i}(-1)^s\binom{i}{s}s^i = (-1)^i i!.$$

Now, using (1.1.34), it is not difficult to find that

$$\deg d'_{si}(\lambda) \le s-i, \quad \deg h'_{si}(\lambda) = m+s-i-1,$$

where $d'_{si}(\lambda)$ and $h'_{si}(\lambda)$ are elements of the matrices $D'_m(\lambda)$ and $N'_m(\lambda)$. Let us estimate the value of ω_i from (1.1.9), which for the Dirichlet problem we denote by ω'_i, and for the Neumann problem by ω''_i. We get

$$\omega'_i = \max_s\Big(\deg d'_{si}(\lambda)+\max_j \deg d_{js}(\lambda)\Big) \le \max_s(s-i+m-s) = m-i,$$

$$\omega''_i = \max_s\Big(\deg h'_{si}(\lambda)+\max_j \deg h_{js}(\lambda)\Big) \le \max_s(m+s-i-1$$
$$+m-s+1) = 2m-i,$$

From here, taking into account that $s_i = \omega_i - \mu$, in view of the equalities (1.1.34), we obtain the necessary estimates for s_i, $i=1,\dots,m$.

Let us apply Theorem 1.2 to the Dirichlet and Neumann problems. By virtue of Theorem 1.2, a solution to the Dirichlet problem exists for any $f_i \in C^{m-i+\varepsilon}$, and a solution to the Neumann problem exists for $f_i \in C^{m-i+\varepsilon}$, satisfying the condition

$$\int_{\partial S}\sum_{i=1}^{m} b_i f_i(\xi)\,d\xi = 0,$$

where $b_s = \det \Delta_s$, for $s = \overline{1, m-1}$, and $b_m = -\det \Delta_m$. The matrix Δ_m is found from the equality

$$\Delta_m = \left((2j)^{[i]}\right)_{i,j=\overline{1,m-1}},$$

and Δ_s is obtained from Δ_m by replacing the row $\left(2^{[s]}, \ldots, (2m-2)^{[s]}\right)$ with the following row $\left(2^{[m]}, \ldots, (2m-2)^{[m]}\right)$.

The solution to the Dirichlet problem is unique, and the solution to the Neumann problem is unique up to a constant, since the first column of the matrix $N_m(0)$ is zero and it has a nonzero minor of $(m-1)$ order.

To construct a solution to the problem $\mathcal{H}_m$, according to Theorem 1.2, it is necessary to construct the function $U(x)$ using the formula (1.1.32). For the Dirichlet problem $U_\nu = 0$ and

$$R(x,\xi) = D'_m(\Lambda_x)\frac{D(x,\xi)}{\det D_m(0)}.$$

For the Neumann problem $\nu = 0$,

$$R(x,\xi) = N'_m(tD_t)\left(\int_0^t \left(\frac{t^2-\tau^2}{2t^2}\right)^{m-1,!} \frac{D(\tau x,\xi)-1}{\det D_m(0)}\frac{d\tau}{\tau}\right)\Big|_{t=1},$$

and A_0 is found from the equality $A_0 = C\cdot(0,\tilde{a})^T$, in which the vector $\tilde{a}$ is determined from the equality $\tilde{a} = \Sigma^{-1}\tilde{f}$, where

$$\Sigma = \left((2j-2)^{[i]}\right)_{i,j=\overline{2,m}}, \qquad \tilde{f} = \left(\int_{\partial S} f_i(\xi)\,d\xi\right)_{i=\overline{2,m}}.$$

If, in the resulting solution to the Neumann problem, we put $m = 1$, then since $N'_1(\lambda) = 1$ we arrive at the result of paper [30]. The Dirichlet and Neumann problems are considered in more detail in Chapters 2 and 3, respectively.

1.2 Navier problem and systems of normalized functions

One of the methods for constructing solutions to various boundary value problems for partial differential equations is the method of normalized systems of functions, described in [89]. This section briefly describes this method. In connection with property 2 of the proposed method (see below), this method is sometimes called the operator method [34]. Variants of the presented method are successfully applied not only to solve classical problems, but also to construct solutions to initial problems for equations with fractional derivatives [172, 178] and solutions to the generalized Dirichlet problem for the Dirac equation with iterated slices [190]. The construction of solutions to the Cauchy problem for linear ODEs using this method is given in [75].

In this section, in Theorem 1.3, a system of 0-normalized functions with respect to the Laplace operator in the starlike domain $\Omega \subset \mathbb{R}^n$ is constructed, which has the most general basis. Then, as an application of the method of normalized systems of functions to polyharmonic problems, Theorem 1.4 on the solvability of the Navier problem [166] in Ω and Lemma 1.6 on solutions of the homogeneous Helmholtz equation are proved.

Let $\mathcal{L}$ be a linear operator acting on functions $f(x)$, $x \in \Omega$ belonging to the set $\mathbb{X}$ such that $\mathcal{L}\mathbb{X} \subset \mathbb{X}$ and defined in some domain $\Omega \subset \mathbb{R}^n$. Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.2. We call an ordered system of functions $\{f_k(x) : k \in \mathbb{N}_0\}$ from $\mathbb{X}$ f-normalized with respect to $(\mathcal{L}_1, \mathcal{L}_2)$ in the domain Ω with base $f_0(x)$ if everywhere in this domain

$$\mathcal{L}_1 f_0(x) = f(x), \quad \mathcal{L}_1 f_k(x) = \mathcal{L}_2 f_{k-1}(x), \quad k \in \mathbb{N}, \quad x \in \Omega.$$

An important special case of the introduced concept is the case when $\mathcal{L}_2 = I$, and I is the identity operator. Then, the f-normalized system of functions $\{f_k(x) : k \in \mathbb{N}_0\}$ with respect to $(\mathcal{L}, I)$ has the property

$$\mathcal{L} f_0(x) = f(x), \quad \mathcal{L} f_k(x) = f_{k-1}(x), \quad k \in \mathbb{N}, \quad x \in \Omega. \tag{1.2.1}$$

This property for $f = 0$ in [34] is also called the normalizability of the system of functions $\{f_k(x) : k \in \mathbb{N}_0\}$ with respect to the operator $\mathcal{L}$. Therefore, keeping with tradition, we will also call a system of functions $\{f_k(x) : k \in \mathbb{N}_0\}$ that has the property (1.2.1) f-normalized with respect to the operator $\mathcal{L}$.

The simplest example of an f-normalized system of functions with respect to the operator $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, where $\alpha \in \ BbbN_0^n$ in $\mathbb{R}^n$ for $f(x) = x^{\beta,!}$, $\beta \in \mathbb{N}_0^n$ is a system of polynomials $\{f_k(x) : k \in \mathbb{N}_0\}$ of the form

$$f_k(x) = x^{(k+1)\alpha+\beta,!}, \quad x^{\alpha,!} = \frac{x_1^{\alpha_1}}{\alpha_1!} \cdots \frac{x_n^{\alpha_n}}{\alpha_n!}.$$

Obviously, if the system $\{f_k(x)\}$ is f-normalized, and the system $\{\varphi_k(x)\}$ is φ-normalized with respect to $\mathcal{L}$ in the domain Ω, then the system $\{f_k(x) + \varphi_k(x)\}$

is $(f+\varphi)$-normalized with respect to $\mathcal{L}$ in the domain Ω. It should also be noted that all elements $f_{k-1}(x)$, $f_{k-2}(x)$, etc., preceding a given element $f_k(x)$ of some f-normalized system of functions with respect to the operator $\mathcal{L}$ are uniquely determined through this element $f_k(x)$, and subsequent elements $f_{k+1}(x)$, $f_{k+2}(x)$, etc., in general, no.

Let us present the main property of normalized systems of functions. Let a linear equation of the form be given

$$(\mathcal{L}_1-\mathcal{L}_2)u(x)=f(x),\quad x\in\Omega, \tag{1.2.2}$$

and let $\{f_k(x) : k\in\mathbb{N}_0\}$ be some f-normalized system of functions with respect to $(\mathcal{L}_1,\mathcal{L}_2)$ in Ω. Let us compose the functional series $\sum_{n=0}^{\infty} f_n(x)$ from the elements of this system $\{f_k(x) : k\in\mathbb{N}_0\}$. We assume that it converges in Ω to the function $u(x)$ from $\mathbb{X}$ and allows term-by-term application of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ under the sum sign.

Property 1. Sum of functional series

$$u(x)=\sum_{n=0}^{\infty} f_n(x),\quad x\in\Omega$$

is a formal solution to the equation (1.2.2) in Ω. If the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ can be entered under the series sign, then $u(x)$ is the usual solution to the equation (1.2.2) in Ω.

Indeed, the equalities are true

$$\begin{aligned}(\mathcal{L}_1-\mathcal{L}_2)u(x)&=\mathcal{L}_1 f_0(x)+\sum_{n=1}^{\infty}\mathcal{L}_1 f_n(x)-\sum_{n=0}^{\infty}\mathcal{L}_2 f_n(x)\\&=f(x)+\sum_{n=1}^{\infty}\mathcal{L}_2 f_{n-1}(x)-\sum_{n=0}^{\infty}\mathcal{L}_2 f_n(x)=f(x).\end{aligned}$$

In accordance with this property, to find a solution to the equation (1.2.2) it is necessary to be able to construct normalized systems of functions with respect to the pair of operators $(\mathcal{L}_1,\mathcal{L}_2)$. In the case of commutability of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, the following connection holds between the concepts of normalized systems introduced above.

Property 2. If an f-normalized system of functions $\{\varphi_k(x) : k \in \mathbb{N}_0\}$ is given, then an f-normalized system of functions with respect to $(\mathcal{L}_1,\ mathcalL_2)$ can be written as

$$u(x) = \sum_{k=0}^{\infty} \mathcal{L}_2^k \varphi_k(x), \quad x \in \Omega. \tag{1.2.3}$$

Indeed, $\mathcal{L}_1(\mathcal{L}_2^0\varphi_0(x)) = f(x)$, and for $k \in \mathbb{N}$

$$\mathcal{L}_1(\mathcal{L}_2^k\varphi_k(x)) = \mathcal{L}_2^k\mathcal{L}_1\varphi_k(x) = \mathcal{L}_2^k\varphi_{k-1}(x) = \mathcal{L}_2(\mathcal{L}_2^{k-1}\varphi_{k-1}(x))$$

and therefore the equality (1.2.3) is a consequence of the equality from the property 1. This property of normalized systems of functions allows one to construct polynomial solutions of partial differential equations with constant coefficients [64, 65, 67–69].

In addition, knowledge of the f-normalized system of functions with respect to $(\mathcal{L}_1, \mathcal{L}_2)$ allows one to construct an f-normalized system of functions with respect to the operator $\mathcal{L}_1 - \mathcal{L}_2$.

Property 3. Let the system of functions $\{f_k(x) : k \in \mathbb{N}_0\}$ be f-normalized with respect to the operator $\mathcal{L}_1$ in Ω, then the system of functions $\{\varphi_k(x) : k \in \mathbb{N}_0\}$, where

$$\varphi_k(x) = \sum_{n=k}^{\infty} \binom{n}{k} \mathcal{L}_2^{n-k} f_n(x)$$

is f-normalized with respect to the operator $\mathcal{L}_1 - \mathcal{L}_2$ in Ω.

Indeed, due to property 1 we have

$$(\mathcal{L}_1 - \mathcal{L}_2)\varphi_0(x) = f(x),$$

and for $k \in \mathbb{N}$ we get

$$\begin{aligned}
(\mathcal{L}_1 - \mathcal{L}_2)\varphi_k(x) &= \sum_{n=k}^{\infty} \binom{n}{k} \mathcal{L}_2^{n-k}\mathcal{L}_1 f_n(x) - \sum_{n=k}^{\infty} \binom{n}{k} \mathcal{L}_2^{n-k+1} f_n(x) \\
&= \mathcal{L}_2 f_{k-1}(x) + \sum_{n=k+1}^{\infty} \binom{n}{k} \mathcal{L}_2^{n-k} f_{n-1}(x) - \sum_{n=k}^{\infty} \binom{n}{k} \mathcal{L}_2^{n-k+1} f_n(x) \\
&= \mathcal{L}_2 f_{k-1}(x) + \sum_{n=k}^{\infty} \left(\binom{n+1}{k} - \binom{n}{k} \right) \mathcal{L}_2^{n-k+1} f_n(x) = \mathcal{L}_2 f_{k-1}(x) \\
&= \sum_{n=k}^{\infty} \binom{n}{k-1} \mathcal{L}_2^{n-k+1} f_n(x) = \sum_{n=k-1}^{\infty} \binom{n}{k-1} \mathcal{L}_2^{n-(k-1)} f_n(x) = \varphi_{k-1}(x).
\end{aligned}$$

Definition 1.3. The generalized Pochhammer symbol is the expression $(a,b)_k = a(a+b)\ldots(a+kb-b)$, with the agreement $(a,b)_0 = 1$. It is clear that $(2,2)_k = (2k)!!$. If $b=1$ then write $(a,1)_k = (a)_k = a(a+1)\cdots(a+k-1)$ and this is the Pochhammer symbol.

Example 1.2. Let us construct a 0-normalized system of polynomials with respect to the Laplace operator having a polynomial base $u(x)$, i.e. the first element of the system is equal to the harmonic polynomial $u(x)$.

Let us first consider the function $f(x) = |x|^p u_m(x)$, where $u_m(x)$ is a homogeneous harmonic polynomial of degree $m \in \mathbb{N}_0$ and $p \geq 0$. System of polynomials $\{\Phi_k(x) : k \in \mathbb{N}_0\}$, where

$$\Phi_k(x) = \frac{|x|^{2k+2}}{(p+2,2)_{k+1}(n+p+2m,2)_{k+1}}|x|^p u_m(x) \tag{1.2.4}$$

is f-normalized with respect to Δ in $\mathbb{R}^n$. Indeed, from the obvious equality

$$\Delta(u_m(x)|x|^k) = k\Big((k+n-2)u_m(x) + 2\sum_{i=1}^n x_i D_{x_i} u_m(x)\Big)|x|^{k-2},$$

taking into account the homogeneity of the polynomial $u_m(x)$, the equality follows

$$\Delta\left(u_m(x)|x|^k\right) = k(k+2m+n-2)u_m(x)|x|^{k-2},$$

which is equivalent to fulfilling (1.2.1) for $f_k(x) = \Phi_k(x)$ and $\mathcal{L}_1 = \Delta$.

Further, from the previous arguments it is clear that the system of polynomials $\{h_k(x) : k \in \mathbb{N}_0\}$, with

$$h_k(x) = \frac{|x|^{2k}}{(2,2)_k(n+2m,2)_k}u_m(x)$$

is 0-normalized with respect to Δ in $\mathbb{R}^n$. Let us transform it. Using equalities

$$\frac{1}{(n+2m,2)_k} = \frac{(n,2)_k}{(n,2)_{k+m}}, \quad (n,2)_s = 2^s\frac{\Gamma(n/2+s)}{\Gamma(n/2)}$$

we find

$$\frac{1}{(n+2m,2)_k} = \frac{1}{2^k\Gamma(k)}\frac{\Gamma(n/2+m)\Gamma(k)}{\Gamma(n/2+m+k)},$$

whence, using the connection between the Γ- and B-functions [15] and the definition of the B-function, we obtain

$$\frac{1}{(n+2m,2)_k} = \frac{1}{2}\int_0^1 \alpha^m\frac{(1-\alpha)^{k-1}}{(2k-2)!!}\alpha^{n/2-1}\,d\alpha.$$

Now, taking into account that $(2,2)_k = 2k!! = 2^k k!$, the polynomial $h_k(x)$ takes the form

$$h_k(x) = \frac{|x|^{2k}}{4^k k!(k-1)!} \int_0^1 (1-\alpha)^{k-1} u_m(\alpha x) \alpha^{n/2-1}\, d\alpha.$$

From the resulting equality it is clear that its right-hand side does not depend on m, except for the index of $u_m(x)$, and therefore this index can be omitted and the 0-normalized system of polynomials $\{h_k(x) : k \in \ BbbN_0\}$ with respect to the operator Δ with polynomial base $u(x)$ takes the form

$$h_k(x) = \frac{|x|^{2k}}{4^k k!(k-1)!} \int_0^1 (1-\alpha)^{k-1} u(\alpha x) \alpha^{n/2-1}\, d\alpha,$$

i.e. $\Delta h_k(x) = h_{k-1}(x)$, for $k > 1$ and $\Delta h_1(x) = u(x)$.

Various examples of using systems of normalized functions in solving initial and boundary value problems can be found in [89].

Now we construct a normalized system of functions with respect to the Laplace operator. Let $\Omega \subset \mathbb{R}^n$ be some open bounded domain with the starlike property: $\forall x \in \Omega$, $\forall t \in [0,1]$, $tx \in \Omega$. To do this, we define in the domain Ω the following system of functions: $G_0(x;u) = u(x)$ and for $k > 0$

$$G_k(x;u) = \frac{1}{4^k} \frac{|x|^{2k}}{k!} \int_0^1 \frac{(1-\alpha)^{k-1} \alpha^{n/2-1}}{(k-1)!} u(\alpha x)\, d\alpha, \tag{1.2.5}$$

where $u(x)$ is some function harmonic in Ω.

Theorem 1.3. *The system of functions $\{G_k(x;u) : k \in \mathbb{N}_0\}$ is 0-normalized with respect to the operator Δ in Ω.*

Proof. In Example 1.2 it is established that the system of polynomials $\{G_k(x;u_m) : k \in \mathbb{N}_0\}$, where $u_m(x)$ is a harmonic the polynomial is 0-normalized with respect to the operator Δ in $\mathbb{R}^n$. Let us sum the polynomials $G_k(x;u_m)$ over m in some closed neighborhood of the origin $x \in \Omega_0 \subset \Omega$, where the series $\sum_m u_m(x)$ and all its derivatives up to the second order converge uniformly in x to the harmonic function $u(x)$ and its derivatives. We get

$$\sum_{m=0}^{\infty} G_k(x;u_m) = G_k(x;u) = \frac{1}{4^k} \frac{|x|^{2k}}{k!} \int_0^1 \frac{(1-\alpha)^{k-1} \alpha^{n/2-1}}{(k-1)!} u(\alpha x)\, d\alpha.$$

Due to the possibility of differentiation and integration under the sum sign in Ω_0 we can write $\Delta G_k(x;u) = 0$ and

$$\Delta G_k(x;u) = \sum_{m=0}^{\infty} \Delta G_k(x;u_m) = \sum_{m=0}^{\infty} G_{k-1}(x;u_m) = G_{k-1}(x;u),\ x \in \Omega_0.$$

Therefore, the system of functions $\{G_k(x;u) : k \in \mathbb{N}_0\}$, determined from (1.2.5) is 0-normalized with respect to the Laplace operator Δ in the domain Ω_0. It is obvious that the functions $G_k(x;u)$ are defined on the entire set Ω and, moreover, $G_k(x;u) \in C^2(\Omega)$ due to the possibility of differentiating with respect to a parameter under the integral sign.

Let us check directly that the equality $\Delta G_k(x;u) = G_{k-1}(x;u)$ holds not only in Ω_0, but also at any point in the set Ω. Keeping in mind the identities

$$\Delta(vw) = v\Delta w + w\Delta v + 2\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}, \quad \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(\alpha x) = \frac{\partial u}{\partial \alpha},$$

and equality

$$\Delta(u_m(x)|x|^l) = l(l + 2m + n - 2)u_m(x)|x|^{l-2}, \quad x \in \mathbb{R}^n$$

for $l = 2k$ and $m = 0$ we get

$$\begin{aligned} \Delta G_k(x;u) = \frac{2}{4^k}\frac{|x|^{2k-2}}{(k-1)!} \\ \times \int_0^1 \frac{(1-\alpha)^{k-1}\alpha^{n/2-1}}{(k-1)!}\left((2k+n-2)u(\alpha x) + 2\alpha\frac{\partial}{\partial \alpha}u(\alpha x)\right) d\alpha. \end{aligned} \tag{1.2.6}$$

Let $k > 1$. Applying integration by parts we obtain

$$\begin{aligned} \Delta G_k(x;u) &= \frac{2}{4^k}\frac{|x|^{2k-2}}{(k-1)!}\int_0^1 \frac{(1-\alpha)^{k-2}\alpha^{n/2-1}}{(k-1)!} \\ &\quad \times ((2k+n-2)(1-\alpha) + 2(k-1)\alpha - n(1-\alpha))\, u(\alpha x)\, d\alpha \\ &= \frac{2}{4^k}\frac{|x|^{2k-2}}{(k-1)!}\int_0^1 \frac{(1-\alpha)^{k-2}\alpha^{n/2-1}}{(k-1)!}(2k-2)u(\alpha x)\, d\alpha = G_{k-1}(x;u). \end{aligned}$$

If $k = 1$ then, in accordance with (1.2.6), we can write

$$\Delta G_k(x;u) = \frac{1}{2}\int_0^1 \alpha^{n/2}\left(nu(\alpha x) + 2\alpha\frac{\partial}{\partial\alpha}u(\alpha x)\right)\,d\alpha$$
$$= \int_0^1 \alpha^{n/2}\frac{\partial}{\partial\alpha}u(\alpha x) + \frac{\partial}{\partial\alpha}\left((\alpha^{n/2}\right)u(\alpha x)\,d\alpha$$
$$= \alpha^{n/2}u(\alpha x)|_0^1 = u(x) \equiv G_0(x;u).$$

Therefore, the system of functions $\{G_k(x;u) : k \in \mathbb{N}_0\}$ is 0-normalized with respect to the operator Δ in the entire domain Ω. The theorem is proved. □

As one of the useful properties of the system of functions $\{G_k(x;u)\}$, which follows from property 2, we prove the following statement.

Lemma 1.6. *Solutions of the homogeneous Helmholtz equation* $\Delta v + \lambda v = 0$ *in* Ω *can be found in the form*

$$v_m(x) = g_{2m+n}(\lambda|x|^2)u_m(x),$$

where $u_m(x)$ *is a homogeneous harmonic polynomial of degree* $m \in \mathbb{N}_0$ *and*

$$g_m(t) = \sum_{k=0}^{\infty}\frac{(-1)^k t^k}{(2,2)_k(m,2)_k}.$$

Proof. Let us take in the equation (1.2.2) $\mathcal{L}_1 = \Delta$, $\mathcal{L}_2 = -\lambda$. Then, by property 2, solutions to the Helmholtz equation can be written in the form

$$v(x) = \sum_{k=0}^{\infty}(-\lambda)^k G_k(x;u).$$

If now instead of the function $u(x)$ we take the homogeneous harmonic polynomial $u_m(x)$, then $G_k(x;u_m) = h_k(x)$ and we get

$$v_m(x) = \sum_{k=0}^{\infty}(-1)^k\frac{(\lambda|x|^2)^k}{(2,2)_k(n+2m,2)_k}u_m(x) = g_{2m+n}(\lambda|x|^2)u_m(x).$$

The lemma is proved. □

The correspondence $u_m(x) \to v_m(x)$ between functions harmonic in Ω and solutions of the Helmholtz equation can be one-to-one extended to all functions harmonic in Ω [68].

Remark 1.4. Note that if $J_m(t)$ is a Bessel function of the first kind, then

$$J_m(t) = \frac{t^m}{2^m\Gamma(m+1)}\, g_{2m+2}(t^2), \quad g_{2m+2}(t^2) = \frac{2^m\Gamma(m+1)}{t^m} J_m(t)$$

and that means for $\lambda \geq 0$

$$v(x) = g_{2m+n}(\lambda|x|^2)u_m(x) = g_{2(m+n/2-1)+2}((\sqrt{\lambda}|x|)^2)u_m(x) \\ = \frac{C}{|x|^{m+n/2-1}} J_{m+n/2-1}(\sqrt{\lambda}|x|)u_m(x),$$

where $C = 2^{m+n/2-1}\Gamma(m+n/2)\lambda^{-m/2-n/4+1/2}$.

Let us consider the following boundary value problem for the polyharmonic equation, called the Navier problem [166] (in the papers [34, 146] it is called the Riquier problem).

Navier problem. *Find a function $u \in C^{2m}(\Omega)$ such that $\Delta^k u \in C(\overline{\Omega})$ for $k = 0, \ldots, m-1$ and satisfying the equalities*

$$\begin{gathered} \Delta^m u = 0, \quad x \in \Omega, \\ \Delta^k u|_{\partial\Omega} = f_k(s), \quad s \in \partial\Omega, \quad k = 0, \ldots, m-1. \end{gathered} \tag{1.2.7}$$

Theorem 1.4. *Let the functions $f_k(s)$, $k = 0, \ldots, m-1$ be continuous in $\partial\Omega$. Then the Navier problem* (1.2.7) *has a unique solution.*

Proof. Let us construct a solution to the problem under consideration (1.2.7) in explicit form. We look for this solution in the form

$$u(x) = \sum_{k=0}^{m-1} G_k(x; u_{(k)}), \tag{1.2.8}$$

where $G_k(x; u)$ is defined in (1.2.5), and the functions $u_{(k)}(x)$ are solutions to the following Dirichlet problems

$$\begin{gathered} \Delta u_{(k)} = 0, \quad x \in \Omega; \quad u_{(k)} \in C(\overline{\Omega}); \\ u_{(k)}(x)|_{\partial\Omega} = f_k(s) - \sum_{i=1}^{m-k-1} G_i(x; u_{(i+k)})|_{\partial\Omega}. \end{gathered} \tag{1.2.9}$$

For example, when $k = m-1$

$$\Delta u_{(m-1)} = 0; \quad u_{(m-1)}(x)|_{\partial\Omega} = f_{m-1}(s)$$

and for $k = m - 2$

$$\Delta u_{(m-2)} = 0; \quad u_{(m-2)}(x)|_{\partial\Omega} = f_{m-2}(s) - G_1(x; u_{(m-1)})|_{\partial\Omega}.$$

Since the functions $u_{(k)}(x)$ are harmonic in Ω, it follows from (1.2.5) that $u \in C^{2m}(\Omega)$ and moreover one can notice that $\Delta^m G_k(x; v)$=0 for an arbitrary harmonic $v(x)$ and $0 \leq k \leq m-1$. Therefore, if we take ν such that $0 \leq \nu \leq m-1$, then we get

$$\Delta^\nu u(x) = \sum_{k=\nu}^{m-1} G_{k-\nu}(x; u_{(k)}) = u_{(\nu)}(x) + \sum_{i=1}^{m-\nu-1} G_i(x; u_{(i+\nu)}). \tag{1.2.10}$$

Letting $x \to \partial\Omega$ and using equalities (1.2.9) with $k = \nu$ we obtain $\Delta^\nu u(x)|_{\partial\Omega} = f_\nu(s)$. The constructions made are legal since all functions from the right side of (1.2.10) are continuous in $\overline{\Omega}$.

Thus, to find a solution to the problem (1.2.7) one need to solve the Dirichlet problem (1.2.9) for $k = m-1, \ldots, 0$ and then substitute the resulting solutions $u_{(k)}(x)$ in (1.2.8). The functions $u_{(k)}(x)$ exist because all functions $G_i(x; u_{(j)})$ under the summation sign in (1.2.9) are continuous on $\partial\Omega$ since $f_j \in C(\partial\Omega)$ for $k+1 \leq j \leq m-1$. Thus, the continuity of functions f_k for $k = 0, \ldots, m-1$ is sufficient for the existence of a solution (1.2.8) of the Navier problem (1.2.7) under consideration.

The uniqueness of the solution to the Navier problem follows from the following reasoning. If $u(x)$ is a solution to the homogeneous Navier problem and k is greatest for which $\Delta^{k+1}u(x) \equiv 0$, and $\Delta^k u(x) \not\equiv 0$, then the function $\Delta^k u(x)$ is harmonic and $\Delta^k u|_{\partial\Omega} = 0$, which means $\Delta^k u(x) \equiv 0$. The theorem is proved. □

Remark 1.5. If we consider the Dirichlet problem for the polyharmonic equation in a ball, studied in Chapter 2, then the continuity of the boundary functions f_k is not sufficient for the existence of a solution to the problem. In this case, $f_k \in C^{m-1-k+\varepsilon}(\partial\Omega)$ for $k = 0, \ldots, m-1$ (see Section 2.4.3). This can be explained by the fact that the smoothness that we require from the solution to the Navier problem $\Delta^k u \in C(\overline{\Omega})$ for $k = 0, \ldots, m-1$ differs from the smoothness of the solution to the Dirichlet problem $u \in C^{m-1}(\overline{\Omega})$.

1.3 On some special polynomials and functions

To approximately solve various boundary value problems for the harmonic and biharmonic equations, among other methods, the Trefz method, the least squares method, the collocation method, etc. are used. The rationale for these computational methods can be found in [142]. To ensure stability of the computational process in these methods, some additional requirements are imposed on the coordinate functions. It is very convenient to use harmonic and biharmonic polynomials as coordinate functions in these methods. The construction of biharmonic polynomials in Cartesian coordinates based on spherical functions using the Almansi formula leads to cumbersome calculations. A convenient algorithm is developed by M.P. Barnett and A.B. Otis [149, 150]. A system of linearly independent homogeneous harmonic polynomials in three variables is constructed by P.P. Teodorescu [167]. A system of linearly independent homogeneous polyharmonic polynomials is also constructed by B.A. Bondarenko [33]. In the multidimensional case, i.e. for $n > 3$ the situation is even more complicated [15]. Harmonic polynomials in n variables are also used by A.V. Bitsadze [28].

Section 1.3 is organized as follows. In 1.3.1, a complete system of homogeneous harmonic polynomials in n variables $\{G_{(\nu)}(x)\}$ is constructed (Theorem 1.5), which are orthogonal in $L_2(\partial S)$, where ∂S is the unit sphere in $\mathbb{R}^n$ (Theorem 1.6) and in $\mathcal{P}$ (Theorem 1.7). As an application of these theorems, in Theorem 1.9 an equality is obtained for finding the value of an expression of the form $G_{(\nu)}(D)u(x)|_{x=0}$, where $u(x)$ is harmonic in S and continuous in $\bar{S}$ function. Theorem 1.10 provides an equality for calculating an integral of the form $\int_{|x|=1} Q_m(x)\, ds_x$, where $Q_m(x)$ is a homogeneous polynomial. In 1.3.2 the concept of a G-function is introduced and a connection between G-functions and the Legendre and Chebyshev polynomials is established (Theorems 1.11 and 1.12), and a Rodrigue's formula is also given (Theorem 1.13). In 1.3.3 the orthogonality of G-functions $G_k^{s,n}(t)$ on the interval $[-1, 1]$ with weight $\rho(t) = (1 - t^2)^{(n-3)/2}$ is proved (Theorem 1.14). The results of Section 1.3 are given in more detail in the papers [65, 68, 69, 79].

1.3.1 G-polynomials

Let $\Omega \subset \mathbb{R}^n$ be some bounded domain with the starlike property: $\forall t \in [0, 1]$, $x \in \Omega \Rightarrow tx \in \Omega$ and $u(x)$ be some harmonic function in Ω. Consider the following system of functions: $G_0(x; u) = u(x)$ and for $k > 0$

$$G_k(x; u) = \frac{1}{4^k} \frac{|x|^{2k}}{k!} \int_0^1 \frac{(1-\alpha)^{k-1}\alpha^{n/2-1}}{(k-1)!} u(\alpha x)\, d\alpha, \quad x \in \Omega.$$

Theorem 1.3 proves that the system of functions $\{G_k(x; u) : k \in \mathbb{N}_0\}$ is 0-normalized with respect to the Laplace operator Δ in Ω. If $u(x) = H_m(x)$ is some homogeneous harmonic polynomial of degree m, then we have

$$G_k(x; H_m) = \frac{|x|^{2k} H_m(x)}{4^k k!(k-1)!} B(k, m+n/2)$$
$$= \frac{|x|^{2k} H_m(x)}{4^k k!(k-1)!} \frac{\Gamma(k)\Gamma(m+n/2)}{\Gamma(k+m+n/2)} = \frac{|x|^{2k} H_m(x)}{(2,2)_k (n+2m,2)_k}, \quad (1.3.1)$$

where $B(\alpha, \beta)$ and $\Gamma(\alpha)$ are the well-known Euler functions B and Γ, and $(a, b)_k$ is the generalized Pochhammer symbol. If we now represent the Laplace equation in the form

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{\partial^2 u}{\partial x_n^2} = L_1 u - L_2 u = 0,$$

then, by Theorem 1.3 and by the main property 2 of normalized systems of functions, the following polynomials are harmonic polynomials in n variables

$$u_{k,s}(x) = \sum_{i=0}^{[k/2]} \frac{|x_{(n-1)}|^{2i} H_s(x_{(n-1)})}{(2,2)_i (n-1+2s,2)_i} \left(-\frac{\partial^2}{\partial x_1^2}\right)^i x_n^{k,!} =$$
$$= G_k^s(x_{(n)}) H_s(x_{(n-1)}), \quad (1.3.2)$$

where denoted $x^{k,!} = x^k/k!$, $x_{(n-1)} = (x_1, \ldots, x_{n-1})$ and

$$G_k^s(x_{(n)}) = \sum_{i=0}^{[k/2]} (-1)^i \frac{|x_{(n-1)}|^{2i} x_n^{k-2i,!}}{(2,2)_i (n-1+2s,2)_i}. \quad (1.3.3)$$

Definition 1.4. A polynomial $G_k^s(x_{(n)})$ of the form (1.3.3) is called a G-polynomial of degree k, order s, and kind n.

Example 1.3. Let us write down some of the G-polynomials

$$G_0^s(x) = 1, \quad G_1^s(x) = x_n, \quad G_2^s(x) = \frac{x_n^2}{2} - \frac{x_1^2 + \cdots + x_{n-1}^2}{2(n-1+2s)},$$
$$G_3^s(x) = \frac{x_n^3}{6} - \frac{(x_1^2 + \cdots + x_{n-1}^2) x_n}{2(n-1+2s)}, \quad (1.3.4)$$
$$G_4^s(x) = \frac{x_n^4}{24} - \frac{(x_1^2 + \cdots + x_{n-1}^2) x_n^2}{4(n-1+2s)} + \frac{(x_1^2 + \cdots + x_{n-1}^2)^2}{8(n-1+2s)(n+1+2s)}.$$

In accordance with this definition above, it is established that the product of a homogeneous harmonic polynomial $H_s(x_{(n-1)})$ in $(n-1)$ variables by the G-polynomial $G_k^s(x_{(n)})$ gives a harmonic polynomial in n variables $u_{k,s}(x) = G_k^s(x_{(n)})H_s(x_{(n-1)})$. For example, the polynomial $G_2^0(x_{(n-1)})$ is harmonic, which means the polynomial $u_{4,2}(x) = G_4^2(x_{(n)})G_2^0(x_{(n-1)})$ is also harmonic, but in n variables. Its degree is 6. The stronger statement is true.

Lemma 1.7. *Any homogeneous harmonic polynomial in n variables can be represented as a sum of polynomials of the form* (1.3.2)*, where $H_s(x_{(n-1)})$ are some harmonic polynomials in* $(n-1)$ *variables* $x_{(n-1)} = (x_1, \dots, x_{n-1})$.

Proof. Let $u(x)$ be some homogeneous harmonic polynomial in n variables of degree l, and $H_s(x_{(n-1)})x_n^{k,!}$ be part of its monomials, containing the highest degree $k \le l$ in the variable x_n. Then the polynomial $H_s(x_{(n-1)})$ must be a harmonic polynomial in $(n-1)$ variables. Consider a harmonic polynomial in n variables of the form $u(x) - u_{k,s}(x)$, where the polynomial $u_{k,s}(x)$ is found from (1.3.2) and let $H_{s_1}(x_{(n-1)})x_n^{k_1,!}$ is the part of its monomials containing the highest degree k_1 in the variable x_n. By construction $k_1 < k$. Now consider a harmonic polynomial in n variables of the form $u(x) - u_{k,s}(x) - u_{k_1,s_1}(x)$, which has the highest degree $k_2 < k_1$ in the variable x_n, etc. The process stops when the degree of the harmonic polynomial in n variables $u(x) - u_{k,s}(x) - \dots - u_{k_{m-1},s_{m-1}}(x)$, by variable x_n is equal to zero, or it itself is equal to zero $H_{s_m}(x_{(n-1)}) = 0$, which means $u(x) - u_{k,s}(x) - \dots - u_{k_{m-1},s_{m-1}}(x) = u_{0,l}(x)$ and hence $u(x) - u_{k,s}(x) - \dots - u_{k_{m-1},s_{m-1}}(x) = u_{0,l}(x)$ and therefore

$$u(x) = u_{k,s}(x) + \dots + u_{k_{m-1},s_{m-1}}(x) + u_{0,l}(x).$$

The lemma is proved. □

The polynomials $G_k^s(x_{(n)})$ for $s = 0$ are harmonic, and for $s > 0$ and $k > 1$ they are non-hamonic. If $n = 2$, then the largest number of homogeneous of degree k, linearly independent harmonic polynomials, as is known, is only two. These polynomials can be chosen in the form

$$H_k^s(x_{(2)}) = \sum_{i=0}^{[(k-s)/2]} (-1)^i x_1^{2i+s,!} x_2^{k-2i-s,!}, \quad s = 0, 1. \tag{1.3.5}$$

For example, $H_2^0(x_{(2)}) = (x_2^2 - x_1^2)/2$, $H_2^1(x_{(2)}) = x_2x_1$. The index s indicates the parity of the polynomial $H_k^s(x_{(2)})$ in the variable x_1, and the index k indicates its degree. It is easy to see that $H_k^s(x_{(2)}) = G_{k-s}^s(x_{(2)})x_1^s$ for $s = 0, 1$.

Theorem 1.5. *For $n \geq 2$ and $\nu_1 \geq \cdots \geq \nu_n$, $\nu_n = 0, 1$ polynomials*

$$G_{(\nu)}(x_{(n)}) = \prod_{i=1}^{n-2} G^{\nu_{i+1}}_{\nu_i - \nu_{i+1}}(x_{(n-i+1)}) H^{\nu_n}_{\nu_{n-1}}(x_{(2)}), \tag{1.3.6}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ form a basis in homogeneous of degree ν_1 harmonic polynomials. The number of polynomials $G_{(\nu)}(x_{(n)})$ for $\nu_1 = m$ is equal to

$$h^{(n)}_m = \binom{m+n-1}{n-1} - \binom{m-2+n-1}{n-1}, \tag{1.3.7}$$

where $\binom{k}{p} = 0$ if $k < p$.

Proof. It is easy to see that the equality (1.3.6) can be rewritten in the form

$$G_{(\nu)}(x_{(n)}) = G^{\nu_2}_{\nu_1 - \nu_2}(x_{(n)}) G_{(\bar{\nu})}(x_{(n-1)}), \tag{1.3.8}$$

where $\bar{\nu} = (\nu_2, \ldots, \nu_n)$. For $n = 2$ we have $G_{(\nu)}(x_{(2)}) = H^{\nu_2}_{\nu_1}(x_{(2)})$ and the statement of the theorem immediately follows from the completeness of the system of polynomials (1.3.5). Next we use induction on n. Let, for some $n-1$, the system of polynomials (1.3.6) form a basis in homogeneous of degree ν_2 harmonic polynomials. Then, applying the result of Lemma 1.7 we conclude that any homogeneous harmonic polynomial $u_{\nu_1}(x_{(n)})$ of degree ν_1 can be written as a sum of polynomials of the form (1.3.2), which means the same is true for polynomials of the form $G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)}) G_{(\bar{\nu})}(x_{(n-1)}) = G_{(\nu)}(x_{(n)})$, for $\nu_2 \leq \nu_1$. This means that the system of polynomials (1.3.6) is a basis of homogeneous degree ν_1 harmonic polynomials. The induction step has been proven, which means the system of polynomials (1.3.6) is a basis for any $n \geq 2$.

Let us prove the equality (1.3.7). For $n = 2$ we have $h^{(2)}_0 = 1$, $h^{(2)}_1 = 2$ and for $m \geq 2$ we get $h^{(2)}_m = 2$, which is correct. Assume that (1.3.7) is true for some $n \geq 3$. Then from the equality (1.3.8) it follows that

$$h^{(n)}_m = h^{(n-1)}_m + h^{(n-1)}_{m-1} + \ldots + h^{(n-1)}_0 = \sum_{k=0}^{m} \binom{k+n-1}{n-1} - \binom{k-2+n-1}{n-1}.$$

It is easy to see that since $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$, then

$$\sum_{k=0}^{m} \binom{k+n-1}{n-1} = \binom{n}{n} + \binom{n}{n-1} + \sum_{k=2}^{m} \binom{k+n-1}{n-1} = \binom{n+1}{n}$$

$$+\binom{n+1}{n-1}+\sum_{k=3}^{m}\binom{k+n-1}{n-1}=\ldots=\binom{n+m-1}{n}+\binom{m+n-1}{n-1}$$
$$=\binom{m+n}{n}$$

Using the resulting equality in the sum for the values of $h_m^{(n)}$ we obtain the equality (1.3.7). The theorem is proved. □

Remark 1.6. Polynomials $G_{(\nu)}$ can also be defined by the following recurrence relation: $G_{(\nu)}(x) = H_{\nu_{n-1}}^{\nu_n}(x)$ for $n = 2$ and $G_{(\nu)}(x_{(n)}) = G_{\nu_1-\nu_2}^{\nu_2}(x_{(n)})G_{(\bar\nu)}(x_{(n-1)})$ for $n > 2$, where $\bar\nu = (\nu_2, \ldots, \nu_n)$.

Example 1.4. In order, for example, to construct a harmonic polynomial $P_5(x)$ of degree 5, it is necessary to construct a vector $\nu = (5, 3, 2, \ldots, 2, 0)$ that satisfies the conditions of Theorem 1.5 and in accordance with the formula (1.3.6) multiply the corresponding G-polynomials

$$\begin{aligned}P_5(x) &= G_{\nu_1-\nu_2}^{\nu_2}(x_{(n)})G_{\nu_2-\nu_3}^{\nu_3}(x_{(n-1)})\cdots G_{\nu_{n-2}-\nu_{n-1}}^{\nu_{n-1}}(x_{(3)})H_{\nu_{n-1}}^{\nu_n}(x_{(2)})\\ &= G_2^3(x_{(n)})G_1^2(x_{(n-1)})G_0^2(x_{(n-2)})\cdots G_0^2(x_{(3)})H_2^0(x_{(2)})\\ &= \left(\frac{x_n^2}{2}-\frac{x_1^2+\cdots+x_{n-1}^2}{2(n+5)}\right)x_{n-1}\frac{x_2^2-x_1^2}{2}.\end{aligned}$$

Here we used the equalities (1.3.4) and (1.3.5).

Let us prove the orthogonality of the system of harmonic polynomials (1.3.6) on the unit sphere $\partial S = \{x \in \mathbb{R}^n : |x| = 1\}$. In the future, where this does not cause ambiguity, instead of $x_{(n)}$ and $x_{(n-1)}$ we will write x and $\tilde x$, respectively, as well as $\partial\tilde S = \{x_{(n-1)} \in \mathbb{R}^{n-1} : |x_{(n-1)}| = 1\}$.

Lemma 1.8. *For function $f \in C(\partial S)$ the following equality holds*

$$\int_{\partial S} f(x)\,ds_x = \int_{-1}^{1}(1-t^2)^{(n-3)/2}\,dt\int_{\partial\tilde S} f\big(\sqrt{1-t^2}\tilde x, t\big)\,ds_{\tilde x}. \tag{1.3.9}$$

Proof. It is easy to verify that $f \in C(\partial S)$ satisfies the equality

$$\int_{\partial S} f(x)\,ds_x = \int_{|\tilde x|\le 1}\big[f(\tilde x, \sqrt{1-|\tilde x|^2}) + f(\tilde x, -\sqrt{1-|\tilde x|^2})\big](1-|\tilde x|^2)^{-1/2}\,d\tilde x,$$

whence it follows that

$$\int_{\partial S} f(x)\,ds_x = \int_0^1 \frac{\tau^{n-2}}{\sqrt{1-\tau^2}}\,d\tau \int_{\partial\tilde S}\left[f(\tau\tilde x,\sqrt{1-\tau^2})+f(\tau\tilde x,-\sqrt{1-\tau^2})\right]ds_{\tilde x}.$$

Changing variables using the formula $t=\sqrt{1-\tau^2}$, we get

$$\int_{\partial S} f(x)\,ds_x = \int_0^1 (1-t^2)^{(n-3)/2}\,dt \int_{\partial\tilde S}\left[f(\sqrt{1-t^2}\tilde x,t)+f(\sqrt{1-t^2}\tilde x,-t)\right]ds_{\tilde x}.$$

If we divide the integral obtained on the right by two and change the variables t to $-t$ in the second integral, we obtain (1.3.9). The lemma is proved. □

Lemma 1.9. *If for the function $f\in C(\partial S)$ the equality holds $f(x)=\varphi(|\tilde x|,x_n)P_k(\tilde x)$, where $P_k(\tilde x)$ is a homogeneous polynomial of degree k, and $\varphi\in C(\partial S)$, then*

$$\int_{\partial S} f(x)\,ds_x = \int_{\partial S}\varphi(|\tilde x|,x_n)\left(1-x_n^2\right)^{k/2}ds_x\,\frac{1}{\omega_{n-1}}\int_{\partial\tilde S}P_k(\tilde x)\,ds_{\tilde x}. \quad (1.3.10)$$

Proof. Let us use the statement of Lemma 1.8 for the function $f(x)$. We get

$$\int_{\partial S} f(x)\,ds_x = \int_{-1}^1 (\sqrt{1-t^2})^{n+k-3}\varphi(\sqrt{1-t^2},t)\,dt\int_{\partial\tilde S}P_k(\tilde x)\,ds_{\tilde x}.$$

Let us multiply the resulting equality by $\omega_{n-1}=\int_{|\tilde x|=1}ds_{\tilde x}$ and again use Lemma 1.8 with the function $f(x)=\varphi(|\tilde x|,x_n)(1-x_n^2)^{k/2}$. We get

$$\omega_{n-1}\int_{\partial S} f(x)\,ds_x = \int_{\partial S}\varphi(|\tilde x|,x_n)(1-x_n^2)^{k/2}\,ds_x\int_{|\tilde x|=1}P_k(\tilde x)\,ds_{\tilde x},$$

hence follows (1.3.10). □

Lemma 1.10. *G-polynomials of the same order s are orthogonal on the unit sphere with weight $\rho(x)=(1-x_n^2)^s$.*

Proof. Let us use Lemma 1.9 for $f(x)=u_{k_1}(x)u_{k_2}(x)$, where $u_{k_i}(x)$ $(i=1,2)$ are homogeneous harmonic polynomials, which, in accordance with the formula (1.3.2), can have the form $u_{k_i}(x) = u_s(\tilde{x})G^s_{k_i-s}(x)$, $(s\ lek_i)$. This is possible because the polynomials $G^s_k(x)$ can be represented in the form $G^s_k(x) = G^s_k(|\tilde{x}|, x_n)$. We have

$$\int_{\partial S} u_{k_1}(x)u_{k_2}(x)\,ds_x$$
$$= \frac{1}{\omega_{n-1}} \int_{\partial S} G^s_{k_1-s}(x)G^s_{k_2-s}(x)(1-x_n^2)^s\,ds_x \int_{\partial\tilde{S}} u_s^2(\tilde{x})\,ds_{\tilde{x}}.$$

Since the left side of the resulting equality is equal to zero for $k_1 \neq k_2$ and $u_s(\tilde{x}) \not\equiv 0$, then the lemma can be considered proved. □

Using the above lemmas, it is not difficult to establish the orthogonality of the system of polynomials $\{G_{(\nu)}\}$ on the unit sphere ∂S.

Theorem 1.6. *Polynomials $\{G_{(\nu)}\}$ for various vectors $\nu \in \mathbb{N}_0^n$ satisfying the condition $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, $\nu_n = 0, 1$, are orthogonal on the unit sphere ∂S.*

Proof. Let $n = 2$. For $x \in \{y \in \mathbb{R}^2 : |y| = 1\}$ the polynomials $H^s_m(x)$ have the form

$$H^0_m(x) = \sum_{k=0}^{[m/2]} (-1)^k x_1^{2k,!} x_2^{m-2k,!} = \frac{\operatorname{Re}(x_2+ix_1)^m}{m!} = \frac{\cos mt}{m!},$$
$$H^1_m(x) = \sum_{k=0}^{[(m-1)/2]} (-1)^k x_1^{2k+1,!} x_2^{m-2k-1,!} = \frac{\operatorname{Im}(x_2+ix_1)^m}{m!} = \frac{\sin mt}{m!},$$

where $x_2 = \cos t$, $x_1 = \sin t$, $t \in [0, 2\pi]$ and therefore they are orthogonal on the unit circle

$$\int_{|x_{(2)}|=1} H^0_m(x)H^1_m(x)\,ds_x = \frac{1}{m!^2}\int_0^{2\pi} \cos mt \sin mt\,dt = 0,$$

since $ds_x = dt$. If now $n > 2$, then using Lemma 1.9 $n-2$ times, it is easy to find

$$\int_{\partial S} G_{(\nu)}(x)G_{(\mu)}(x)ds_x = \prod_{i=1}^{n-2} \frac{1}{\omega_{n-i}} \int_{|y_{(n-i+1)}|=1} G^{\nu_{i+1}}_{\nu_i-\nu_{i+1}}(y)G^{\mu_{i+1}}_{\mu_i-\mu_{i+1}}(y)$$
$$\times \left(1-y^2_{n-i+1}\right)^{(\nu_{i+1}+\mu_{i+1})/2} ds_y \int_{|y_{(2)}|=1} H^{\nu_n}_{\nu_{n-1}}(y)H^{\mu_n}_{\mu_{n-1}}(y)ds_y. \quad (1.3.11)$$

Suppose that $\nu \neq \mu$, and therefore $\exists i$, $\nu_i \neq \mu_i$. Let j be the maximal index that satisfies this condition. If $j = n$ or $j = n-1$, then equality the right side of (1.3.11) to zero follows from the validity of the theorem for $n = 2$ and if $j < n - 1$, then from Lemma 1.10, since $\nu_{j+1} = \mu_{j+1}$ and $\nu_j \neq \mu_j$ which means

$$\int_{|y_{(n-j+1)}|=1} G^{\nu_{j+1}}_{\nu_j-\nu_{j+1}}(y) G^{\mu_{j+1}}_{\mu_j-\mu_{j+1}}(y) \left(1 - y^2_{n-j+1}\right)^{\nu_{j+1}} ds_y = 0.$$

The theorem is proved. □

Consider the polynomials $G_{(\nu)}(x)$ in the space of polynomials

$$\mathcal{P} = \left\{P(x) = \sum_{\alpha} p_\alpha x^\alpha : p_\alpha \in \mathbb{R},\ x \in \mathbb{R}^n\right\}$$

with scalar product [89]

$$\langle P(x), Q(x)\rangle = P(D)Q(x)|_{x=0}.$$

The norm of the polynomial $P(x) \in \mathcal{P}$ is denoted by $|P|_D$. For example,

$$|x_1^2 + \cdots + x_n^2|_D = \Delta|x|^2|_{x=0} = 2n.$$

The system of polynomials $G_{(\nu)}(x)$ has another interesting property.

Theorem 1.7. *Polynomials $G_{(\nu)}(x)$ for various vectors $\nu \in \mathbb{N}_0^n$ satisfying the condition $\nu_1 \geq \cdots \geq \nu_n$ ($\nu_n = 0, 1$) are orthogonal in $\mathcal{P}$.*

Proof. We need to check the validity of the following statement: $\nu \neq \mu \Rightarrow \langle G_{(\nu)}, G_{(\mu)}\rangle = 0$. Let us apply induction on the dimension n. Consider the case $\nu_1 = \mu_1$ because if $\nu_1 \neq \mu_1$ then the harmonic polynomials $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal, as having different degrees. This follows from the divergence theorem [28]

$$\begin{aligned} 0 = \int_{\partial S} \left(H_p(x)\frac{\partial}{\partial \nu}H_m(x) - H_m(x)\frac{\partial}{\partial \nu}H_p(x)\right) ds_x \\ = (m-p)\int_{\partial S} H_p(x)H_m(x)\, ds_x, \end{aligned}$$

where $H_p(x)$ and $H_m(x)$ are homogeneous harmonic polynomials of degrees $p \neq m$, respectively.

Let $n = 2$. The polynomials $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal because they have different parities in x_1: if $\nu_2 = 0$ then $G_{(\nu)}(x)$ is even, and if $\nu_2 = 1$ then it is odd.

Let $n > 2$. Consider the case $\nu_2 < \mu_2$. We define G-polynomials for $k \in \mathbb{Z} \setminus \mathbb{N}_0$ as $G_k^s \equiv 0$ and denote $\tilde{\Delta} = \Delta - \partial^2/\partial x_n^2$. It is easy to check that

$$\frac{\partial}{\partial x_n} G_{(\mu)}(x) = G_{(\mu_-)}(x), \quad \tilde{\Delta} G_{(\mu)}(x) = -G_{(\mu_=)}(x), \tag{1.3.12}$$

where $\mu_- = (\mu_1 - 1, \mu_2, \dots, \mu_n)$ and $\mu_= = (\mu_-)_-$. Therefore, using Remark 1.6 to Theorem 1.5, we obtain

$$G_{(\nu)}(D) G_{(\mu)}(x) = G_{(\bar{\nu})}(\tilde{D}) \big(G^{\nu_2}_{\nu_1 - \nu_2}(D) G^{\mu_2}_{\mu_1 - \mu_2}(x) G_{(\bar{\mu})}(\tilde{x}) \big). \tag{1.3.13}$$

Since according to the equality (1.3.1) for $k \ge m$ and the homogeneous harmonic polynomial $H_s(x)$ we have

$$\Delta^m \frac{|x|^{2k} H_s(x)}{(2,2)_k (n+2s,2)_k} = \frac{|x|^{2k-2m} H_s(x)}{(2,2)_{k-m}(n+2s,2)_{k-m}},$$

then we get

$$\begin{aligned}
&G^{\nu_2}_{k_1}(D) G^{\mu_2}_{k_2}(x) H_{\mu_2}(\tilde{x}) \\
&\qquad = \sum_{i=0}^{[k_1/2]} \frac{(-1)^i \tilde{\Delta}^i D_n^{k_1-2i}}{(2,2)_i (n-1+2\nu_2,2)_i (k_1-2i)!} \\
&\qquad\quad \times \sum_{j=0}^{[k_2/2]} \frac{(-1)^j |\tilde{x}|^{2j} x_n^{k_2-2j,!}}{(2,2)_j (n-1+2\mu_2,2)_j} H_{\mu_2}(\tilde{x}) \\
&\qquad = \sum_{i=0}^{[k_1/2]} \frac{1}{(2,2)_i (n-1+2\nu_2,2)_i (k_1-2i)!} \\
&\qquad\quad \times \sum_{j=i}^{[(k_2-k_1)/2]+i} \frac{(-1)^{j-i} |\tilde{x}|^{2j-2i} x_n^{k_2-k_1+2i-2j,!}}{(2,2)_{j-i} (n-1+2\mu_2,2)_{j-i}} H_{\mu_2}(\tilde{x}) \\
&\qquad = C(k_1,\nu_2) \sum_{j=0}^{[(k_2-k_1)/2]} \frac{(-1)^j |\tilde{x}|^{2j} x_n^{k_2-k_1-2j,!}}{(2,2)_j (n-1+2\mu_2,2)_j} H_{\mu_2}(\tilde{x}) \\
&\qquad = C(k_1,\nu_2) G^{\mu_2}_{k_2-k_1}(x) H_{\mu_2}(\tilde{x}),
\end{aligned}$$

where $C(k_1,\nu_2) = \sum_{i=0}^{[k_1/2]} 1/((2,2)_i (n-1+2\nu_2,2)_i (k_1-2i)!)$. So from (1.3.13) with $k_1 = \nu_1 - \nu_2$ and $k_1 = \mu_1 - \mu_2$ we derive

$$
\begin{aligned}
&G_{(\nu)}(D)G_{(\mu)}(x)\\
&\qquad\qquad = C(\nu_1-\nu_2,\nu_2)G_{(\bar\nu)}(\tilde D)\big[G^{\mu_2}_{(\mu_1-\mu_2)-(\nu_1-\nu_2)}(x)G_{(\bar\mu)}(\tilde x)\big]. \quad (1.3.14)
\end{aligned}
$$

Since $\mu_1-\mu_2-\nu_1+\nu_2=\nu_2-\mu_2<0$, we can conclude that $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal. Due to the symmetry of the scalar product, this statement is also true for $\nu_2>\mu_2$. Let us consider the last case $\nu_2=\mu_2$. Using the equality (1.3.14) and taking into account that $G^s_0=1$ we obtain $\langle G_{(\nu)},G_{(\mu)}\rangle = C(\nu_1-\nu_2,\nu_2)\langle G_{(\bar\nu)},G_{(\bar\mu)}\rangle$. Since $\nu_1=\mu_1$ and $\nu\neq\mu$ we have $\bar\nu\neq\bar\mu$. Therefore, by induction, the polynomials $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal in $\mathcal{P}$. The theorem is proved. □

It is known [165] that the maximum number of linearly independent homogeneous degree k harmonic polynomials is equal to

$$
\begin{aligned}
h_k &= \binom{k+n-1}{n-1}-\binom{k-2+n-1}{n-1}=\frac{(k+n-3)!}{(n-1)!k!}\\
&\times((k+n-1)(k-1+n-1)-k(k-1))=\frac{(k+n-3)!}{(n-1)!k!}\\
&\times(k(n-1)+k(k-1)+(n-1)(k-1)+(n-1)^2-k(k-1))\\
&=(2k+n-2)\frac{(k+n-3)!}{(n-2)!k!}=\frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}. \quad (1.3.15)
\end{aligned}
$$

Definition 1.5. We renumber the polynomials $G_{(\nu)}(x)$ for $\nu_1=k$ so that we obtain a complete system of orthogonal harmonic polynomials of degree k. We denote this system as $\{H^{(i)}_k(x)\,:\,i=1,\dots,h_k\}$ and normalize these polynomials so that $\int_{\partial S}(H^{(i)}_k(x))^2\,ds_x=\omega_n$. Note that the dimension h_k coincides with the number of distinct polynomials $G_{(\nu)}(x)$ for $\nu_1=k$ found in (1.3.7).

Theorem 1.8. *Let $u(x)$ be a function harmonic in S and continuous in $\bar S$, then the following representation holds*

$$
u(x)=\sum_{k=0}^{\infty}\sum_{i=1}^{h_k}h^{(i)}_kH^{(i)}_k(x),\quad |x|\le a<1, \qquad (1.3.16)
$$

where

$$
h^{(i)}_k=\frac{1}{\omega_n}\int_{\partial S}H^{(i)}_k(\xi)u(\xi)\,ds_\xi,
$$

and the functional series converges uniformly in a certain neighborhood of the origin.

Proof. Let $n > 2$ and $E(x,\xi) = (n-2)^{-1}|\xi - x|^{2-n}$ be an elementary solution of the Laplace equation [28]. It is easy to see that for the operator $\Lambda = x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$ and for $x \neq \xi \in \partial S$ the equality is true

$$\Lambda_x E(x,\xi) - \Lambda_\xi E(x,\xi) = -\sum_{i=1}^{n} \frac{x_i(x_i - \xi_i) - \xi_i(\xi_i - x_i)}{|x-\xi|^n} = \frac{1-|x|^2}{|x-\xi|^n}.$$

Let us use Lemma 2.7 on page 88 from which it follows that the representation is true

$$E(x,\xi) = \sum_{k=0}^{\infty} \frac{|\xi|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi),$$

moreover, the functional series converges uniformly in x for $|x| < a < |\xi|$ and fixed ξ (see Lemma 3.34 on page 299). Therefore, for $|x| < |\xi| = 1$ the Poisson kernel has the form

$$\frac{1-|x|^2}{|x-\xi|^n} = \Lambda_x E(x,\xi) - \Lambda_\xi E(x,\xi) = \sum_{k=0}^{\infty} \frac{k - k + (2k+n-2)}{2k+n-2} \\ \times \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi) = \sum_{k=0}^{\infty}\sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi). \quad (1.3.17)$$

Differentiation and passage to the limit under the summation sign are legal, since the series in (1.3.17) converges uniformly in $\xi \in \partial S$ and in x for $|x| \le a < 1$. Substituting the obtained value of the Poisson kernel into the well-known formula for solving the Dirichlet problem in the unit ball and using the uniform convergence of the series in $\xi \in \partial S$ we arrive at (1.3.16)

$$u(x) = \frac{1}{\omega_n}\int_{\partial S} \frac{1-|x|^2}{|x-\xi|^n} u(\xi)\, ds_\xi = \frac{1}{\omega_n}\int_{\partial S} \sum_{k=0}^{\infty}\sum_{i=1}^{h_k} H_k^{(i)}(\xi) u(\xi)\, ds_\xi H_k^{(i)}(x) \\ = \sum_{k=0}^{\infty}\sum_{i=1}^{h_k} \frac{1}{\omega_n}\int_{\partial S} H_k^{(i)}(\xi) u(\xi)\, ds_\xi H_k^{(i)}(x) = \sum_{k=0}^{\infty}\sum_{i=1}^{h_k} h_k^{(i)} H_k^{(i)}(x).$$

If $n = 2$, then the proof is similar, but one need to use Lemma 2.7 from Section 2.2.1. The theorem is proved. □

Theorem 1.9. *Let $u(x)$ be a function harmonic in S and continuous in $\bar{S}$, then the equality holds*

$$\int_{\partial S} G_{(\nu)}(\xi) u(\xi)\, ds_\xi = g_\nu G_{(\nu)}(D) u(x)|_{x=0}, \quad (1.3.18)$$

where $g_\nu = |G_{(\nu)}|^2_{L_2}/|G_{(\nu)}|^2_D$.

Proof. Let us use the equality (1.3.16) from Theorem 1.8. Since for each i and k there is a vector ν such that $\nu_1 = k \geq \cdots \geq \nu_n$ ($\nu_n = 0, 1$) and

$$H_k^{(i)}(x) = \sqrt{\omega_n}\frac{G_{(\nu)}(x)}{|G_{(\nu)}|_{L_2(\partial S)}},$$

then we have

$$u(x) = \sum_{\nu_1=0}^{\infty} \frac{G_{(\nu)}(x)}{|G_{(\nu)}|^2_{L_2(\partial S)}} \int_{\partial S} G_{(\nu)}(\xi)u(\xi)\, ds_\xi.$$

Applying the operator $G_{(\mu)}(D)$ to this equality (the differentiation can be used under the integral since the series converges uniformly in $|x| \leq \alpha < 1$) and then setting $x = 0$ we obtain

$$G_{(\mu)}(D)u(x)|_{x=0}$$
$$= \sum_{\nu_1=\mu_1} \frac{\langle G_{(\mu)}, G_{(\nu)}\rangle}{|G_{(\nu)}|^2_{L_2(\partial S)}} \int_{\partial S} G_{(\nu)}(\xi)u(\xi)\, ds_\xi = \frac{1}{g_\nu}\int_{\partial S} G_{(\nu)}(\xi)u(\xi)\, ds_\xi.$$

Here we used Theorem 1.7 on the orthogonality of polynomials $G_{(\mu)}(x)$ in $\mathcal{P}$. From here we immediately obtain the equality (1.3.18). The theorem is proved. □

To use Theorem 1.9 one need an equality for calculating integrals of the form $\int_{|x|=1} Q_m(x)\, ds_x$, where $Q_m(x)$ is a homogeneous polynomial of degree m.

Theorem 1.10. *The equality holds true*

$$\int_{\partial S} Q_m(x)\, ds_x = \begin{cases} 0, & m \in 2\mathbb{N} - 1 \\ \dfrac{\Delta^{m/2} Q_m(x)}{m!!\, n \cdots (n+m-2)}\, \omega_n, & m \in 2\mathbb{N} \end{cases}. \tag{1.3.19}$$

Proof. Let us use the following statement.

Lemma 1.11. *[73, Lemma 4] Harmonic polynomials $R_{m-2k}(x)$ in the expansion of the homogeneous polynomial $Q_m(x)$ according to the Almansi formula*

$$Q_m(x) = R_m(x) + |x|^2 R_{m-2}(x) + \cdots + |x|^{2l} R_{m-2l}(x)$$

look like

$$R_{m-2k}(x) = \frac{2m - 4k + n - 2}{(2,2)_k} \sum_{s=0}^{\infty} \frac{(-1)^s |x|^{2s} \Delta^{s+k} Q_m(x)}{(2,2)_s (2m - 4k - 2s + n - 2, 2)_{s+k+1}}.$$

From the mean value theorem for harmonic functions and Almansi formula it follows that

$$\int_{\partial S} Q_m(x)\, ds_x$$
$$= \int_{\partial S} \big(R_m(x) + R_{m-2}(x) + \cdots + R_{m-2l}(x)\big)\, ds_x = \begin{cases} 0, & 2l < m \\ \omega_n R_0, & 2l = m \end{cases}.$$

For odd m, the integral is equal to zero since $m-2l > 0$ and therefore $R_{m-2l}(0) = 0$. For even m, using Lemma 1.11 we write ($s = 0$, $k = m/2$)

$$\int_{\partial S} Q_m(x)\, ds_x = \frac{n-2}{(2,2)_{m/2}} \frac{\Delta^{m/2} Q_m(x)}{(n-2,2)_{m/2+1}}\, \omega_n = \frac{\Delta^{m/2} Q_m(x)}{m!!\, n \cdots (n+m-2)}\, \omega_n.$$

The theorem is proved. □

Remark 1.7. For the m-harmonic function the values of $\Delta^k u(0)$ are found in Theorem 3.4 on page 174.

Corollary 1.2. If $u(x)$ is a harmonic in S function and continuous in $\bar{S}$, then the equality

$$\frac{1}{\omega_n} \int_{\partial S} \bar{G}_{(\nu)}(\xi) u(\xi)\, ds_\xi = \frac{\Delta^{\nu_1} \bar{G}^2_{(\nu)}(x)}{(2\nu_1)!!(n,2)_{\nu_1}}\, \bar{G}_{(\nu)}(D) u(x)\big|_{x=0}, \qquad (1.3.20)$$

where $\bar{G}_{(\nu)}(x) = G_{(\nu)}(x)/|G_{(\nu)}|_D$ holds true.

Proof. It is easy to see that in the notation of Theorem 1.9 and using Theorem 1.10 ($m = 2\nu_1$) we can write

$$g_\nu = \frac{|G_{(\nu)}|^2_{L_2}}{|G_{(\nu)}|^2_D} = |\bar{G}_{(\nu)}|^2_{L_2} = \frac{\Delta^{\nu_1} \bar{G}^2_{(\nu)}(x)}{(2\nu_1)!!(n,2)_{\nu_1}}\, \omega_n. \qquad (1.3.21)$$

Substituting the found value of the constant g_ν into (1.3.18) and dividing both sides of the resulting equality by $|G_{(\nu)}|_D$ and ω_n we obtain (1.3.20). □

Example 1.5. Let $u(x)$ be a function harmonic in S and continuous in $\bar{S}$, then the equalities hold

$$\frac{1}{\omega_n} \int_{\partial S} \xi_i u(\xi)\, ds_\xi = \frac{1}{n} u_{x_i}(0), \quad \frac{1}{\omega_n} \int_{\partial S} \xi_i \xi_j u(\xi)\, ds_\xi = \frac{1}{n(n+2)} u_{x_i x_j}(0),$$

$$\frac{1}{\omega_n}\int_{\partial S}\xi_i^2 u(\xi)\,ds_\xi = \frac{1}{n(n+2)}u_{x_ix_i}(0) + \frac{1}{n}u(0).$$

I. The first formula is easy to obtain from (1.3.18). If we take $\nu = e_1 + \cdots + e_{n-i+1}$, where $e_i = (\delta_{1,i}, \ldots, \delta_{n,i})$, then similarly to Example 1.4 and taking into account Example 1.3 we get

$$\begin{aligned} G_{(\nu)}(x) &= G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)})G^{\nu_3}_{\nu_2-\nu_3}(x_{(n-1)})\cdots G^{\nu_{n-1}}_{\nu_{n-2}-\nu_{n-1}}(x_{(3)})H^{\nu_n}_{\nu_{n-1}}(x_{(2)}) \\ &= G^1_0(x_{(n)})G^1_0(x_{(n-1)})\cdots G^0_1(x_{(i)})\cdots G^0_0(x_{(3)})H^0_0(x_{(2)}) = x_i. \end{aligned}$$

Since

$$|x_i|^2_{L_2} = \int_{\partial S} x_i^2\,ds_x = \frac{1}{n}\int_{\partial S}\left(x_1^2 + \cdots + x_n^2\right)ds_x = \frac{\omega_n}{n}$$

and $|x_i|^2_D = D_{x_i}x_i = 1$, then $g_\nu = |x_i|^2_{L_2}/|x_i|^2_D = \omega_n/n$ and that means the first formula is correct

$$\int_{\partial S}\xi_i u(\xi)\,ds_\xi = \frac{\omega_n}{n}D_{x_i}u(x)|_{x=0}.$$

The last equality implies the well-known property of $u(x)$

$$\int_{\partial S}\sum_{i=1}^n \xi_i u_{\xi_i}(\xi)\,ds_\xi = \frac{\omega_n}{n}\sum_{i=1}^n D^2_{x_i}u(x)|_{x=0} = 0.$$

II. Next we set $\nu = 2e_1 + \cdots + 2e_{n-j+1} + e_{n-j+2} + \cdots + e_{n-i+1}$ if $i > j$. Then the equalities hold

$$\begin{aligned} G_{(\nu)}(x) &= G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)})G^{\nu_3}_{\nu_2-\nu_3}(x_{(n-1)})\cdots G^{\nu_{n-1}}_{\nu_{n-2}-\nu_{n-1}}(x_{(3)})H^{\nu_n}_{\nu_{n-1}}(x_{(2)}) \\ &= G^2_0(x_{(n)})G^2_0(x_{(n-1)})\cdots G^1_1(x_{(i)})G^1_0(x_{(i-1)})\cdots \\ &\qquad \cdots G^1_0(x_{(j+1)})G^0_1(x_{(j)})\cdots G^0_0(x_{(3)})H^0_0(x_{(2)}) = x_ix_j. \end{aligned}$$

To calculate the integral

$$|x_ix_j|^2_{L_2} = \int_{\partial S} x_i^2x_j^2\,ds_x$$

we use the equality (1.3.19) with $Q_m(x) = x_i^2x_j^2$, $m = 4$. We get

$$\frac{\Delta^{m/2}Q_m(x)}{m!!\,n\cdots(n+m-2)} = \frac{1}{4!!(n,2)_2}\Delta^2\left(x_i^2x_j^2\right)$$

$$= \frac{1}{8n(n+2)}\Delta\big(2x_i^2 + 2x_j^2\big) = \frac{1}{n(n+2)}.$$

That is why

$$\int_{\partial S} x_i^2 x_j^2\, ds_x = \frac{\omega_n}{n(n+2)}.$$

Since $|x_i x_j|_D^2 = D_{x_i} D_{x_j} x_i x_j = 1$, then

$$g_\nu = \frac{|x_i|_{L_2}^2}{|x_i|_D^2} = \frac{\omega_n}{n(n+2)}$$

and that means the second formula is also correct

$$\int_{\partial S} \xi_i \xi_j u(\xi)\, ds_\xi = \frac{\omega_n}{n(n+2)} D_{x_i} D_{x_j} u(x)|_{x=0}.$$

III. To prove the third equality, we choose $\nu = 2e_1$ and therefore

$$G_{(\nu)}(x) = G_2^0(x_{(n)})G_0^0(x_{(n-1)})\cdots H_0^0(x_{(2)}) = \frac{x_n^2}{2} - \frac{x_1^2 + \cdots + x_{n-1}^2}{2(n-1)}.$$

It follows that

$$G_{(\nu)}(x) = \frac{x_n^2}{2} - \frac{x_1^2 + \cdots + x_{n-1}^2}{2(n-1)} = \frac{n x_n^2}{2(n-1)} - \frac{|x|^2}{2(n-1)},$$

or

$$x_n^2 = 2\frac{n-1}{n} G_{(\nu)}(x) + \frac{1}{n}|x|^2,$$

and therefore, using the formula (1.3.18) and the mean value theorem, we write

$$\begin{aligned}\int_{\partial S} \xi_n^2 u(\xi)\, ds_\xi &= 2\frac{n-1}{n}\int_{\partial S} G_{(\nu)}(\xi) u(\xi)\, ds_\xi + \frac{1}{n}\int_{\partial S} u(\xi)\, ds_\xi \\ &= 2\frac{n-1}{n} g_\nu G_{(\nu)}(D) u(x)|_{x=0} + \frac{\omega_n}{n} u(0) = g_\nu\Big(2\frac{n-1}{n} G_{(\nu)}(D) \\ &\quad + \frac{1}{n}\Delta\Big) u(x)|_{x=0} + \frac{\omega_n}{n} u(0) = g_\nu D_{x_n}^2 u(x)|_{x=0} + \frac{\omega_n}{n} u(0).\end{aligned}$$

Let us calculate g_ν. It is easy to calculate that

$$|G_{(\nu)}|_D^2 = G_{(\nu)}(D) G_{(\nu)}(x) = \frac{n}{2n-2},$$

and therefore, using the equality (1.3.21) ($\nu_1 = 2$) we write

$$
\begin{aligned}
\frac{g_\nu}{\omega_n} = \frac{2n-2}{8n(n,2)_2}\Delta^2 G^2_{(\nu)}(x) = \frac{\Delta^2\big(nx_n^2 - |x|^2\big)^2}{16n^2(n-1)(n+2)} \\
= \frac{\Delta^2\big(n^2x_n^4 - 2nx_n^2|x|^2 + |x|^4\big)}{16n^2(n-1)(n+2)} = \frac{24n^2 - 16n(n+2) + 8n(n+2)}{16n^2(n-1)(n+2)} \\
= \frac{16n(n-1)}{16n^2(n-1)(n+2)} = \frac{1}{n(n+2)}.
\end{aligned}
$$

Substituting the calculated value g_ν into the previous formula and dividing by ω_n we obtain the third formula for $i = n$. Due to symmetry, this formula is true for any $i = 1, \ldots, n$.

1.3.2 Connection of G-functions with Legendre and Chebyshev polynomials

Consider the trace of a G-polynomial on the unit sphere. In this case, the concept of a G-function arises. Let $[m]$ denote the integer part of m.

Definition 1.6. Function

$$
G_k^{s,n}(t) = \sum_{i=0}^{[(k-s)/2]} (-1)^i \frac{t^{k-s-2i,!}(1-t^2)^{i+s/2}}{(2,2)_i(n-1+2s,2)_i}
$$

is called a G-function of degree k, order s, and genus n. Here $s, k \in \mathbb{N}_0$, $s \le k$, $n \in \mathbb{N}$.

It is easy to see that if we denote $t_j = x_j/|x_{(j)}|$, then $|x_{(j-1)}|^2/|x_{(j)}|^2 = 1 - t_j^2$ and we have

$$
|x_{(j-1)}|^s G^s_{k-s}(x_{(j)}) = \sum_{i=0}^{[(k-s)/2]} (-1)^i \frac{x_j^{k-s-2i,!}|x_{(j-1)}|^{2i+s}}{(2,2)_i(j-1+2s,2)_i} = |x_{(j)}|^k G_k^{s,n}(t_j).
$$

So we can write

$$
\begin{aligned}
G_{(\nu)}(x_{(n)}) = x_1^{\nu_n} G^{\nu_n}_{\nu_{n-1}-\nu_n}(x_{(2)}) G^{\nu_{n-1}}_{\nu_{n-2}-\nu_{n-1}}(x_{(3)}) G^{\nu_{n-2}}_{\nu_{n-3}-\nu_{n-2}}(x_{(4)}) \cdots \\
\cdots G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)}) = G^{\nu_n,2}_{\nu_{n-1}-\nu_n}(t_2)|x_{(2)}|^{\nu_{n-1}} G^{\nu_{n-1}}_{\nu_{n-2}-\nu_{n-1}}(x_{(3)}) G^{\nu_{n-2}}_{\nu_{n-3}-\nu_{n-2}}(x_{(4)}) \cdots \\
\cdots G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)}) = G^{\nu_n,2}_{\nu_{n-1}-\nu_n}(t_2) G^{\nu_{n-1}}_{\nu_{n-2}-\nu_{n-1}}(t_3)|x_{(3)}|^{\nu_{n-2}} G^{\nu_{n-2}}_{\nu_{n-3}-\nu_{n-2}}(x_{(4)}) \cdots \\
\cdots G^{\nu_2}_{\nu_1-\nu_2}(x_{(n)}) = G^{\nu_n,2}_{\nu_{n-1}-\nu_n}(t_2) G^{\nu_{n-1},3}_{\nu_{n-2}-\nu_{n-1}}(t_3) G^{\nu_{n-2},4}_{\nu_{n-3}-\nu_{n-2}}(t_4) \cdots
\end{aligned}
$$

$$\cdots |x_{(n-1)}|^{\nu_2} G^{\nu_2,n}_{\nu_1-\nu_2}(x_{(n)}) = |x|^{\nu_1} \prod_{i=1}^{n-1} G^{\nu_{i+1},n-i+1}_{\nu_i}(t_{n-i+1}),$$

or for $\varphi_i = \arccos(x_i/|x_{(i)}|)$, $\nu_1 \geq \cdots \geq \nu_n$ and $\nu_n = 0, 1$ we have

$$G_{(\nu)}(x) = |x|^{\nu_1} \prod_{i=1}^{n-1} G^{\nu_{i+1},n-i+1}_{\nu_i}(\cos\varphi_{n-i+1}),$$

where $\varphi_i \in [0, \pi]$. Let us explore the connection between G-functions $G^{s,n}_k(t)$ and known special functions.

Theorem 1.11. *For a G-function of odd genus n the following equality is true*

$$G^{s,2m+3}_k(t) = \frac{2(s+m)!!}{(k+s+2m)!}(1-t^2)^{-m/2} P^{s+m}_{k+m}(t),$$

where $m, s, k \in \mathbb{N}_0$, $k \geq s$ and $P^s_k(t)$ are the associated Legendre function.

Proof. Let us prove that

$$D^m_t = m! \sum_{k=0}^{[m/2]} (\sqrt{y})^{m-2k,!} \frac{(2D_y)^{m-k}}{(2,2)_k}\Big|_{y=t^2}. \tag{1.3.22}$$

It is easy to see that for $m \in \mathbb{N}_0$ and some a^m_k the equality holds true

$$(\sqrt{y}D_y)^m = \sqrt{y}D_y(\sqrt{y}D_y)\cdots(\sqrt{y}D_y) = \sum_{k=0}^{[m/2]} a^m_k (\sqrt{y})^{m-2k,!} D^{m-k}_y. \tag{1.3.23}$$

From the resulting equality for $m = 0$ it follows that $a^0_0 = 1$. For $m = 1$ we have $k = 0$ and therefore we get $\sqrt{y}D_y = a^1_0\sqrt{y}D_y$, where from $a^1_0 = 1$. In addition, the degree of the operator D_y on the right cannot be greater than m, which means $a^m_{-1} = 0$. Let us find a^m_k. It is easy to calculate that the equalities are true

$$\begin{aligned}
(\sqrt{y}D_y)^{m+1} &= \sqrt{y}D_y \sum_{0\leq 2k\leq m} a^m_k (\sqrt{y})^{m-2k,!} D^{m-k}_y \\
&= \sum_{0\leq 2k\leq m-1} \left(\frac{1}{2}a^m_k(\sqrt{y})^{m-2k-1,!}D^{m-k}_y + a^m_k(m-2k+1)(\sqrt{y})^{m-2k+1,!}D^{m-k+1}_y\right) \\
&= \sum_{2\leq 2k\leq m+1} \frac{1}{2}a^m_{k-1}(\sqrt{y})^{m-2k+1,!}D^{m-k+1}_y
\end{aligned}$$

$$+\sum_{0\le 2k\le m-1} a_k^m(m-2k+1)(\sqrt{y})^{m-2k+1,!}D_y^{m-k+1}$$
$$=\sum_{0\le 2k\le m+1}\Big(\frac{1}{2}a_{k-1}^m+(m-2k+1)a_k^m\Big)(\sqrt{y})^{m-2k+1,!}D_y^{m-k+1}$$

and therefore the recurrent equation is satisfied

$$a_k^{m+1}=\frac{1}{2}a_{k-1}^m+(m-2k+1)a_k^m \qquad (1.3.24)$$

in the domain $(k,m)\in Q=\{k,m\in\mathbb{N}_0 : 2k\le m\}$.

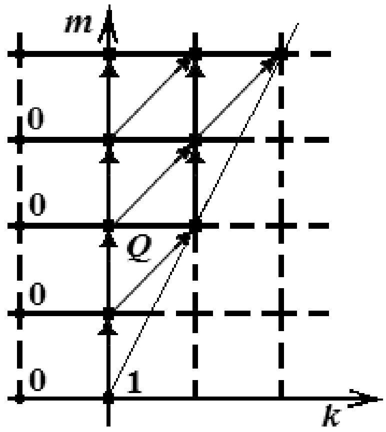

Figure 1.1: Dependencies in (1.3.24)

As boundary conditions we take $a_{-1}^m=0$, $m\in\mathbb{N}_0$ and $a_0^0=1$. It is interesting to note that there are no boundary conditions on the right boundary of the domain Q, and yet this equation has a unique solution! Indeed, if $(k,m+1)\in Q$ and $m+1>2k$, then the value of a_k^{m+1} is uniquely determined from the equation through a_k^m, a_{k-1}^m and $(k,m),(k-1,m)\in Q$. If $(k,m+1)\in Q$ and $m+1=2k$, then the value of a_k^{m+1} is also uniquely expressed, but only through a_{k-1}^m, since the coefficient of a_k^m (it is clear that $(k,m)\notin Q$) is equal to zero and $(k-1,m)\in Q$ (see Fig.1.1). The solution to the recurrent equation (1.3.24) has the form $a_k^m=m!/(4^kk!)$. Let us check this solution

$$\frac{1}{2}a_{k-1}^m+(m-2k+1)a_k^m=\frac{m!}{4^kk!}(2k+m-2k+1)=\frac{(m+1)!}{4^kk!}=a_k^{m+1}.$$

Substituting the found value of a_k^m into (1.3.23) and noting that $\sqrt{y}D_y=2^{-1}D_t$ for $y=t^2$ we get

$$2^{-m}D_t^m=(\sqrt{y}D_y)^m|_{y=t^2}=2^{-m}m!\sum_{k=0}^{[m/2]}(\sqrt{y})^{m-2k,!}\frac{(2D_y)^{m-k}}{(2,2)_k}\Big|_{y=t^2}.$$

Let us apply the operator equality (1.3.22) for $m = m + s$ to the function $(1 - t^2)^m$. Then we get

$$
\begin{aligned}
&D_t^{m+s}(1-t^2)^m \\
&\quad = (-1)^m (m+s)! \sum_{k=0}^{[(m+s)/2]} (\sqrt{y})^{m+s-2k,!} \frac{(2D_y)^{m+s-k}}{(2,2)_k} (y-1)^m{}_{|y=t^2} \\
&\quad = (-1)^m (m+s)! \sum_{k=s}^{[(m+s)/2]} t^{m+s-2k,!} \frac{2m \cdots (2k-2s+2)}{(2,2)_k} (t^2-1)^{k-s} \\
&\quad = (-1)^m (m+s)! \sum_{k=0}^{[(m-s)/2]} t^{m-s-2k,!} \frac{2m \cdots (2k+2)}{(2,2)_{k+s}} (t^2-1)^k \\
&\quad = \frac{(-1)^m (m+s)!(2,2)_m}{(2,2)_s} \sum_{k=0}^{[(m-s)/2]} (-1)^k \frac{t^{m-s-2k,!}(1-t^2)^k}{(2,2)_k (2+2s,2)_k}. \qquad (1.3.25)
\end{aligned}
$$

Hence, using the known equalities [15]

$$
P_k^s(t) = (1-t^2)^{s/2} D_t^s P_k(t), \quad P_k(t) = 2^{-k} D_t^k (t^2-1)^{k,!}
$$

we find

$$
\begin{aligned}
G_k^{s,3}(t) \equiv &\sum_{i=0}^{[(k-s)/2]} (-1)^i t^{k-s-2i,!} \frac{(1-t^2)^{i+s/2}}{(2,2)_i (2+2s,2)_i} \\
&= \frac{(2,2)_s (1-t^2)^{s/2}}{(k+s)!(2,2)_k} D_t^{k+s} (t^2-1)^k \\
&= \frac{2s!!}{(k+s)!} (1-t^2)^{s/2} D_t^s \Big(2^{-k} D_t^k (t^2-1)^{k,!} \Big) = \frac{2s!!}{(k+s)!} P_k^s(t).
\end{aligned}
$$

Finally, it is easy to see that the equalities are true

$$
G_k^{s,2m+3}(t) = (1-t^2)^{-m/2} G_{k+m}^{s+m,3}(t) = \frac{2(s+m)!!}{(2m+k+s)!} (1-t^2)^{-m/2} P_{k+m}^{s+m}(t),
$$

which complete the proof of the theorem. □

By analogy with associated Legendre functions, we introduce associated Chebyshev functions.

Definition 1.7. The following functions

$$T_k^s(t) = (1-t^2)^{s/2} D_t^s T_k(t), \quad t \in [-1, 1],$$

where $s, k \in \mathbb{N}$, $s \le k$, and $T_k^0(t) = T_k(t)$ are Chebyshev polynomials we call associated Chebyshev functions.

Theorem 1.12. *For G-functions of even genus the following equality is true*

$$G_k^{s,2m+2}(t) = \frac{(2s+2m-1)!!}{(k+s+2m-1)!(k+m)} (1-t^2)^{-m/2} T_{k+m}^{s+m}(t),$$

where $m \in \mathbb{N}_0$, $s, k \in \mathbb{N}$, but $k \ge s$ and $G_k^{0,2}(t) = \frac{1}{k!} T_k^0(t)$.

Proof. Let us prove the auxiliary equality

$$\begin{aligned}
&(\sqrt{y} D_y)^s \sum_{i=0}^{[k/2]} (-1)^i (\sqrt{y})^{k-2i,!} \frac{(1-y)^i}{(2,2)_i (1,2)_i} \\
&\quad = \frac{2^{-s}(k+s-1)!}{(k-1)!(2s-1)!!} \sum_{i=0}^{[(k-s)/2]} (-1)^i (\sqrt{y})^{k-s-2i,!} \frac{(1-y)^i}{(2,2)_i (1+2s,2)_i}
\end{aligned} \tag{1.3.26}$$

for $s \in \mathbb{N}$ and $y > 0$. Let $s = 1$. Then we have

$$\begin{aligned}
\sqrt{y} D_y \sum_{0 \le 2i \le k} (\sqrt{y})^{k-2i,!} \frac{(y-1)^i}{(2,2)_i (1,2)_i} &= \sum_{0 \le 2i \le k-1} (\sqrt{y})^{k-2i-1,!} \frac{(y-1)^i}{2(2,2)_i (1,2)_i} \\
+ \sum_{2 \le 2i \le k} (\sqrt{y})^{k-1-2i+2,!} \frac{(k-2i+1)(y-1)^{i-1}}{2(2,2)_{i-1}(1,2)_i} &= \sum_{0 \le 2i \le k-1} (\sqrt{y})^{k-2i-1,!} \\
\times \frac{(y-1)^i}{2(2,2)_i (1,2)_i} + \sum_{0 \le 2i \le k-2} (\sqrt{y})^{k-2i-1,!} &\frac{(k-2i-1)(y-1)^i}{2(2,2)_i (1,2)_{i+1}} \\
= \sum_{0 \le 2i \le k-1} \frac{(\sqrt{y})^{k-2i-1,!}(y-1)^i}{2(2,2)_i (1,2)_i} &\left(1 + \frac{k-2i-1}{2i+1}\right) \\
= \frac{k}{2} \sum_{i=0}^{(k-1)/2]} \frac{(\sqrt{y})^{k-2i-1,!}(y-1)^i}{(2,2)_i (3,2)_i} &= \frac{2^{-1}(k+1-1)!}{(k-1)!(2-1)!!} \\
&\times \sum_{i=0}^{[(k-1)/2]} (\sqrt{y})^{k-1-2i,!} \frac{(y-1)^i}{(2,2)_i (1+2,2)_i}.
\end{aligned}$$

Let the equality (1.3.26) be true for some $s \in \mathbb{N}$. Let us prove that it is true for $s+1$ as well. Let us denote

$$A_{s,k} = \frac{2^{-s}(k+s-1)!}{(k-1)!(2s-1)!!}.$$

Then we have

$$\begin{aligned}
&\frac{(\sqrt{y}D_y)^{s+1}}{A_{s,k}} \sum_{i=0}^{[k/2]} (\sqrt{y})^{k-2i,!} \frac{(y-1)^i}{(2,2)_i(1,2)_i} \\
&\qquad = \sqrt{y}D_y \sum_{0\le 2i\le k-s} (\sqrt{y})^{k-s-2i,!} \frac{(y-1)^i}{(2,2)_i(1+2s,2)_i} \\
&\qquad = \sum_{0\le 2i\le k-s-1} (\sqrt{y})^{k-s-2i-1,!} \frac{(y-1)^i}{2(2,2)_i(1+2s,2)_i} \\
&\qquad + \sum_{2\le 2i\le k-s} (\sqrt{y})^{k-s-2i+1,!} \frac{(k-s-2i+1)(y-1)^{i-1}}{2(2,2)_{i-1}(1+2s,2)_i} \\
&\qquad = \sum_{0\le 2i\le k-s-1} (\sqrt{y})^{k-s-2i-1,!} \frac{(y-1)^i}{2(2,2)_i(1+2s,2)_i} \\
&\qquad + \sum_{0\le 2i\le k-s-2} (\sqrt{y})^{k-s-2i-1,!} \frac{(k-s-2i-1)(y-1)^i}{2(2,2)_i(1+2s,2)_{i+1}} \\
&\qquad = \sum_{0\le 2i\le k-s-1} \frac{(\sqrt{y})^{k-s-2i-1,!}(y-1)^i}{2(2,2)_i(1+2s,2)_i}\left(1+\frac{k-s-2i-1}{2i+2s+1}\right) \\
&\qquad = \frac{s+k}{2(1+2s)} \sum_{i=0}^{[(k-s-1)/2]} \frac{(\sqrt{y})^{k-s-1-2i,!}(y-1)^i}{(2,2)_i(1+2s+2,2)_i}.
\end{aligned}$$

Since

$$A_{s,k}\frac{s+k}{2(1+2s)} = \frac{2^{-s}(k+s-1)!}{(k-1)!(2s-1)!!}\frac{s+k}{2(1+2s)} = A_{s+1,k},$$

then the equality (1.3.26) is also true for $s+1$.

Let $s=0$. Using the formula (1.3.12) we write

$$\begin{aligned}
G_k^{0,2}(t) &= \sum_{j=0}^{[k/2]} (-1)^j \frac{t^{k-2j,!}(1-t^2)^j}{(2,2)_j(1,2)_j} \\
&= \frac{1}{k!}\sum_{j=0}^{[k/2]} (-1)^j \binom{k}{2j} t^{k-2j}\left(\pm\sqrt{1-t^2}\right)^{2j}
\end{aligned}$$

$$= \frac{1}{k!} H_k^0(t, \pm\sqrt{1-t^2}) = \frac{1}{k!} \cos(k \arccos t) = \frac{1}{k!} T_k(t). \quad (1.3.27)$$

If we put $y = t^2$ in the equality (1.3.26), then we get

$$2^{-s} D_t^s \sum_{i=0}^{[k/2]} (-1)^i t^{k-2i,!} \frac{(1-t^2)^i}{(2,2)_i (1,2)_i}$$
$$= \frac{2^{-s}(k+s-1)!}{(k-1)!(2s-1)!!} \sum_{i=0}^{[(k-s)/2]} \frac{(-1)^i t^{k-s-2i,!} (1-t^2)^i}{(2,2)_i (1+2s,2)_i}.$$

After reducing by 2^{-s} and multiplying both sides by $(1-t^2)^{s/2}$ we get

$$(1-t^2)^{s/2} D_t^s G_k^{0,2}(t) = \frac{(k+s-1)!}{(k-1)!(2s-1)!!}$$
$$\times \sum_{i=0}^{[(k-s)/2]} \frac{(-1)^i t^{k-s-2i,!} (1-t^2)^{i+s/2}}{(2,2)_i (1+2s,2)_i} = \frac{(k+s-1)!}{(k-1)!(2s-1)!!} G_k^{s,2}(t),$$

from where, taking into account (1.3.27), we obtain

$$G_k^{s,2}(t) = \frac{(k-1)!(2s-1)!!}{(k+s-1)!} (1-t^2)^{s/2} D_t^s G_k^{0,2}(t)$$
$$= \frac{(2s-1)!!}{k(k+s-1)!} (1-t^2)^{s/2} D_t^s T_k(t).$$

If we now take into account that $T_k^s(t) \equiv (1-t^2)^{s/2} D_t^s T_k(t)$ then we write

$$G_k^{s,2}(t) = \frac{(2s-1)!!}{k(k+s-1)!} T_k^s(t).$$

Using this equality, for the final proof of the theorem it is enough to note that

$$G_k^{s,2m+2}(t) = \sum_{i=0}^{[(k-s)/2]} (-1)^i \frac{t^{k-s-2i,!} (1-t^2)^{i+s/2}}{(2,2)_i (2m+2s+1,2)_i}$$
$$= (1-t^2)^{-m/2} \sum_{i=0}^{[(k+m-s-m)/2]} (-1)^i \frac{t^{k+m-s-m-2i,!} (1-t^2)^{i+(s+m)/2}}{(2,2)_i (2m+2s+1,2)_i}$$
$$= (1-t^2)^{-m/2} G_{k+m}^{s+m,2}(t) = \frac{(2s+2m-1)!!}{(k+m)(k+s+2m-1)!} (1-t^2)^{-m/2} T_{k+m}^{s+m}(t).$$

The theorem is proved. □

From the proven theorems it follows that the polynomials $G_{(\nu)}(x)$ for $n = 2, 3$ are reduced to the known spherical functions $r^n P_n(\cos\theta)$ and

$$r^n P_n^m(\cos\theta)\cos m\varphi, \quad r^n P_n^m(\cos\theta)\sin m\varphi.$$

Theorem 1.13. *For G-functions, Rodrigues formula is true*

$$(-1)^k D_t^k \left(1-t^2\right)^{k+s+(n-3)/2} = k!(n-1+2s,2)_k \left(1-t^2\right)^{(s+n-3)/2} G_{k+s}^{s,n}(t) \quad (1.3.28)$$

and for $n \geq 3$, $k, s \in \mathbb{N}_0$ and $k \geq s$ the following connection between G-functions and Gegenbauer polynomials $C_k^{n/2-1}[t]$

$$G_k^{s,n}(t) = \frac{(2s+n-3)!}{(k+s+n-3)!}(1-t^2)^{s/2} C_{k-s}^{s+n/2-1}[t]. \quad (1.3.29)$$

Proof. Let n be even. Similar to the derivation of the formula (1.3.25), let us apply the operator equality (1.3.22) to the function $(1-t^2)^{m+s-1/2}$. Then we get

$$\begin{aligned}
D_t^m(1-t^2)^{m+s-1/2} &= m! \sum_{k=0}^{[m/2]} (\sqrt{y})^{m-2k,!} \frac{(2D_y)^{m-k}}{(2,2)_k}(1-y)^{m+s-1/2}|_{y=t^2} \\
&= (-1)^m m! \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(2m+2s-1)\cdots(2s+2k+1)}{(2,2)_k}(1-t^2)^{s+k-1/2} \\
&= (-1)^m m! \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(2m+2s-1)\cdots(2s+1)}{(2s+2k-1)\cdots(2s+1)(2,2)_k}(1-t^2)^{s+k-1/2} \\
&= (-1)^m m!(2s+1,2)_m \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(1-t^2)^{k+s-1/2}}{(2,2)_k(1+2s,2)_k}.
\end{aligned}$$

Let us denote by s' the integer such that $s' - 1/2 = s + (n-3)/2$ and therefore $s' = s + (n-2)/2$. Then, by virtue of the previous equality, we have

$$\begin{aligned}
D_t^m(1-t^2)^{m+s+(n-3)/2} &= D_t^m(1-t^2)^{m+s'-1/2} \\
&= (-1)^m m!(2s'+1,2)_m \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(1-t^2)^{k+s'-1/2}}{(2,2)_k(1+2s',2)_k}
\end{aligned}$$

$$= (-1)^m m!(n-1+2s,2)_m (1-t^2)^{(s+n-3)/2} \sum_{k=0}^{[m/2]} (-1)^k \frac{t^{m-2k,!}(1-t^2)^{k+s/2}}{(2,2)_k (n-1+2s,2)_k}$$
$$= (-1)^m m!(n-1+2s,2)_m (1-t^2)^{(s+n-3)/2} G_{m+s}^{s,n}(t)$$

and therefore the formula (1.3.28) is true for even n.

Let n be odd. Let us apply the operator equality (1.3.22) to the function $(1-t^2)^{m+s}$. Then we get

$$D_t^m (1-t^2)^{m+s} = m! \sum_{k=0}^{[m/2]} (\sqrt{y})^{m-2k,!} \frac{(2D_y)^{m-k}}{(2,2)_k} (1-y)^{m+s}|_{y=t^2}$$
$$= (-1)^m m! \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(2m+2s)\cdots(2s+2k+2)}{(2,2)_k} (1-t^2)^{s+k}$$
$$= (-1)^m m! \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(2m+2s)\cdots(2s+2)}{(2k+2s)\cdots(2s+2)(2,2)_k} (1-t^2)^{s+k}$$
$$= (-1)^m m!(2s+2,2)_m \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(1-t^2)^{k+s}}{(2,2)_k (2+2s,2)_k}.$$

Let us denote by s'' an integer such that $s'' = s + (n-3)/2$, then

$$D_t^m (1-t^2)^{m+s+(n-3)/2} = D_t^m (1-t^2)^{m+s''} = (-1)^m m!(2s''+2,2)_m$$
$$\times \sum_{k=0}^{[m/2]} (-1)^k t^{m-2k,!} \frac{(1-t^2)^{k+s''}}{(2,2)_k (2+2s'',2)_k} = (-1)^m m!(n-1+2s,2)_m$$
$$\times (1-t^2)^{(s+n-3)/2} \sum_{k=0}^{[m/2]} (-1)^k \frac{t^{m-2k,!}(1-t^2)^{k+s/2}}{(2,2)_k (n-1+2s,2)_k}$$
$$= (-1)^m m!(n-1+2s,2)_m (1-t^2)^{(s+n-3)/2} G_{m+s}^{s,n}(t)$$

and therefore the formula (1.3.28) is also true for odd n.

To prove the formula (1.3.29) we use Rodrigues formula [15] for Gegenbauer polynomials

$$2^k k!(\lambda+1/2)_k (1-t^2)^{\lambda-1/2} C_k^\lambda[t] = (-1)^k (2\lambda)_k D_t^k (1-t^2)^{k+\lambda-1/2},$$

from which for $\lambda - 1/2 = s + (n-3)/2$, and therefore for $\lambda = s + (n-2)/2$ it follows

$$(-1)^k(2s+n-2)_k D_t^k(1-t^2)^{k+s+(n-3)/2} \\ = 2^k k!(s+(n-1)/2)_k(1-t^2)^{s+(n-3)/2}C_k^{s+(n-2)/2}[t].$$

The equality (1.3.28) gives

$$(-1)^k D_t^k\left(1-t^2\right)^{k+s+(n-3)/2} = k!(n-1+2s,2)_k(1-t^2)^{(s+n-3)/2}G_{k+s}^{s,n}(t),$$

which means

$$k!(n-1+2s,2)_k(1-t^2)^{(s+n-3)/2}G_{k+s}^{s,n}(t) \\ = \frac{k!(n-1+2s,2)_k}{(2s+n-2)_k}(1-t^2)^{s+(n-3)/2}C_k^{s+(n-2)/2}[t].$$

Reducing both sides of the equality by equal factors we find

$$G_{k+s}^{s,n}(t) = \frac{(1-t^2)^{s/2}}{(2s+n-2)_k}C_k^{s+n/2-1}[t] \\ = \frac{(2s+n-3)!}{(k+2s+n-3)!}(1-t^2)^{s/2}C_k^{s+n/2-1}[t].$$

Replacing $k+s$ with k we get (1.3.29), which is true for $n \geq 3$ and $k, s \in \mathbb{N}_0$, but $k \geq s$. The theorem is proved. □

1.3.3 Orthogonality of G-functions

Let us study the properties of the G-function in the space $L_2(-1,1)$.

Theorem 1.14. *G-functions of genus n ($n \geq 2$), of the same order s, but of different degrees k, are orthogonal on $[-1,1]$ with weight $\rho_n(t) = (1-t^2)^{(n-3)/2}$ and*

$$\int_{-1}^{1}\left(G_k^{s,n}(t)\right)^2(1-t^2)^{(n-3)/2}\,dt \\ = \frac{2^{2s+n-2}\Gamma^2(s+(n-1)/2)}{(2k+n-2)(k+s+n-3)!(k-s)!}. \tag{1.3.30}$$

Proof. Let $k \neq m \in \mathbb{N}_0$. It is easy to see that according to Lemma 1.8

$$\int_{\partial S}\varphi(|\tilde{x}|, x_n)\,ds_x = \omega_{n-1}\int_{-1}^{1}(1-t^2)^{(n-3)/2}\varphi(\sqrt{1-t^2}, t)\,dt.$$

Bearing in mind that $G_k^s(x) = G_k^s(|\tilde{x}|, x_n)$ and on the unit sphere the equality holds

$$G_k^s(x)G_m^s(x)(1-x_n^2)^s = \sum_{i=0}^{[k/2]} \frac{(-1)^i(1-x_n^2)^{i+s/2}x_n^{k-2i,!}}{(2,2)_i(n-1+2s,2)_i} \times \sum_{j=0}^{[m/2]} \frac{(-1)^j(1-x_n^2)^{j+s/2}x_n^{m-2j,!}}{(2,2)_j(n-1+2s,2)_j} = G_{k+s}^{s,n}(x_n)G_{m+s}^{s,n}(x_n),$$

then by Lemma 1.10 we get

$$0 = \int_{\partial S} G_k^s(x)G_m^s(x)(1-x_n^2)^s\, ds_x = \omega_{n-1}\int_{-1}^1 G_{k+s}^{s,n}(t)G_{m+s}^{s,n}(t)\rho_n(t)\,dt,$$

where $k, s \in \mathbb{N}_0$. Denoting $k' = k+s$ and $m' = m+s$ we have $k' \geq s$, $m' \geq s$, $k' \neq m'$ and

$$\int_{-1}^1 G_{k'}^{s,n}(t)G_{m'}^{s,n}(t)\rho_n(t)\,dt = 0.$$

This proves the orthogonality of G-functions of the same kind and order, but of different degrees. From Theorem 1.11 for odd $n \geq 3$ it is easy to find

$$G_k^{s,n}(t) = \frac{(2s+n-3)!!}{(k+s+n-3)!}(1-t^2)^{-(n-3)/4}P_{k+(n-3)/2}^{s+(n-3)/2}(t),$$

and from Theorem 1.12 for even $n \geq 2$ it follows

$$G_k^{s,n}(t) = \frac{(2s+n-3)!!}{(k+(n-2)/2)(k+s+n-3)!}(1-t^2)^{-(n-2)/4}T_{k+(n-2)/2}^{s+(n-2)/2}(t).$$

It is known [15] that

$$\int_{-1}^1 (P_k^s(t))^2\,dt = \frac{2}{2k+1}\frac{(k+s)!}{(k-s)!},$$

and therefore, since for odd $n \geq 3$ the number $2s+n-3$ is even, then

$$\begin{aligned}\int_{-1}^1 (G_k^{s,n}(t))^2\rho_n(t)\,dt &= \frac{((2s+n-3)!!)^2}{((k+s+n-3)!)^2}\int_{-1}^1 \left(P_{k+(n-3)/2}^{s+(n-3)/2}(t)\right)^2 dt \\ &= \frac{2((2s+n-3)!!)^2}{(2k+n-2)(k+s+n-3)!(k-s)!} \\ &= \frac{2^{2s+n-2}\Gamma^2(s+(n-1)/2)}{(2k+n-2)(k+s+n-3)!(k-s)!}. \qquad (1.3.31)\end{aligned}$$

If now $n \geq 2$ is even, then based on the connection of the Gegenbauer polynomials and Chebyshev polynomials [15], for $k \geq s$ we write

$$T_k^s(t) = k(2s-2)!!(1-t^2)^{s/2} C_{k-s}^s[t].$$

Because according to [154]

$$\int_{-1}^{1} (C_k^s[t])^2 (1-t^2)^{s-1/2}\, dt = \frac{(2s)_k}{(k+s)k!} \frac{\Gamma(1/2)\Gamma(s+1/2)}{\Gamma(s)},$$

then we easily deduce that

$$\int_{-1}^{1} (T_k^s(t))^2 (1-t^2)^{-1/2}\, dt = k\frac{(k+s-1)!}{(k-s)!}\frac{\pi}{2}. \tag{1.3.32}$$

This means, taking into account that $\Gamma^2(1/2) = \pi$ and

$$(2s+n-3)!!\sqrt{\pi} = 2^{s+(n-2)/2}\big(s+\frac{n-3}{2}\big)\cdots\frac{1}{2}\Gamma(1/2) = 2^{s+(n-2)/2}\Gamma\big(s+\frac{n-1}{2}\big),$$

that

$$\begin{aligned}
&\int_{-1}^{1} (G_k^{s,n}(t))^2 \rho_n(t)\, dt \\
&\quad= \frac{((2s+n-3)!!)^2}{(k+(n-2)/2)^2((k+s+n-3)!)^2} \int_{-1}^{1} \Big(T_{k+(n-2)/2}^{s+(n-2)/2}(t)\Big)^2 (1-t^2)^{-1/2}\, dt \\
&\quad= \frac{((2s+n-3)!!)^2}{(k+(n-2)/2)(k+s+n-3)!(k-s)!}\frac{\pi}{2} \\
&\quad= \frac{2^{2s+n-2}\Gamma^2(s+(n-1)/2)}{(2k+n-2)(k+s+n-3)!(k-s)!}.
\end{aligned} \tag{1.3.33}$$

Combining the formulas (1.3.31) and (1.3.33) we arrive at (1.3.30). □

Corollary 1.3. Associated Chebyshev functions of the same order are orthogonal on the interval $[-1;1]$ with weight $\rho(t) = (1-t^2)^{-1/2}$ and the equality is true

$$\int_{-1}^{1} \left(T_k^s(t)\right)^2 (1-t^2)^{-1/2}\, dt = k\frac{(k+s-1)!}{(k-s)!}\frac{\pi}{2}.$$

Chapter 2

Dirichlet problem for the polyharmonic equation

In this chapter we study the Dirichlet boundary value problem for the polyharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^m u(x) = Q(x), \quad x \in S; \tag{2.0.1}$$

$$u|_{\partial S} = f_0(s), \ \frac{\partial u}{\partial \nu}\Big|_{\partial S} = f_1(s), \ \ldots, \ \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\partial S} = f_{m-1}(s), \ s \in \partial S, \tag{2.0.2}$$

where ν is the outer normal to the sphere ∂S.

In Section 1.1, as a corollary of Theorem 1.2, the solvability of the Dirichlet problem for the homogeneous polyharmonic equation in the unit ball is proved, but the resulting representation of the solution to this problem is not entirely convenient. For this problem, we apply a method based on the mean value property on the unit sphere for normal derivatives of polyharmonic functions (see Section 3.1) and the Green's function method.

A few words about the Green's function method. One of the effective methods for presenting solutions to various boundary value problems is a method based on finding the Green's function of these problems. The explicit form of Green's functions for various boundary value problems is presented in many works. We cite only a few of them, excluding the author's works. For example, in the two-dimensional case, in [20], based on the well-known harmonic Green's function, the Green's functions of various biharmonic problems are presented. An explicit form of the Green's function the Robin boundary value problem is found in [22, 107, 156], and the Green's function in the sector for the biharmonic and 3-harmonic equations in [187, 188]. The articles [32, 56, 57] are devoted to the construction of the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball. As the most general results on the generalized Neumann problem containing powers of normal derivatives in boundary conditions, we note [164]. We also present some recently published papers on the construction of the Green's function for various boundary value problems [3, 23, 24, 139]. The application of Green's functions in problems of mechanics and physics can be found in [51, 53]. We also note here the papers [190, 191], where the generalized Dirichlet problem for the iterated slice Dirac equation and the Dirichlet and Neumann problem for the polyharmonic Dunkl equation are studied. In [26], boundary value problems for the polyharmonic equation with the Hadamard-Marchot boundary operator is considered.

The results of Chapter 2 are as follows. Section 2.1 provides a method for solving the homogeneous Dirichlet problem (2.0.1)-(2.0.2) in the case when the right-hand side of the equation – the function $f(x)$, is a polynomial. First, Theorem 2.1 gives a solution to the homogeneous Dirichlet problem for $f(x) = Q_l(x)$, where $Q_l(x)$ is homogeneous polynomial of degree l, and then Theorem 2.2 gives the operator form of the explicit solution to the considered problem with an arbitrary polynomial $f(x)$ on the right side of the equation. To find a polynomial solution to the Dirichlet problem, it is necessary to act on the polynomial $f(x)$ with a special operator containing powers of the Laplace operator and obtain the desired solution to the homogeneous Dirichlet problem in explicit form. Examples 2.1-2.3 are given. The results of Section 2.1 are based on the following papers [73, 74, 76, 76, 77, 80, 83, 86, 91].

In Section 2.2 the biharmonic equation is considered and the Green's function of the Dirichlet problem in the unit ball is constructed. Lemmas 2.7 and 2.8 contain power series expansions of the elementary solution $E(x, \xi)$ and the Green's function $G(x, \xi)$ of the Dirichlet problem for the Poisson equation. The equality (2.2.7) defines the elementary solution of the biharmonic equation $E_4(x, \xi)$, Theorem 2.3 gives the representation of $E_4(x, \xi)$ in the form of a series, and then, in Theorem 2.4 for $n \geq 3$ the Green's function $G_4(x, \xi)$ of the Dirichlet problem for the biharmonic equation is determined. Using this result the solution of the homogeneous Dirichlet problem is found. Theorem 2.5 for $n > 4$ contains a series expansion of the function $G_4(x, \xi)$ and based on this expansion, Theorem 2.6 obtains a solution to the homogeneous Dirichlet problem for $f(x) = |x|^{2l} H_k(x)$. The Green's function for the case $n = 2$ is found in Theorem 2.7.

In Section 2.3, based on the results of Section 2.2, the Green's function of the polyharmonic equation is constructed. First, the fundamental solution of the polyharmonic equation $\mathcal{E}_{2m}(x, \xi)$ is determined and its properties are studied in Lemmas 2.15 and 2.16. Theorem 2.8 defines the Green's function $G_{2m}(x, \xi)$ of the Dirichlet problem, Theorem 2.9 finds a solution to the homogeneous Dirichlet problem for a polyharmonic equation in the unit ball, and in Theorem 2.10 we consider a special case of the Dirichlet problem with $f(x) = |x|^{2l} H_k(x)$.

Section 2.4 contains an explicit form of the solution to the Dirichlet problem for a homogeneous polyharmonic equation in the unit ball. Using Theorem 2.11 and Lemmas 2.18-2.20, based on the properties of the matrix $\mathbb{H}_l$, Theorem 2.12 on the representation of the solution to the Dirichlet problem is proved. It is noteworthy that the resulting solution is not related to the Green's function of the problem. The matrix $\mathbb{H}_4$ is given in Example 2.4. Theorem 2.13 formulates the main result for solving the Dirichlet boundary value problem. Examples 2.5-2.7 of solving the Dirichlet problem for 3-harmonic and 4-harmonic equations are given. Theorem 2.14 provides an equality for recursively finding a solution to the Dirichlet problem. The uniqueness of the solution to the Dirichlet problem is given in Theorem 2.16. Theorem 2.15 makes it easier to find a solution to the Dirichlet problem for the Poisson equation with polynomial data.

2.1 Polynomial solutions of the homogeneous Dirichlet problem

To construct a solution to a specific Dirichlet problem, for example, for a 3-harmonic equation in the traditional way (see, for example, [182, p.200]) with polynomial boundary data (f_0, f_1 and f_2 are traces of polynomials of degree k) adhere to the following scheme. Take a complete system of orthonormal harmonic polynomials $G^i_j(x)$ on the unit sphere ∂S, $j = 1, \dots, h_i$, where $h_i = (1 + \frac{2i}{n-2})\binom{i+n-3}{n-3}$ (see equality (1.3.15)). For example, one can take G-polynomials from Section 1.3. Then they construct 3-harmonic polynomials of the form $G^i_j(x)$, $|x|^2 G^i_j(x)$ and $|x|^4 G^i_j(x)$ and look for a solution to the Dirichlet problem in the form [77]

$$u(x) = \sum_{i=0}^{k}\sum_{j=1}^{h_i}\left(C^i_j + D^i_j|x|^2 + E^i_j|x|^4\right)G^i_j(x).$$

The unknown coefficients C^i_j, D^i_j and E^i_j for each i and j are easily determined from the system of algebraic equations

$$C^i_j + D^i_j + E^i_j = \int_{\partial S} G^i_j(s) f_0(s)\, ds,$$

$$iC^i_j + (i+2)D^i_j + (i+4)E^i_j = \int_{\partial S} G^i_j(s) f_1(s)\, ds,$$

$$i(i-1)C^i_j + (i+2)(i+1)D^i_j + (i+4)(i+3)E^i_j = \int_{\partial S} G^i_j(s) f_2(s)\, ds,$$

where $j = 1, \ldots, h_i$ and $0 \le i \le k$. The determinant of this system is the Vandermonde determinant $W[i, i+2, i+4]$ with factorial powers. With a large dimension of space n and degree of polynomials k, this procedure is quite complicated, even with simple polynomials f_0, f_1 and f_2 since $h_k \sim 2k^{n-2}/(n-2)!$, $k \to \infty$ and one need to calculate many surface integrals over ∂S. Therefore, in this section, we propose another way to construct a polynomial solution to the Dirichlet problem, which requires only finding the powers of the Laplace operator from some auxiliary polynomials (see equality (2.1.24)).

Let us consider the homogeneous boundary value problem (2.0.1)-(2.0.2) for inhomogeneous m-harmonic equation in the unit ball S

$$\Delta^m u(x) = Q(x), \quad x \in S; \tag{2.1.1}$$

$$u|_{\partial S} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0, \quad \ldots, \quad \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\partial S} = 0, \tag{2.1.2}$$

with polynomial right-hand side $Q(x)$ and for $n \ge 2$. Let us introduce the following operator

$$\Delta_{-m} Q(x) = \frac{|x|^{2m}}{2(2m-2)!!} \sum_{k=0}^{\infty} \frac{|x|^{2k}}{(2k)!!(2k+2m)!!} \times \int_0^1 (1-\alpha)^{k+m-1} \alpha^{k+n/2-1} (-\Delta)^k Q(\alpha x)\, d\alpha. \tag{2.1.3}$$

In [72, Theorem 3] it is established that the polynomial $u(x) = \Delta_{-m} Q(x)$ is a particular solution of the polyharmonic equation (2.1.1), i.e. the equality $\Delta^m(\Delta_{-m} Q(x)) = Q(x)$ is true. Let us slightly correct the found solution of the inhomogeneous polyharmonic equation so that it additionally satisfies the homogeneous boundary conditions.

Suppose that $Q(x) = Q_l(x)$ is a homogeneous polynomial of degree l. In [72, Theorem 4] it is shown that in this case the solution (2.1.3) can be written in the form

$$u(x) = \sum_{s=0}^{[l/2]} (-1)^s \binom{s+m-1}{m-1} \frac{|x|^{2s+2m} \Delta^s Q_l(x)}{(2,2)_{s+m} (2l-2s+n, 2)_{s+m}}, \tag{2.1.4}$$

where $(a, b)_k = a(a+k) \cdots (a+(k-1)b)$ is the generalized Pochhammer symbol (1.3), and $[c]$ is the integer part of the number c. In the denominator of the fraction under the sum sign in (2.1.4) there is the expression $(2l-2s+n, 2)_{s+m} = (2l-2s+n) \cdots (2l+2m+n-2)$, which does not vanish because $2s \le l$.

Let us expand the homogeneous polynomial $Q_l(x)$ using the Almansi formula into a sum of terms of the form $|x|^{2s}R_{l-2s}(x)$

$$Q_l(x) = R_l(x) + |x|^2 R_{l-2}(x) + \cdots + |x|^{2s} R_{l-2s}(x), \tag{2.1.5}$$

where $l-2s \geq 0$, and the homogeneous harmonic polynomials $R_{l-2k}(x)$ are given in Lemma 1.11

$$R_{l-2k}(x) = \frac{2l-4k+n-2}{(2,2)_k} \sum_{s=0}^{[l/2]-k} \frac{(-1)^s |x|^{2s} \Delta^{s+k} Q_l(x)}{(2,2)_s (2l-4k-2s+n-2,2)_{s+k+1}}. \tag{2.1.6}$$

It is easy to verify (see also [77, Lemma 2]) that the equality holds

$$\Delta_{-m}\big(|x|^{2l} \cdot P_s(x)\big) = \frac{|x|^{2l+2m} \cdot P_s(x)}{(2l+2,2)_m (2l+2s+n,2)_m}. \tag{2.1.7}$$

Therefore, the solution to the equation $\Delta^m v(x) = Q_l(x)$, given by the formula (2.1.4) has the form

$$v(x) = \sum_{s=0}^{[l/2]} \frac{|x|^{2s+2m} R_{l-2s}(x)}{(2s+2,2)_m (2l-2s+n,2)_m}. \tag{2.1.8}$$

Consider the operator $\Lambda u(x) = \sum_{k=1}^n x_k u_{x_k}(x)$, which is defined on functions from $C^1(S)$. As noted above, it has easily verifiable properties: if the function u is harmonic in S, then the function Λu is also harmonic in S; the equality $\Lambda(uv) = v\Lambda u + u\Lambda v$ is true; if $P_l(x)$ is a homogeneous polynomial of degree l, then $\Lambda P_l(x) = lP_l(x)$.

Let $P_k(t) = \sum_{s=0}^k c_s t^s$ be some polynomial. Consider the factorial polynomial from Definition 1.1 on page 13 corresponding to the polynomial $P_k(t)$. It is given by the equality

$$P_{[k]}(t) = \sum_{s=0}^k c_s t^{[s]},$$

where the degree $t^{[k]}$ is determined by the equality $t^{[k]} = t(t-1)\cdots(t-k+1)$. Let us recall another important property of the operator Λ, which is already used in Theorem 1.1.

Lemma 2.1. *On the unit sphere ∂S the equality holds*

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial S} = \Lambda^{[k]} u|_{\partial S}.$$

Proof. It is not hard to see that

$$\frac{\partial u(x)}{\partial \nu}\Big|_{\partial S} = \sum_{k=1}^{n} \frac{x_k}{|x|}\frac{\partial u(x)}{\partial x_k}\Big|_{\partial S} = \frac{1}{|x|}\Lambda u(x)_{|\partial S},$$

that is why

$$\frac{\partial^k u(x)}{\partial \nu^k}\Big|_{\partial S} = \Big(\frac{1}{|x|}\Lambda\Big)^k u(x)_{|\partial S}. \tag{2.1.9}$$

Let us prove that

$$\Big(\frac{1}{|x|}\Lambda\Big)^k = \frac{1}{|x|^k}\Lambda^{[k]}. \tag{2.1.10}$$

For $k = 1$ this equality is obvious. Let it be true for $k = m$. Let us prove its correctness for $k = m + 1$. Indeed, since $\Lambda(uv) = v\Lambda u + u\Lambda v$ and $\Lambda|x|^\alpha = \alpha|x|^\alpha$, then

$$\Big(\frac{1}{|x|}\Lambda\Big)^{m+1} = \frac{1}{|x|}\Lambda\Big(\frac{1}{|x|^m}\Lambda^{[m]}\Big) = \frac{1}{|x|^{m+1}}\Big(-m\Lambda^{[m]} + \Lambda\Lambda^{[m]}\Big)$$
$$= \frac{1}{|x|^{m+1}}\Lambda^{[m]}(\Lambda - m) = \frac{1}{|x|^{m+1}}\Lambda^{[m+1]}.$$

Induction is proved. Based on (2.1.10) from (2.1.9) we find

$$\frac{\partial^k u(x)}{\partial \nu^k}_{|\partial S} = \Big(\frac{1}{|x|}\Lambda\Big)^k u(x)_{|\partial S} = \Lambda^{[k]}u(x)_{|\partial S}.$$

The lemma is proved. □

Corollary 2.1. The following equality takes place

$$P_k\Big(\frac{\partial}{\partial \nu}\Big)u|_{\partial S} = P_{[k]}(\Lambda)u|_{\partial S}.$$

Proof. To prove the corollary, it is necessary to apply Lemma 2.1 to each term of the polynomial $P_k\big(\frac{\partial}{\partial\nu}\big)u|_{\partial S}$. □

Let us study the Dirichlet problem (2.1.1)-(2.1.2) for $Q(x) = |x|^{2s}R_{l-2s}(x)$. Let us prove the following statement.

Lemma 2.2. *The solution $v_s(x)$ of the homogeneous Dirichlet problem* (2.1.1)-(2.1.2) *for $Q(x) = |x|^{2s}R_{l-2s}(x)$ has the form*

$$v_s(x) = \Big(|x|^{2s+2m} - m\binom{s+m}{m}\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{m-1}{i}\frac{|x|^{2i}}{s+m-i}\Big)\frac{R_{l-2s}(x)}{C_{l,s,m}}, \quad (2.1.11)$$

where $C_{l,s,m} = (2s+2,2)_m(2l-2s+n,2)_m$.

Proof. Let the polynomial $v_s(x)$ be defined by the equality

$$v_s(x) = \frac{1}{C_{l,s,m}}\Big(|x|^{2s+2m}R_{l-2s}(x) - \sum_{i=0}^{m-1}|x|^{2i}H^i_{l-2s}(x)\Big), \quad (2.1.12)$$

where $H^i_{l-2s}(x)$ are homogeneous harmonic polynomials of degree $l-2s$. Using the equality (2.1.7) and the definition of $C_{l,s,m}$ from the conditions of the theorem we obtain

$$\Delta^m v_s(x) = \frac{1}{C_{l,s,m}}\Delta^m\left(|x|^{2s+2m}R_{l-2s}(x)\right) = |x|^{2s}R_{l-2s}(x),$$

i.e. polynomial $v_s(x)$ is a solution to the equation (2.1.1) with $Q(x) = |x|^{2s}R_{l-2s}(x)$.

Now we choose homogeneous polynomials $H^i_{l-2s}(x)$ so that the homogeneous boundary conditions (2.1.2) are satisfied. Then we have

$$R_{l-2s}(x) - \sum_{i=0}^{m-1}H^i_{l-2s}(x) = 0 \Rightarrow v_{s|\partial S} = 0.$$

Further, using Lemma 2.1 and, for brevity, the notations $\delta_i = l + 2i - 2s$ and $\delta = l + 2m$ we obtain

$$\delta R_{l-2s}(x) - \sum_{i=0}^{m-1}\delta_i H^i_{l-2s}(x) = 0 \Rightarrow (\Lambda^{[1]}v_s)_{|\partial S} = 0 \Rightarrow \frac{\partial v_s}{\partial \nu}_{|\partial S} = 0,$$

$$\dots$$

$$\delta^{[m-1]}R_{l-2s}(x) - \sum_{i=0}^{m-1}\delta_i^{[m-1]}H^i_{l-2s}(x) = 0 \Rightarrow (\Lambda^{[m-1]}v_s)_{|\partial S} = 0 \Rightarrow \frac{\partial^{m-1}v_s}{\partial \nu^{m-1}}_{|\partial S} = 0,$$

and therefore, to determine $H^i_{l-2s}(x)$ it is necessary to solve the following system of equations

$$\sum_{i=0}^{m-1} \delta_i^{[j]} H^i_{l-2s}(x) = \delta^{[j]} R_{l-2s}(x), \quad j = 0, \dots, m-1. \tag{2.1.13}$$

The determinant $\mathcal{D}$ of this system of equations for $H^i_{l-2s}(x)$ has the form

$$\mathcal{D} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_0 & \delta_1 & \cdots & \delta_{m-1} \\ \dots & & & \\ \delta_0^{[m-1]} & \delta_1^{[m-1]} & \cdots & \delta_{m-1}^{[m-1]} \end{vmatrix} = W[\delta_0, \delta_1, \dots, \delta_{m-1}] = \prod_{0 \le j < i \le m-1} (\delta_i - \delta_j),$$

where $W[\delta_0, \delta_1, \dots, \delta_{m-1}]$ is the Vandermonde determinant of order m. Since $\delta_i - \delta_j = 2(i - j)$, we get

$$\mathcal{D} = \prod_{i=1}^{m-1} (2i)!!.$$

Let us denote by $\mathcal{D}_{i-1}$ the determinant obtained from the determinant $\mathcal{D}$ by replacing the ith column with the column of free terms of the system (2.1.13). Then we have

$$\mathcal{D}_{i-1} = R_{l-2s}(x) \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \delta_0 & \cdots & \delta_{i-2} & \delta & \delta_i & \cdots & \delta_{m-1} \\ \dots & & & & & & \\ \delta_0^{[m-1]} & \cdots & \delta_{i-2}^{[m-1]} & \delta^{[m-1]} & \delta_i^{[m-1]} & \cdots & \delta_{m-1}^{[m-1]} \end{vmatrix} =$$
$$= R_{l-2s}(x) W[\delta_0, \dots, \delta_{i-2}, \delta, \delta_i, \dots, \delta_{m-1}], \quad i = 1, \dots, m.$$

Let us calculate the Vandermonde determinant $W[\delta_0, \dots, \delta_{i-2}, \delta, \delta_i, \dots, \delta_{m-1}]$. We get

$$W[\delta_0, \dots, \delta_{i-2}, \delta, \delta_i, \dots, \delta_{m-1}] =$$
$$= W[\delta_0, \dots, \delta_{m-1}] \frac{\prod_{j=0}^{i-2} (\delta - \delta_j) \prod_{j=i}^{m-1} (\delta_j - \delta)}{\prod_{j=0}^{i-2} (\delta_{i-1} - \delta_j) \prod_{j=i}^{m-1} (\delta_j - \delta_{i-1})} =$$
$$= W[\delta_0, \dots, \delta_{m-1}] \frac{(-1)^{m-i} \prod_{j=0,\, j \ne i-1}^{m-1} (\delta - \delta_j)}{(2i-2)!!(2m-2i)!!} =$$

$$= \mathcal{D}\frac{(-1)^{m-i}}{(m-1)!}\binom{m-1}{i-1}\prod_{j=1,\, j\neq m-i+1}^{m}(s+j),$$

where it is taken into account that $\delta_i = l + 2i - 2s$ and $\delta = l + 2m$. It follows from this that

$$H_{l-2s}^{i-1}(x) = \frac{\mathcal{D}_{i-1}}{\mathcal{D}} = R_{l-2s}(x)m\binom{s+m}{m}(-1)^{m-i}\binom{m-1}{i-1}\frac{1}{s+m-i+1}$$

for $i = 1, \dots, m$. Substituting the obtained values $H_{l-2s}^{i}(x)$ into the formula (2.1.12) we get

$$v_s(x) = \frac{R_{l-2s}(x)}{C_{l,s,m}}\left(|x|^{2s+2m} - m\binom{s+m}{m}\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{m-1}{i}\frac{|x|^{2i}}{s+m-i}\right).$$

This implies the equality (2.1.11) stated in the lemma. □

Let us transform the resulting solution.

Lemma 2.3. *The solution* (2.1.11) *of the homogeneous Dirichlet problem* (2.1.1)-(2.1.2) *with* $Q(x) = |x|^{2s}R_{l-2s}(x)$ *can be written in the form*

$$v_s(x) = (|x|^2 - 1)^m \frac{R_{l-2s}(x)}{C_{l,s,m}}\sum_{i=0}^{s}\binom{s+m}{i+m}(|x|^2-1)^i, \tag{2.1.14}$$

where $C_{l,s,m} = (2s+2,2)_m(2l-2s+n,2)_m$.

Proof. Consider the polynomial

$$P(\tau) = \tau^{s+m} - m\binom{s+m}{m}\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{m-1}{i}\frac{\tau^i}{s+m-i}.$$

Then, in accordance with (2.1.11), the equality holds

$$v_s(x) = \frac{R_{l-2s}(x)}{C_{l,s,m}}P(|x|^2). \tag{2.1.15}$$

Let us transform the polynomial $P(\tau)$. Since the equality holds true

$$\frac{\tau^i}{s+m-i} = \tau^i\int_0^1 t^{s+m-1-i}\,dt = \int_0^1 \tau^i t^{m-1-i}t^s\,dt,$$

then

$$P(\tau) = \tau^{s+m} - m\binom{s+m}{m}\int_0^1 t^s \sum_{i=0}^{m-1}\binom{m-1}{i}\tau^i(-t)^{m-1-i}dt$$
$$= \tau^{s+m} - m\binom{s+m}{m}\int_0^1 t^s(\tau-t)^{m-1}dt.$$

It is not hard to see that

$$m\int_0^1 (\tau-t)^{m-1}t^s dt = m\int_0^1(\tau-t)^{m-1}d\frac{t^{s+1}}{s+1} = \frac{m}{s+1}(\tau-t)^{m-1}t^{s+1}\Big|_0^1$$
$$+\frac{m(m-1)}{s+1}\int_0^1(\tau-t)^{m-2}t^{s+1}dt = \frac{m}{s+1}(\tau-1)^{m-1} + \frac{m(m-1)}{(s+1)(s+2)}(\tau-t)^{m-2}t^{s+2}\Big|_0^1$$
$$+\frac{m(m-1)(m-2)}{(s+1)(s+2)}\int_0^1(\tau-t)^{m-3}t^{s+2}dt = \cdots = m!s!\sum_{k=0}^{m-2}\frac{(\tau-1)^{m-1-k}}{(s+k+1)!(m-1-k)!}$$
$$+\frac{m!s!}{(s+m-1)!}\int_0^1(\tau-t)^0 t^{s+m-1}dt = m!s!\sum_{k=0}^{m-1}\frac{(\tau-1)^{m-1-k}}{(s+k+1)!(m-1-k)!}$$
$$= m!s!\sum_{j=0}^{m-1}\frac{(\tau-1)^j}{(s+m-j)!j!} = \frac{m!s!}{(s+m)!}\sum_{j=0}^{m-1}\binom{s+m}{j}(\tau-1)^j.$$

Therefore

$$P(\tau) = \tau^{s+m} - \sum_{j=0}^{m-1}\binom{s+m}{j}(\tau-1)^j$$
$$= \sum_{j=m}^{s+m}\binom{s+m}{j}(\tau-1)^j = (\tau-1)^m\sum_{k=0}^{s}\binom{s+m}{k+m}(\tau-1)^k.$$

Substituting the resulting value $P(\tau)$ into the formula (2.1.15) we get (2.1.14). The lemma is proved. □

The proven Lemma 2.3 already allows us to construct solutions to the simplest Dirichlet problems.

Example 2.1. Consider the Dirichlet problem (2.1.1)-(2.1.2) for $Q(x) = 1$. Its solution, found using the equality (2.1.14), in which $R_{l-2s}(x) = 1$, $l = 0$ and $s = 0$ has the form

$$v_0(x) = \frac{(|x|^2-1)^m}{(2,2)_m(n,2)_m}. \tag{2.1.16}$$

Let $u_0(x)$ be a polynomial solution to the Dirichlet problem (2.1.1)-(2.1.2), when the polynomial $Q(x)$ is a homogeneous polynomial of degree l, i.e. $Q(x) = Q_l(x)$. Let us construct $u_0(x)$.

Lemma 2.4. *Solution of the homogeneous Dirichlet problem* (2.1.1)-(2.1.2) *with* $Q(x) = Q_l(x)$ *can be written as*

$$u_0(x) = (|x|^2-1)^m \sum_{s=0}^{[l/2]} \frac{\Delta^s Q_l(x)}{4^{s+m}} \sum_{i=0}^{s} |x|^{2i} \binom{m-1+s-i}{m-1} \\ \times \sum_{j=0}^{i} (-1)^j \frac{l-2s+2j+n/2-1}{j!(m+s-j)!(l-2s+j+n/2-1)_{s+m+1}}. \tag{2.1.17}$$

Proof. Let us expand the polynomial $Q_l(x)$ according to the equality (2.1.5), and then apply Lemma 2.3 to each term, where s is replaced by k and the solution to each such homogeneous problem is denoted by $v_k(x)$. After applying the equality (2.1.6) we obtain the solution $u_0(x)$ of the problem under consideration in the form

$$u_0(x) = \sum_{k=0}^{[l/2]} v_k(x) = (|x|^2-1)^m \sum_{k=0}^{[l/2]} \frac{R_{l-2k}(x)}{C_{l,k,m}} \sum_{i=0}^{k} \binom{k+m}{i+m} (|x|^2-1)^i \\ = (|x|^2-1)^m \sum_{k=0}^{[l/2]} \frac{2l-4k+n-2}{(2,2)_k(2k+2,2)_m(2l-2k+n,2)_m} \\ \times \sum_{s=0}^{[l/2]-k} \frac{(-1)^s |x|^{2s} \Delta^{s+k} Q_l(x)}{(2,2)_s(2l-4k-2s+n-2,2)_{s+k+1}} \sum_{i=0}^{k} \binom{k+m}{i+m} (|x|^2-1)^i \\ = (|x|^2-1)^m \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2l-4k+n-2)|x|^{2s} \Delta^{s+k} Q_l(x)}{(2,2)_s(2,2)_{k+m}(2l-4k-2s+n-2,2)_{s+k+m+1}} \\ \times \sum_{i=0}^{k} \binom{k+m}{i+m} (|x|^2-1)^i = (|x|^2-1)^m \sum_{r=0}^{\infty} \frac{\Delta^r Q_l(x)}{4^{r+m}} \\ \times \sum_{s=0}^{r} \frac{(-1)^s (l-2r+2s+n/2-1)|x|^{2s}}{s!(l-2r+s+n/2-1)_{r+m+1}} \sum_{i=0}^{r-s} \frac{(|x|^2-1)^i}{(i+m)!(r-s-i)!}. \tag{2.1.18}$$

The resulting expression involves the following polynomial

$$R_l(t) = \sum_{i=0}^{l} \frac{(t-1)^i}{(i+m)!(l-i)!}.$$

Let us transform it. We use the notation $\tau^{i,!} = \tau^i/i!$ and the following equality

$$\frac{1}{(i+m)!} = \frac{1}{i!}\int_0^1 \tau^i (1-\tau)^{m-1,!} d\tau.$$

Then, using the properties of the Euler beta function $\mathrm{B}(x)$ we get

$$\begin{aligned}
R_l(t) &= \sum_{i=0}^{l} \int_0^1 \frac{((t-1)\tau)^i}{i!(l-i)!}(1-\tau)^{m-1,!} d\tau \\
&= \frac{1}{l!}\int_0^1 (1-\tau)^{m-1,!} \sum_{i=0}^{l} \binom{l}{i} ((t-1)\tau)^i d\tau \\
&= \frac{1}{l!}\int_0^1 (1-\tau)^{m-1,!} ((t-1)\tau+1)^l d\tau \\
&= \frac{1}{l!}\int_0^1 (1-\tau)^{m-1,!} (t\tau+1-\tau)^l d\tau \\
&= \frac{1}{(m-1)!}\sum_{i=0}^{l} \frac{t^i}{i!(l-i)!}\int_0^1 \tau^i (1-\tau)^{m-1+l-i} d\tau \\
&= \frac{1}{(m-1)!}\sum_{i=0}^{l} \frac{\mathrm{B}(i+1, m+l-i)}{i!(l-i)!} t^i \\
&= \frac{1}{(m-1)!(m+l)!}\sum_{i=0}^{l} \frac{(m-1+l-i)!}{(l-i)!} t^i.
\end{aligned}$$

Substituting the value of the polynomial $R_{r-s}(|x|^2)$ into (2.1.18) we find

$$u_0(x) = (|x|^2-1)^m \sum_{r=0}^{\infty} \frac{\Delta^r Q_l(x)}{4^{r+m}} P_r(|x|^2), \qquad (2.1.19)$$

where denoted

$$P_r(|x|^2) = \frac{1}{(m-1)!}\sum_{s=0}^{r}\frac{(-1)^s(l-2r+2s+n/2-1)|x|^{2s}}{s!(l-2r+s+n/2-1)_{r+m+1}(m+r-s)!} \times\sum_{i=0}^{r-s}\frac{(m-1+r-s-i)!}{(r-s-i)!}|x|^{2i}.$$

Now transform the polynomial $P_r(|x|^2)$. We have

$$P_r(|x|^2) = \frac{1}{(m-1)!}\sum_{s=0}^{r}\frac{(-1)^s(l-2r+2s+n/2-1)}{s!(l-2r+s+n/2-1)_{r+m+1}(m+r-s)!} \times\sum_{i=0}^{r-s}\frac{(m-1+i)!}{i!}|x|^{2r-2i} = \frac{1}{(m-1)!}\sum_{i=0}^{r}\frac{(m-1+i)!}{i!} \times\sum_{s=0}^{r-i}\frac{(-1)^s(l-2r+2s+n/2-1)}{s!(l-2r+s+n/2-1)_{r+m+1}(m+r-s)!}|x|^{2r-2i} = \sum_{i=0}^{r}\frac{(m-1+r-i)!}{(m-1)!(r-i)!}|x|^{2i}\sum_{s=0}^{i}\frac{(-1)^s(l-2r+2s+n/2-1)}{s!(l-2r+s+n/2-1)_{r+m+1}(m+r-s)!}.$$

Replacing r with s and s with j we get

$$P_s(|x|^2) = \sum_{i=0}^{s}\binom{m-1+s-i}{m-1}|x|^{2i} \times\sum_{j=0}^{i}(-1)^j\frac{l-2s+2j+n/2-1}{j!(l-2s+j+n/2-1)_{s+m+1}(m+s-j)!}.$$

Substituting the resulting value $P_s(|x|^2)$ into (2.1.19), where r is replaced by s, we get (2.1.17). □

Let us transform the resulting solution $u_0(x)$ a little more.

Theorem 2.1. *The solution to the homogeneous Dirichlet problem* (2.1.1)-(2.1.2) *for $Q(x) = Q_l(x)$, where $Q_l(x)$ is a homogeneous polynomial of degree l is written as*

$$u_0(x) = (|x|^2-1)^m\sum_{s=0}^{[l/2]}\binom{s+m-1}{m-1}\frac{\Delta^s Q_l(x)}{4^{s+m}(s+m)!} \times\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{|x|^{2k}}{(l-2s+k+n/2)_{s+m}}. \quad (2.1.20)$$

Proof. Let us use Lemma 2.4 to write the solution $u_0(x)$ to the problem (2.1.1)-(2.1.2) for $Q(x) = Q_l(x)$. In the equality (2.1.17) we use the constant

$$A_{s,i} = \sum_{j=0}^{i}(-1)^j \frac{A_s + 2j - 1}{j!(m+s-j)!(A_s+j-1)_{s+m+1}},$$

where $A_s = l - 2s + n/2$ and transform it to a simpler form. To do this, we use the equality from [77, Remark 1], in which an assumption is made about the form of the solution $u_0(x)$. From this equality it follows that

$$A_{s,i} = (-1)^i \frac{(m+s-i)_i}{i!(m+s)!(A_s+i)_{s+m}}. \tag{2.1.21}$$

Let us prove this formula by mathematical induction. Let $i = 0$. Then

$$A_{s,0} = \sum_{j=0}^{0}(-1)^j \frac{A_s + 2j - 1}{j!(m+s-j)!(A_s+j-1)_{s+m+1}} \\ = \frac{A_s - 1}{(m+s)!(A_s-1)_{s+m+1}} = \frac{(m+s)_0}{(m+s)!(A_s)_{s+m}}$$

and therefore (2.1.21) is correct in this case.

Let the equality (2.1.21) be true for some $i \in \mathbb{N}_0$. Let us prove its validity for $i + 1$. Using the induction hypothesis we find

$$\begin{aligned} A_{s,i+1} &= \sum_{j=0}^{i+1} \frac{(-1)^j (A_s + 2j - 1)}{j!(m+s-j)!(A_s+j-1)_{s+m+1}} = A_{s,i} \\ &+ \frac{(-1)^{i+1}(A_s + 2i + 1)}{(i+1)!(m+s-i-1)!(A_s+i)_{s+m+1}} \underset{\text{Ind.}}{=} \frac{(-1)^i (m+s-i)_i}{i!(m+s)!(A_s+i)_{s+m}} \\ &+ \frac{(-1)^{i+1}(A_s + 2i + 1)}{(i+1)!(m+s-i-1)!(A_s+i)_{s+m+1}} \\ &= (-1)^{i+1} \frac{(A_s + 2i + 1)(m+s) - (A_s + i + s + m)(i+1)}{(A_s+i)_{s+m+1}(m+s-i-1)!(i+1)!} \\ &= (-1)^{i+1} \frac{(A_s + i)(m+s) - (A_s + i)(i+1)}{(A_s+i)_{s+m+1}(m+s-i-1)!(i+1)!} \\ &= \frac{(-1)^{i+1}(A_s + i)(m+s-i-1)}{(A_s+i)_{s+m+1}(m+s-i-1)!(i+1)!} \\ &= \frac{(-1)^{i+1}(m+s-i-1)}{(A_s+i+1)_{s+m}(m+s-i-1)!(i+1)!} \end{aligned}$$

$$= (-1)^{i+1} \frac{(m+s-i-1)_{i+1}}{(i+1)!(m+s)!(A_s+i+1)_{s+m}}.$$

Substituting the value $A_{s,i}$ from (2.1.21) into the equality (2.1.17) and and considering that

$$\binom{m-1+s-i}{m-1} \frac{(m+s-i)_i}{i!} = \frac{(m-1+s)!}{(m-1)!s!} \frac{s!}{i!(s-i)!} = \binom{m-1+s}{m-1} \binom{s}{i}$$

and $A_s = l - 2s + n/2$ and replacing the index i with k we get

$$\begin{aligned} u_0(x) &= (|x|^2-1)^m \sum_{s=0}^{[l/2]} \frac{\Delta^s Q_l(x)}{4^{s+m}(m+s)!} \\ &\qquad \times \sum_{i=0}^{s} (-1)^i |x|^{2i} \binom{m-1+s-i}{m-1} \frac{(m+s-i)_i}{i!(A_s+i)_{s+m}} \\ &= (|x|^2-1)^m \sum_{s=0}^{[l/2]} \binom{s+m-1}{m-1} \frac{\Delta^s Q_l(x)}{4^{s+m}(s+m)!} \\ &\qquad \times \sum_{k=0}^{s} (-1)^k \binom{s}{k} \frac{|x|^{2k}}{(l-2s+k+n/2)_{s+m}}. \end{aligned}$$

The theorem is proved. □

Lemma 2.5. *Let $v(x) = (|x|^2-1)^m S(x)$, where $S(x) \in C^{m-1}(\overline{S})$ and $m \in \mathbb{N}$. Then $v(x)$ satisfies the condition*

$$P_{m-1}\Big(\frac{\partial}{\partial \nu}\Big) v(x)|_{\partial S} = 0,$$

where $P_{m-1}(t)$ is an arbitrary polynomial of degree $m-1$, which means $v(x)$ satisfies the homogeneous Dirichlet conditions on ∂S

$$v|_{\partial S} = 0, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial S} = 0, \ldots, \frac{\partial^{m-1} v}{\partial \nu^{m-1}}\Big|_{\partial S} = 0.$$

Proof. Indeed, due to the equality $\Lambda(uv) = v\Lambda u + u\Lambda v$ we can write

$$\begin{aligned} &\Lambda v(x) \\ &\quad = \Lambda(|x|^2-1)^m S(x) = mS(x)(|x|^2-1)^{m-1}\Lambda(|x|^2-1) + (|x|^2-1)^m \Lambda S(x) \end{aligned}$$

$$= (|x|^2-1)^{m-1}(2mS(x)|x|^2 + (|x|^2-1)\Lambda S(x)) \equiv (|x|^2-1)^{m-1}S_1(x),$$

where $S_1(x) \in C^{m-2}(\overline{S})$ and that means $\Lambda v(x)|_{\partial S} = 0$. Continuing similarly, we find $\Lambda^{m-1}v(x) = \Lambda^{m-1}(|x|^2-1)^m S(x) = (|x|^2-1)S_{m-1}(x)$, where $S_{m-1}(x) \in C(\overline{S})$ and therefore $\Lambda^{m-1}v(x)|_{\partial S} = 0$. Therefore, if $P_{m-1}(t)$ is an arbitrary polynomial of degree $m-1$, then by Corollary 2.1 from Lemma 2.1 we have

$$P_{m-1}\Big(\frac{\partial}{\partial \nu}\Big)v(x)_{|\partial S} = P_{[m-1]}(\Lambda)v(x)|_{\partial S} = 0.$$

The lemma is proved. □

Based on Lemma 2.5, the polynomial $u_0(x)$ from Theorem 2.1 satisfies all homogeneous Dirichlet conditions (2.1.2).

Example 2.2. The solution to the Dirichlet problem (2.1.1)-(2.1.2) for $m = 4$ and $Q_6(x) = x_1^3x_2x_3^2$, written in the form (2.1.20) is easily calculated using “Mathematica” and has the form

$$\begin{aligned} u(x_1,x_2,x_3) = &-\frac{x_1x_2(x_1^2+x_2^2+x_3^2-1)^4}{83805321600}\big(-190 + 102x_1^4 - 30x_2^4 \\ &- 616x_3^2 + 366x_3^4 + x_1^2(-112 + 72x_2^2 - 1677x_3^2) + 28x_2^2(5+12x_3^2)\big). \end{aligned}$$

Let us slightly further transform the polynomial $u_0(x)$, which is a solution to the Dirichlet problem (2.1.1)-(2.1.2) with $Q(x) = Q_l(x)$ so that we can then obtain an equality for the case of an inhomogeneous polynomial $Q(x)$.

Lemma 2.6. *The following equality holds true*

$$\begin{aligned} &u_0(x) \\ &\quad = \frac{(|x|^2-1)^m}{2(2m-2)!!}\sum_{s=0}^{[l/2]}\int_0^1 \frac{(1-t|x|^2)^s(1-t)^{s+m-1}}{(2s)!!(2s+2m)!!}\Delta^s Q_l(tx)t^{n/2-1}\,dt. \quad (2.1.22) \end{aligned}$$

Proof. Using the equality (2.1.20) we write

$$\begin{aligned} u_0(x) = \frac{(|x|^2-1)^m}{2(2m-2)!!}&\sum_{s=0}^{[l/2]}\frac{\Delta^s Q_l(x)}{2^s(2s+2m)!!} \\ &\times \sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{(s+1)(s+2)\cdots(s+m-1)}{(A-2s+k)_{s+m}}\,|x|^{2k}, \quad (2.1.23) \end{aligned}$$

where $A = l + n/2$. Let us transform the internal sum in the resulting expression. Using the definition of the Pochhammer symbol $(a)_k$ given in Definition 1.3, the property of the gamma function $\Gamma(x+1) = x\Gamma(x)$ and the connection between gamma $\Gamma(x)$ and beta $\mathrm{B}(x)$ Euler functions we can write

$$\frac{1}{(A-2s+k)_{s+m}} = \frac{1}{(A-2s+k)\cdots(A-s+k+m-1)}$$
$$= \frac{\Gamma(l+n/2-2s+k)}{\Gamma(l+n/2-s+k+m)} = \frac{\mathrm{B}(s+m, l+n/2-2s+k)}{\Gamma(s+m)}$$
$$= \frac{1}{(s+m-1)!}\int_0^1 (1-t)^{s+m-1} t^{l+n/2+k-2s-1}\, dt.$$

Using this equality, we write the internal sum in (2.1.23) in the form

$$\frac{1}{s!}\int_0^1 (1-t)^{s+m-1} t^{l+n/2-2s-1} \sum_{k=0}^{s} (-1)^k \binom{s}{k} |x|^{2k} t^k\, dt$$
$$= \frac{1}{s!}\int_0^1 (1-t)^{s+m-1}(1-t|x|^2)^s\, t^{l-2s} t^{n/2-1}\, dt.$$

Therefore, the polynomial $u_0(x)$ can be written in the form

$$u_0(x) = \frac{(|x|^2-1)^m}{2(2m-2)!!}\sum_{s=0}^{[l/2]}\int_0^1 \frac{(1-t)^{s+m-1}(1-t|x|^2)^s}{(2s+2m)!!(2s)!!}\Delta^s Q_l(tx) t^{n/2-1}\, dt,$$

which coincides with the equality (2.1.22). The lemma is proved. □

Now we can find a solution to the Dirichlet problem (2.1.1)-(2.1.2) with the inhomogeneous polynomial $Q(x)$.

Theorem 2.2. *The solution to the homogeneous Dirichlet problem* (2.1.1)-(2.1.2) *can be written in the form*

$$u(x) = \frac{(|x|^2-1)^m}{2(2m-2)!!}$$
$$\times \int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^{s+m-1}}{(2s)!!(2s+2m)!!}\Delta^s Q(\alpha x)\, \alpha^{n/2-1}\, d\alpha, \quad (2.1.24)$$

where the upper limit of summation over s is finite and limited to the value $[(\deg Q(x))/2]$.

Proof. Let $Q(x)$ be an arbitrary polynomial. Let us represent it as a sum of homogeneous terms $Q(x) = \sum_l Q_l(x)$ and denote by $u_l(x)$ the polynomial solution of the Dirichlet problem (2.1.1)-(2.1.2) with the right-hand side $Q(x) = Q_l(x)$. Then it is obvious that the desired solution has the form $u(x) = \sum_l u_l(x)$. For uniformity of notation, let us replace the upper summation limit in the equality (2.1.22) with ∞. Then we can write

$$\begin{aligned} u(x) &= \sum_l u_l(x) \\ &= \frac{(|x|^2-1)^m}{2(2m-2)!!}\sum_{s=0}^{\infty}\int_0^1 \frac{(1-\alpha|x|^2)^s(1-\alpha)^{s+m-1}}{(2s)!!(2s+2m)!!}\alpha^{n/2-1}\Delta^s\sum_l Q_l(\alpha x)\,d\alpha \\ &\quad = \frac{(|x|^2-1)^m}{2(2m-2)!!}\int_0^1\sum_{s=0}^{\infty}\frac{(1-\alpha|x|^2)^s(1-\alpha)^{s+m-1}}{(2s)!!(2s+2m)!!}\Delta^s Q(\alpha x)\,\alpha^{n/2-1}\,d\alpha. \end{aligned}$$

It is clear that $s \le [(\deg Q(x))/2]$. The theorem is proved. □

Remark 2.1. The Green's function (operator) of the Dirichlet problem (2.1.1)-(2.1.2) in the unit ball in the case of polynomial functions $Q(x)$ can be written in the form

$$G[Q](x;\alpha) = \frac{(|x|^2-1)^m}{2(2m-2)!!}\sum_{s=0}^{\infty}\frac{(1-\alpha|x|^2)^s(1-\alpha)^{s+m-1}}{(2s)!!(2s+2m)!!}\,\alpha^{n/2-1}\Delta^s Q(\alpha x)$$

and then the solution (2.1.24) of the Dirichlet problem in the ball (2.1.1)-(2.1.2) has the form

$$u(x) = \int_0^1 G[Q](x;\alpha)\,d\alpha.$$

Remark 2.2. The solution (2.1.24) of the Dirichlet problem (2.1.1)-(2.1.2) in the unit ball is obtained for the polynomial right-hand side $Q(x)$, but this circumstance is not significantly used in the derivation of this formula. This solution will be valid even if $Q(x)$ is not a polynomial, but a function for which the operator series in (2.1.24) converges. For example, if $Q(x) = e^{\omega\cdot x}$, where $\omega \in \mathbb{R}^n$, then since $\Delta^s Q(x) = |\omega|^{2s}e^{\omega\cdot x}$ we have

$$u(x) = \frac{(|x|^2-1)^m}{2(2m-2)!!}\int_0^1\sum_{s=0}^{\infty}\frac{(1-\alpha|x|^2)^s(1-\alpha)^{m+s-1}}{(2s)!!(2s+2m)!!}|\omega|^{2s}e^{\alpha\omega\cdot x}\alpha^{n/2-1}\,d\alpha.$$

If we introduce an entire function of the form

$$h_m(t) = \sum_{s=0}^{\infty}\frac{t^s}{(2s)!!(2s+2m)!!},$$

then we obtain a solution to the homogeneous Dirichlet problem (2.1.1)-(2.1.2) in the form

$$u(x)=\frac{(|x|^2-1)^m}{2(2m-2)!!}\int_0^1 h_m\left(|\omega|^2(1-\alpha|x|^2)(1-\alpha)\right)e^{\alpha\omega\cdot x}(1-\alpha)^{m-1}\alpha^{n/2-1}\,d\alpha.$$

Example 2.3. Let us check the equality (2.1.24). Let $Q(x)=x_i$ in the Dirichlet problem (2.1.1)-(2.1.2). Then the sum from the equality (2.1.24) has only one term at $s=0$. We get

$$\begin{aligned}u_0(x)&=\frac{(|x|^2-1)^m}{2(2m-2)!!}\int_0^1\frac{(1-\alpha)^{m-1}}{(2m)!!}(\alpha x_i)\alpha^{n/2-1}d\alpha\\&=x_i\frac{\mathrm{B}(m,n/2+1)(|x|^2-1)^m}{2(2m-2)!!(2m)!!}=x_i\frac{(|x|^2-1)^m}{(2m)!!}\frac{\Gamma(n/2+1)}{2^m\Gamma(m+n/2+1)}\\&=x_i\frac{(|x|^2-1)^m}{(2m)!!}\frac{1}{2^m(n/2+1)_m}=x_i\frac{(|x|^2-1)^m}{(2,2)_m(n+2,2)_m},\end{aligned}$$

where $\mathrm{B}(x)$ is Euler beta function. Let us check this solution. By Lemma 2.5 the homogeneous conditions (2.1.2) are satisfied. Next, using the equality

$$\Delta\left(|x|^{2s}P(x)\right)=2s|x|^{2s-2}\left(2\Lambda+2s+n-2\right)P(x)+|x|^{2s}\Delta P(x)\tag{2.1.25}$$

for $P(x)=x_i$, $s=m$, using the properties of the operator Λ and the generalized Pochhammer symbol $(n+2,2)_m$ we calculate

$$\begin{aligned}\Delta^m u_0(x)&=\Delta^{m-1}\frac{\Delta(|x|^{2m}x_i)}{(2,2)_m(n+2,2)_m}=\Delta^{m-1}\frac{2m(n+2m)|x|^{2m-2}x_i}{(2,2)_m(n+2,2)_m}\\&=\Delta^{m-1}\frac{|x|^{2m-2}x_i}{(2,2)_{m-1}(n+2,2)_{m-1}}=\cdots=\frac{\Delta(|x|^2x_i)}{2(n+2)}=x_i.\end{aligned}$$

Note that the equality (2.1.4) is conceptually similar to the equality for representing the solution to the Cauchy problem for an ordinary differential equation with constant coefficients from the paper [75].

Polynomial solutions to the inhomogeneous Dirichlet problem for the homogeneous equation are given in [80, 86, 91]. They are obtained using the idea from Theorem 2.15, and can also be easily obtained using the formulas from Section 2.4.

2.2 Green's function for the biharmonic equation

In this section, a representation of the Green's function for the biharmonic equation in the unit ball, which is used in the next section, is obtained. First, in Lemmas 2.7 and 2.8 from Section 2.2.1 a representation of the Green's function of the Dirichlet problem for the Poisson equation is given in the form of series in the complete system of harmonic polynomials $\{H_m^{(i)}(x) : i = 1, \dots, h_m, m \in \mathbb{N}_0\}$ orthogonal on the unit sphere. Then, in Theorem 2.3 from Section 2.2.2, using the representation of the generating function for the Gegenbauer polynomials $C_k^\lambda[t]$ of order $\lambda = n/2 - 2$ in the form (2.2.9) and connections between the Gegenbauer polynomials of orders $n/2-2$ and $n/2-1$ from Lemma 2.11 and the expansion (2.2.12), a representation of the elementary solution of the biharmonic equation $E_4(x, \xi)$ in the form (2.2.11) is obtained. After this, in Theorem 2.4 from Section 2.2.3, based on Lemmas 2.12-2.14, the Green's function $G_4(x, \xi)$ of the Dirichlet problem (2.2.3),(2.2.4) in the form (2.2.20) is constructed. Theorem 2.5 yields the expansion of the Green's function $G_4(x, \xi)$ in the system $\{H_m^{(i)}(x) : i = 1, \dots, h_m, m \in \mathbb{N}_0\}$. Theorem 2.6 calculates the integral over the unit ball S with the kernel of the Green's function $G_4(x, \xi)$ of a function of the form $f(x) = |x|^{2l} H_m(x)$, where $H_m(x)$ is a homogeneous harmonic polynomial of degree $m \in \mathbb{N}_0$ and $l \in \mathbb{R}_+$. Finally, Theorem 2.7 constructs the Green's function for $n = 2$. The main results of Section 2.2 are published in papers [70, 71, 91, 105, 106, 112, 119].

In connection with the biharmonic equation, we note the results of Sections 3.5 and 3.6 presented below and devoted to the solvability conditions for some interesting boundary value problems in a ball for the biharmonic equation.

It is well known (see, for example, [28]) that the Green's function of the Dirichlet problem for the Poisson equation in a ball for $n \geq 2$ has the form

$$G(x, \xi) = E(x, \xi) - E\left(\frac{x}{|x|}, |x|\xi\right), \tag{2.2.1}$$

where

$$E(x, \xi) = \begin{cases} \dfrac{1}{n-2}|x - \xi|^{2-n}, & n > 2 \\ -\ln|x - \xi|, & n = 2 \end{cases} \tag{2.2.2}$$

is the elementary solution to the Laplace equation [28].

2.2.1 Auxiliary statements

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$ and $f \in C^1(\bar{S})$. Consider in S the homogeneous Dirichlet problem for the biharmonic equation

$$\Delta^2 u(x) = f(x), \quad x \in S, \tag{2.2.3}$$

$$u|_{\partial S} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0, \tag{2.2.4}$$

where ν is the unit outer normal to the sphere ∂S.

Let $\{H_k^{(i)}(x) : i = 1, \ldots, h_k, k \in \mathbb{N}_0\}$ be a complete system of homogeneous degree $k \in \mathbb{N}_0$ orthogonal spherical harmonics from the definition of 1.5, normalized so that

$$\int_{\partial S} (H_k^{(i)}(\xi))^2 \, ds_\xi = \omega_n,$$

where $h_k = \frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}$, $n > 2$; $h_k = 2$, $n = 2$ is the dimension of the basis of homogeneous harmonic polynomials of degree k, and ω_n is the area of the unit sphere ∂S.

Lemma 2.7. *The following representation of the elementary solution $E(x,\xi)$ of the Laplace equation from* (2.2.2) *is valid*

$$E(x,\xi) = \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad n > 2,$$

$$E(x,\xi) = -\ln|x| + \sum_{k=1}^{\infty} \frac{|x|^{-2k}}{2k} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad n = 2,$$

for $|\xi| < |x|$. Here the series 1) converge uniformly in ξ for $|\xi| < a < |x|$ and fixed x; 2) converge uniformly in x for $|\xi| < a < |x|$ and fixed ξ. If $|x| < |\xi|$, then in the resulting representation on the right, the variables x and ξ must be swapped.

Proof. Let us prove the validity of the first equality of the lemma. Let $|\xi| < |x|$ and $C_n^\nu[t]$ be the Gegenbauer polynomial. Then, if $|x| = 1$ and $|\xi| < 1$ then the equality holds true [15]

$$|\xi - x|^{2-n} = \sum_{k=0}^{\infty} C_k^{n/2-1}\left[\left(\frac{\xi}{|\xi|}, x\right)\right] |\xi|^k.$$

It is known [15] that if $\left\{S_k^{(i)}(x) : i = 1, \ldots, h_k\right\}$ is an arbitrary complete orthonormal system of spherical harmonics of degree k, then

$$\frac{C_k^{n/2-1}[(x,\eta)]}{C_k^{n/2-1}[1]} = \frac{\omega_n}{h_k} \sum_{i=1}^{h_k} S_k^{(i)}(x) S_k^{(i)}(\eta),$$

where $C_k^{n/2-1}[1] = C_{k+n-3}^k$ and $|x| = |\eta| = 1$. Hence, putting $S_k^{(i)}(x) = (1/\sqrt{\omega_n}) H_k^{(i)}(x)$, we obtain

$$|x-\xi|^{2-n} = \sum_{k=0}^{\infty} \frac{n-2}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi). \tag{2.2.5}$$

If we now notice that $E(x,\xi) = |x|^{2-n} E(x/|x|, \xi/|x|)$ and $|\xi/|x|| < 1$, then the equality (2.2.5) is easily reduced to the required form. The statement about the uniform convergence of series can be found in Lemma 3.34 on page 299.

Let us prove the second representation of the lemma. Let us transform the expression

$$F(x,\xi) = -\ln|x| + \sum_{k=1}^{\infty} \frac{|x|^{-2k}}{2k} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi),$$

for $n = 2$ and $|\xi| < |x|$. In this case $h_k = 2$. Assuming $(x_1, x_2) = \rho(\cos\varphi, \sin\varphi)$ and $(\xi_1, \xi_2) = r(\cos\psi, \sin\psi)$ we have $H_k^{(1)}(x) = \sqrt{2}\rho^k \cos k\varphi$, $H_k^{(2)}(x) = \sqrt{2}\rho^k \sin k\varphi$ and, accordingly, $H_k^{(1)}(\xi) = \sqrt{2} r^k \cos k\psi$, $H_k^{(2)}(\xi) = \sqrt{2} r^k \sin k\psi$, since these polynomials form a basis and are properly normalized

$$\int_{\partial S} (H_k^{(1)}(x))^2 \, ds_x = 2 \int_0^{2\pi} \cos^2 k\varphi = 2\pi = \omega_2.$$

Therefore, for $r < \rho$ we have

$$F(x,\xi) = -\ln\rho + \sum_{k=1}^{\infty} \frac{\rho^{-2k}}{k} \rho^k r^k (\cos k\varphi \cos k\psi + \sin k\varphi \sin k\psi)$$

$$= -\ln\rho + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{r}{\rho}\right)^k \cos k(\varphi - \psi).$$

The following Fourier series expansion is known

$$\ln(1 - 2q\cos\varphi + q^2) = -2 \sum_{n=1}^{\infty} \frac{q^n}{n} \cos n\varphi$$

and is true for $|q| < 1$. From it we find

$$
\begin{aligned}
F(x,\xi) &= -\ln\rho - \frac{1}{2}\ln\Big(1 - 2\frac{r}{\rho}\cos(\varphi-\psi) + \frac{r^2}{\rho^2}\Big) = -\ln\rho \\
&\quad - \frac{1}{2}\ln\frac{1}{\rho^2}\big(\rho^2 - 2r\rho\cos(\varphi-\psi) + r^2\big) = -\ln\rho + \ln\rho \\
&\quad - \frac{1}{2}\ln\big(\rho^2 - 2r\rho\cos(\varphi-\psi) + r^2\big) = -\frac{1}{2}\ln\big(\rho^2\cos^2\varphi - 2r\rho\cos\varphi\cos\psi \\
&\quad + r^2\cos^2\psi + \rho^2\sin^2\varphi - 2r\rho\sin\varphi\sin\psi + r^2\sin^2\psi\big) \\
&= -\frac{1}{2}\ln\big((\rho\cos\varphi - r\cos\psi)^2 + (\rho\sin\varphi - r\sin\psi)^2\big) \\
&= -\ln\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2} = -\ln|x-\xi| = E(x,\xi).
\end{aligned}
$$

If we repeat the derivation of this equality in the case of $|x| < |\xi|$ (in this case $q = \rho/r < 1$), we obtain the symmetry of $E(x,\xi)$. □

Lemma 2.8. *For $x,\xi \in S$ and $|\xi| < |x|$ the following representation of the Green's function $G(x,\xi)$ of the Dirichlet problem for the Poisson equation holds*

$$
\begin{aligned}
G(x,\xi) &= \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)} - 1}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad n > 2, \\
G(x,\xi) &= -\ln|x| + \frac{1}{2}\sum_{k=1}^{\infty} \frac{|x|^{-(2k)} - 1}{k} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad n = 2.
\end{aligned}
$$

For $|\xi| > |x|$ in the resulting representation on the right, the variables x and ξ must be swapped.

Proof. Since $\big|x/|x|\big| = 1$, and $\big||x|\xi\big| = |x||\xi| < 1$, then by Lemma 2.7

$$
E\Big(\frac{x}{|x|}, |x|\xi\Big) = \sum_{k=0}^{\infty} \frac{1}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \tag{2.2.6}
$$

therefore from the equality (2.2.1) for $n > 2$ and $|\xi| < |x|$ we find

$$
G(x,\xi) = \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)} - 1}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi).
$$

The second formula is derived similarly. The lemma is proved. □

Let us now study the biharmonic equation. Consider the following function defined for $n \geq 2$

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n > 4,\ n = 3 \\ -\dfrac{1}{4}\ln|x-\xi|, & n = 4 \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi| - 1\big), & n = 2 \end{cases}, \tag{2.2.7}$$

which, by analogy with the function $E(x,\xi)$, we call the elementary solution of the biharmonic equation.

Lemma 2.9. *The function $E_4(x,\xi)$, defined for $\xi \neq x$, satisfies the equality*

$$\Delta_\xi E_4(x,\xi) = -E(x,\xi)$$

and therefore is biharmonic for $\xi \neq x$.

Proof. Let $n > 4$ or $n = 3$. It is easy to check that

$$\frac{\partial}{\partial \xi_i}|x-\xi|^{4-n} = \left(2 - \frac{n}{2}\right) 2\,(\xi_i - x_i)\,|x-\xi|^{2-n} = (4-n)\,(\xi_i - x_i)\,|x-\xi|^{2-n}$$

and that means

$$\frac{\partial^2}{\partial \xi_i^2}|x-\xi|^{4-n} = (4-n)\left(|x-\xi|^{2-n} + (2-n)\,(\xi_i - x_i)^2\,|x-\xi|^{-n}\right).$$

That is why

$$\Delta_\xi |x-\xi|^{4-n} = (4-n)\sum_{i=1}^{n}\left(|x-\xi|^{2-n} + (2-n)\,(\xi_i - x_i)^2\,|x-\xi|^{-n}\right)$$
$$= (4-n)\left(n|x-\xi|^{2-n} + (2-n)|x-\xi|^{-n}\sum_{i=1}^{n}(\xi_i - x_i)^2\right) = 2(4-n)|x-\xi|^{2-n}.$$

This implies the equality to be proved

$$\Delta_\xi E_4(x,\xi) = -\frac{1}{(n-2)}|x-\xi|^{2-n} = -E(x,\xi).$$

For $n = 4$ we have

$$\frac{\partial}{\partial \xi_i}\ln|x-\xi| = \frac{(\xi_i - x_i)}{|x-\xi|^2}, \quad \frac{\partial^2}{\partial \xi_i^2}\ln|x-\xi| = \frac{|x-\xi|^2 - 2(\xi_i - x_i)^2}{|x-\xi|^4}$$

and that means

$$\Delta_\xi\big(-\frac{1}{4}\ln|x-\xi|\big)=-\frac{1}{4}\frac{2|x-\xi|^2}{|x-\xi|^4}=-\frac{1}{2}\frac{1}{|x-\xi|^2}=-E(x,\xi).$$

For $n=2$ we have

$$\frac{\partial}{\partial\xi_i}|x-\xi|^2\ln|x-\xi|=2(\xi_i-x_i)\ln|x-\xi|+|x-\xi|^2\frac{\xi_i-x_i}{|x-\xi|^2},$$
$$\frac{\partial^2}{\partial\xi_i^2}|x-\xi|^2\ln|x-\xi|=2\ln|x-\xi|+4\frac{(\xi_i-x_i)^2}{|x-\xi|^2}$$
$$+|x-\xi|^2\frac{|x-\xi|^2-2(\xi_i-x_i)^2}{|x-\xi|^4}$$

and that means

$$\Delta_\xi\big(|x-\xi|^2\ln|x-\xi|\big)=4\ln|x-\xi|+4.$$

That is why

$$\Delta_\xi\Big(\frac{|x-\xi|^2}{4}\big(\ln|x-\xi|-1\big)\Big)=\ln|x-\xi|+1-1=-E(x,\xi).$$

The lemma is proved. □

Let $C_k^\lambda[t]$ denote the Gegenbauer polynomial of degree k. It can be represented as [15]

$$C_k^\lambda[t]=\sum_{m=0}^{[k/2]}\frac{(-1)^m(\lambda)_{k-m}}{m!(k-2m)!}(2t)^{k-2m}, \tag{2.2.8}$$

where $(\lambda)_k=\lambda(\lambda+1)\dots(\lambda+k-1)$, and $(\lambda)_0=1$. It is easy to calculate that $C_0^\lambda[t]=1$ and

$$C_1^\lambda[t]=\sum_{m=0}^{0}\frac{(-1)^0(\lambda)_1}{0!1!}(2t)^1=2\lambda t.$$

In what follows we assume that $C_k^\lambda[t]=0$ for $k<0$. For $|\xi|=1$, $|x|<1$ and $m>2$ the following representation is true [73]

$$|x-\xi|^{2-m}=\sum_{k=0}^{\infty}C_k^{m/2-1}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k,$$

where (x,ξ) denotes the scalar product in $\mathbb{R}^n$. If we put $m = n-2$ here and assume that $n > 4$, then we get

$$|x-\xi|^{4-n} = \sum_{k=0}^{\infty} C_k^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k. \tag{2.2.9}$$

Lemma 2.10. *Next function*

$$C_k^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k$$

for $n > 4$ and $|\xi| = 1$ is a biharmonic polynomial of degree $k \in \mathbb{N}_0$ in x.

Proof. Using the representation of the Gegenbauer polynomial (2.2.8), we can write

$$\begin{aligned} C_k^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k &= \sum_{m=0}^{[k/2]} \frac{(-1)^m (n/2-2)_{k-m}}{m!(k-2m)!}|x|^{2m}(2(x,\xi))^{k-2m} \\ &= \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m)!}|x|^{2m}(x,\xi)^{k-2m}, \end{aligned}$$

where $(n,2)_k = n(n+2)\dots(n+2k-2)$ is the generalized Pochhammer symbol (Definition 1.3). This shows that this is a polynomial of degree k in x. Using equalities

$$\Delta(x,\xi)^m = m(m-1)|\xi|^2(x,\xi)^{m-2}, \quad \Delta|x|^{2l} = 2l(2l+n-2)|x|^{2l-2}$$

it is easy to see that

$$\Delta|x|^{2l}(x,\xi)^m = m(m-1)|\xi|^2|x|^{2l}(x,\xi)^{m-2} + 2l(2l+n-2+2m)|x|^{2l-2}(x,\xi)^m.$$

It follows that

$$\begin{aligned} \Delta|x|^{2m}(x,\xi)^{k-2m} &= (k-2m)(k-2m-1)|\xi|^2|x|^{2m}(x,\xi)^{k-2m-2} \\ &\quad + 2m(2k-2m+n-2)|x|^{2m-2}(x,\xi)^{k-2m} \end{aligned}$$

and therefore the following chain of equalities is true

$$\Delta C_k^{n/2-2}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k = \Delta \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m)!}|x|^{2m}(x,\xi)^{k-2m}$$

$$
\begin{aligned}
&= \sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m-2)!} |\xi|^2 |x|^{2m} (x,\xi)^{k-2m-2} \\
&\quad + \sum_{m=1}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m+1}}{(2m-2)!!(k-2m)!} |x|^{2m-2} (x,\xi)^{k-2m} \\
&\quad + 2\sum_{m=1}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m-2)!!(k-2m)!} |x|^{2m-2} (x,\xi)^{k-2m} \\
&= (|\xi|^2-1) \sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m)!!(k-2m-2)!} |x|^{2m} (x,\xi)^{k-2m-2} \\
&\quad - 2\sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-4,2)_{k-m-1}}{(2m)!!(k-2m-2)!} |x|^{2m} (x,\xi)^{k-2m-2} \\
&= -2(n-4) \sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-2,2)_{k-m-2}}{(2m)!!(k-2m-2)!} |x|^{2m} (x,\xi)^{k-2m-2} \\
&\qquad\qquad = -2(n-4) C_{k-2}^{n/2-1}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^{k-2}.
\end{aligned}
$$

Similarly, it is not difficult to obtain

$$
\begin{aligned}
\Delta C_k^{n/2-1}\Big[\Big(\frac{x}{|x|},\xi\Big)\Big]|x|^k &= \Delta \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-2,2)_{k-m}}{(2m)!!(k-2m)!} |x|^{2m} (x,\xi)^{k-2m} \\
&= (|\xi|^2-1) \sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-2,2)_{k-m}}{(2m)!!(k-2m-2)!} |x|^{2m} (x,\xi)^{k-2m-2} = 0.
\end{aligned}
$$

Therefore, the polynomial $C_k^{n/2-2}\big[(x/|x|,\xi)\big]|x|^k$ is biharmonic for $|\xi| = 1$. The lemma is proved. □

Lemma 2.11. *For $n > 4$ and $k \in \mathbb{N}_0$ the equality holds true*

$$
C_k^{n/2-2}[t] = \frac{n-4}{2k+n-4}\big(C_k^{n/2-1}[t] - C_{k-2}^{n/2-1}[t]\big). \tag{2.2.10}
$$

Proof. Let $n \ge 2$ first. Equalities are valid

$$
C_k^{n/2-1}[t] - C_{k-2}^{n/2-1}[t] = \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-2,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m}
$$

$$
\begin{aligned}
-\sum_{m=0}^{[k/2-1]} \frac{(-1)^m (n-2,2)_{k-m-2}}{(2m)!!(k-2m-2)!} t^{k-2m-2} &= \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-2,2)_{k-m}}{(2m)!!(k-2m)!} t^{k-2m} \\
&+ \sum_{m=1}^{[k/2]} \frac{(-1)^m (n-2,2)_{k-m-1}}{(2m-2)!!(k-2m)!} t^{k-2m} \\
&= \frac{2k+n-4}{n-4} \sum_{m=1}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m-2)!!(k-2m)!} t^{k-2m} + \frac{(n-2,2)_k}{k!} t^k .
\end{aligned}
$$

The last term can be converted to the form

$$
\frac{(n-2,2)_k}{k!} t^k = \frac{2k+n-4}{n-4} \frac{(n-4,2)_k}{k!} t^k
$$

and that means

$$
\begin{aligned}
&C_k^{n/2-1}[t] - C_{k-2}^{n/2-1}[t] \\
&\quad = \frac{2k+n-4}{n-4} \sum_{m=0}^{[k/2]} \frac{(-1)^m (n-4,2)_{k-m}}{(2m-2)!!(k-2m)!} t^{k-2m} = \frac{2k+n-4}{n-4} C_k^{n/2-2}[t].
\end{aligned}
$$

This implies the equality (2.2.10). If we assume, as noted above, that $C_{-2}^{n/2-1}[t] = 0$ and $C_{-1}^{n/2-1}[t] = 0$, then the equality (2.2.10) is true for $k = 0$ (the values of $C_0^{\lambda}[t]$ and $C_1^{\lambda}[t]$ are calculated above)

$$
C_0^{n/2-2}[t] = 1 = \frac{n-4}{n-4} C_0^{n/2-1}[t]
$$

and for $k = 1$

$$
\begin{aligned}
C_1^{n/2-2}[t] = 2(n/2-2)t = (n-4)t &= \frac{n-4}{2+n-4}(n-2)t \\
&= \frac{n-4}{2+n-4} 2(n/2-1)t = \frac{n-4}{2+n-4} C_1^{n/2-1}[t].
\end{aligned}
$$

The lemma is completely proved. □

2.2.2 Representation of the elementary solution $E_4(x, \xi)$

Now we can construct an important representation of the function $E_4(x, \xi)$.

Theorem 2.3. *Let $n > 4$. For the function $E_4(x,\xi)$ for $|x| < |\xi|$ the following representation is valid*

$$E_4(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|\xi|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|\xi|^2}{2k+n-4} - \frac{|x|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi). \quad (2.2.11)$$

For $|\xi| < |x|$, the corresponding equality for $E_4(x,\xi)$ is obtained from the equality (2.2.11) *by interchanging the variables x and ξ.*

Proof. It is known [15] that if $\{S_k^{(i)}(\xi) : i = 1,\dots,h_k\}$ is an arbitrary complete orthonormal system of spherical harmonics of degree k and $|x| = |\xi| = 1$, then the equality

$$\frac{C_k^{n/2-1}[(x,\xi)]}{C_k^{n/2-1}[1]} = \frac{\omega_n}{h_k}\sum_{i=1}^{h_k} S_k^{(i)}(x)S_k^{(i)}(\xi),$$

holds true, where $C_k^{n/2-1}[1] = C_{k+n-3}^k$. From here, putting $S_k^{(i)}(x) = (1/\sqrt{\omega_n})H_k^{(i)}(x)$ we get

$$\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi) = \frac{2k+n-2}{n-2}C_k^{n/2-1}[(x,\xi)]. \quad (2.2.12)$$

Therefore, for $|\xi| = 1$ and $|x| < 1$ we have

$$C_k^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \frac{n-2}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi).$$

Consider the biharmonic polynomial $C_k^{n/2-2}\big[\big(x/|x|,\xi\big)\big]|x|^k$ from Lemma 2.10. Let us transform it using the equality (2.2.10) from Lemma 2.11. We get

$$C_k^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \frac{n-4}{2k+n-4}\left(C_k^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k - |x|^2 C_{k-2}^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^{k-2}\right).$$

Let us recall the equality (2.2.9). From it, for $|\xi| = 1$ and $|x| < 1$, we obtain

$$|x-\xi|^{4-n} = \sum_{k=0}^{\infty} C_k^{n/2-2}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k = \sum_{k=0}^{\infty}\frac{n-4}{2k+n-4}C_k^{n/2-1}\left[\left(\frac{x}{|x|},\xi\right)\right]|x|^k$$

$$
\begin{aligned}
&- |x|^2 \sum_{k=2}^{\infty} \frac{n-4}{2k+n-4} C_{k-2}^{n/2-1}\left[\left(\frac{x}{|x|}, \xi\right)\right] |x|^{k-2} \\
&= \sum_{k=0}^{\infty} \frac{n-4}{2k+n-4} C_{k}^{n/2-1}\left[\left(\frac{x}{|x|}, \xi\right)\right] |x|^{k} - |x|^2 \sum_{k=0}^{\infty} \frac{n-4}{2k+n} C_{k}^{n/2-1}\left[\left(\frac{x}{|x|}, \xi\right)\right] |x|^{k} \\
&= \sum_{k=0}^{\infty} \frac{n-2}{2k+n-2}\left(\frac{n-4}{2k+n-4} - |x|^2 \frac{n-4}{2k+n}\right) \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi).
\end{aligned}
$$

Let us remove the constraint $|\xi| = 1$, but at the same time we assume that $|x| < |\xi|$. In this case, $|x/|\xi|| < 1$ and $|\xi/|\xi|| = 1$ and therefore

$$
\begin{aligned}
|x-\xi|^{4-n} &= |\xi|^{4-n}\left|\frac{x}{|\xi|} - \frac{\xi}{|\xi|}\right|^{4-n} = (n-2)(n-4) \\
&\times \sum_{k=0}^{\infty} \frac{|\xi|^{-(2k+n-4)}}{2k+n-2}\left(\frac{1}{2k+n-4} - \frac{|x|^2/|\xi|^2}{2k+n}\right) \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi) \\
&= (n-2)(n-4) \sum_{k=0}^{\infty} \frac{|\xi|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|\xi|^2}{2k+n-4} - \frac{|x|^2}{2k+n}\right) \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi),
\end{aligned}
$$

whence, taking into account the definition (2.2.7) of the function $E_4(x, \xi)$, it follows (2.2.11).

For the final proof of the theorem, we note that in the equality (2.2.12) the variables x and ξ have equal rights and, therefore, if we swap them, then when deriving the equality for $E_4(x, \xi)$, provided that $|\xi| < |x|$, we get the equality (2.2.11) in which the variables x and ξ are swapped. The theorem is proved. □

2.2.3 Representation of the Green's function $G_4(x, \xi)$

Let us begin constructing the Green's function for the problem (2.2.3)-(2.2.4).

Lemma 2.12. *Let $n > 4$. For the biharmonic function $E_4(x/|x|, |x|\xi)$, where $x, \xi \in S$ the following representation holds*

$$
E_4\left(\frac{x}{|x|}, |x|\xi\right) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2k+n-2}\left(\frac{1}{2k+n-4} - \frac{|x|^2|\xi|^2}{2k+n}\right) \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi), \quad (2.2.13)
$$

in which the series converge uniformly in $x \in \bar{S}$ as $\xi \in S$.

Proof. Let us denote $\hat{x} = x/|x|$ and $\hat{\xi} = |x|\xi$. Then $|\hat{x}| = 1$ and $|\hat{\xi}| = |x||\xi| < |\hat{x}|$. We use the equality (2.2.11) to represent the function $E_4(\hat{x}, \hat{\xi})$

$$
\begin{aligned}
E_4\Big(\frac{x}{|x|}, |x|\xi\Big) &= E_4(\hat{x}, \hat{\xi}) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2k+n-2}\left(\frac{1}{2k+n-4} - \frac{|\hat{\xi}|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(\hat{x})H_k^{(i)}(\hat{\xi}) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2k+n-2}\left(\frac{1}{2k+n-4} - \frac{|x|^2|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi).
\end{aligned}
$$

By Lemma 2.7 the resulting series converge uniformly in $|x| \le 1$ for $|\xi| \le a < 1$ since $|\hat{\xi}| = |x||\xi| < a < 1 = |\hat{x}|$. The biharmonicity of the function $E_4(x/|x|, |x|\xi)$ in both variables for $\xi \in S$ and $x \in \bar{S}$ follows from the equality

$$
\big|x/|x| - |x|\xi\big|^{4-n} = \big|x/|x| - |x|\xi\big|^{2-n}\big|x/|x| - |x|\xi\big|^2,
$$

since the function $\big|x/|x| - |x|\xi\big|^{2-n}$ is harmonic (it is the Kelvin transform of the harmonic function [183]) and

$$
\left|\frac{x}{|x|} - \xi|x|\right|^2 = \frac{|x|^2}{|x|^2} - 2\Big(\frac{x}{|x|}, \xi|x|\Big) + |\xi|^2|x|^2 = 1 - 2(x, \xi) + |\xi|^2|x|^2, \qquad (2.2.14)
$$

and the product of such functions is a biharmonic function. For $n = 3$, the biharmonicity of $E_4(x/|x|, |x|\xi)$ is proved in a similar way. The lemma is proved. □

Remark 2.3. The function on the right side of the equality (2.2.13) is symmetric with respect to x and ξ, but this is not immediately obvious from the left side. However, due to (2.2.14) there is symmetry.

It is easy to see that the function

$$
\hat{E}_4(x, \xi) = E_4(x, \xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) \qquad (2.2.15)
$$

has the property $\hat{E}_4(x, \xi)|_{\partial S} = 0$. In the harmonic case, this difference give the Green's function (2.2.1) of the Dirichlet problem. In the biharmonic case this is not enough. Let us calculate the normal derivative of the function $\hat{E}_4(x, \xi)$ on ∂S. To do this, we use the homogeneous operator $\Lambda u = \sum_{k=1}^{n} x_i u_{x_i}$, which obviously has the property $\big(\Lambda u - \frac{\partial u}{\partial \nu}\ big)|_{\partial S} = 0$.

Lemma 2.13. *For $\xi \in S$ and $x \in \partial S$ the following equality holds*

$$\Lambda \hat{E}_4(x,\xi) = \frac{|\xi|^2-1}{2}\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi). \tag{2.2.16}$$

Proof. Let the point $\xi \in S$ be fixed. If we use Theorem 2.3 for $|\xi| < |x|$ and Lemma 2.12, then we have

$$\begin{aligned}\hat{E}_4(x,\xi) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|x|^2}{2k+n-4}-\frac{|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi)\\ &\quad -\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\left(\frac{1}{2k+n-4}-\frac{|x|^2|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi),\end{aligned}$$

and for $|x| < |\xi|$ the variables x and ξ should be swapped. Let us apply the operator Λ to this equality. Due to the uniform convergence of series for $|\xi| < |x|$ and $\xi \in S$, $x \in \bar{S}$ (see Lemmas 2.7 and 2.12), differentiation can be entered under the sum sign. Then we go to the limit at $x \to \partial S$ under the sum sign, which is again possible due to the uniform convergence of series in x. We have

$$\begin{aligned}\Lambda_x \hat{E}_4(x,\xi) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\left(-\frac{(k+n-4)}{2k+n-4}+\frac{(k+n-2)|\xi|^2}{2k+n}\right.\\ &\quad\left.-\frac{k}{2k+n-4}+\frac{(k+2)|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi)\\ &= \frac{|\xi|^2-1}{2}\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi).\end{aligned}$$

The lemma is proved. □

Remark 2.4. For fixed $\xi \in S$ the following equality holds

$$\frac{\partial}{\partial\nu}\hat{E}_4(x,\xi)\Big|_{\partial S} = \frac{|\xi|^2-1}{2}E\left(\frac{x}{|x|},|x|\xi\right)\Big|_{\partial S}. \tag{2.2.17}$$

Indeed, if we use the equality (2.2.6) from Lemma 2.8 in the equality (2.2.16) and the properties of the operator Λ, then we obtain the equality (2.2.17).

Lemma 2.14. *Let $w(x)$ be a function harmonic in S, continuous in $\bar{S}$, with derivatives bounded on ∂S, then the biharmonic function*

$$v(x) = \frac{|x|^2 - 1}{2} w(x) \tag{2.2.18}$$

has the property

$$v|_{\partial S} = 0, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial S} = w(x)|_{\partial S}. \tag{2.2.19}$$

Proof. From (2.2.18) it is clear that $v(x)$ is a biharmonic function in S and $v|_{\partial S} = 0$. Besides,

$$\Lambda v(x) = \Lambda \frac{|x|^2 - 1}{2} w(x) = |x|^2 w(x) + \frac{|x|^2 - 1}{2} \Lambda w(x)$$

and therefore, since the function $\Lambda w(x)$ is bounded on ∂S, the second condition from (2.2.19) is also satisfied. The lemma is proved. □

Theorem 2.4. *Let $n \geq 3$. The function*

$$G_4(x, \xi) = E_4(x, \xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2 - 1}{2} \frac{|\xi|^2 - 1}{2} E\Big(\frac{x}{|x|}, |x|\xi\Big) \tag{2.2.20}$$

is the Green's function of the Dirichlet problem (2.2.3)-(2.2.4). *The Green's function $G_4(x, \xi)$ is biharmonic for $x, \xi \in S$ and $x \neq \xi$. The solution to the problem* (2.2.3), (2.2.4) *for $f \in C^1(\bar{S})$ can be written in the form*

$$u(x) = \frac{1}{\omega_n} \int_S G_4(x, \xi) f(\xi)\, d\xi.$$

Proof. Let $n > 4$ or $n = 3$ and $\xi \in S$ be fixed. Let us prove that the function $G_4(x, \xi)$ is biharmonic for $x \in S$ and $x \neq \xi$ and satisfies the homogeneous conditions (2.2.4). The biharmonicity of the function $G_4(x, \xi)$ for $x \in S$ and $x \neq \xi$ follows from Lemma 2.9, Lemma 2.12 and Lemma 2.14 (harmonicity $E(x/|x|, |x|\xi)$ is known [28]). Further, as noted above, for $\xi \in S$ the equality $\hat{E}_4(x, \xi)|_{\partial S} = 0$ holds, which means that since the function $E(x/|x|, |x|\xi)$ is bounded by $x \in \bar{S}$, the first condition from (2.2.4) is satisfied

$$G_4(x, \xi)|_{\partial S} = 0. \tag{2.2.21}$$

Let us prove the second condition. Consider the function harmonic in $x \in S$

$$w(x) = \frac{|\xi|^2 - 1}{2} E\Big(\frac{x}{|x|}, |x|\xi\Big).$$

In these notations, according to Remark 2.4 from Lemma 2.13 and according to Lemma 2.14, taking into account (2.2.15) we find

$$\frac{\partial}{\partial\nu}G_4(x,\xi)\Big|_{\partial S} = \frac{\partial}{\partial\nu}\hat{E}_4(x,\xi)\Big|_{\partial S} - \frac{\partial}{\partial\nu}\Big(\frac{|x|^2-1}{2}w(x)\Big)\Big|_{\partial S}$$
$$= w(x)|_{\partial S} - w(x)|_{\partial S} = 0. \quad (2.2.22)$$

The case $n > 4$ or $n = 3$ is proved. Let $n = 4$. The biharmonicity of $E_4(x,\xi)$ for $x \neq \xi$ is proved in Lemma 2.9. If we designate (to within a factor of -4)

$$E_4\Big(\frac{x}{|x|}, |x|\xi\Big) = \ln\Big|\frac{x}{|x|} - |x|\xi\Big| = \frac{1}{2}\ln\Big|1 - 2(x,\xi) + |x|^2|\xi|^2\Big| \equiv \frac{1}{2}\ln t,$$

then

$$\frac{\partial}{\partial x_i}E_4\Big(\frac{x}{|x|}, |x|\xi\Big) = \frac{-\xi_i + x_i|\xi|^2}{t}, \quad \frac{\partial^2}{\partial x_i^2}E_4\Big(\frac{x}{|x|}, |x|\xi\Big)$$
$$= -2\frac{(-\xi_i + x_i|\xi|^2)^2}{t^2} + \frac{|\xi|^2}{t}$$

for $i = 1, \ldots, 4$ and, therefore, we get

$$\Delta E_4\Big(\frac{x}{|x|}, |x|\xi\Big) = -2\frac{|\xi|^2(1 - 2(x,\xi) + |x|^2|\xi|^2)}{t^2} + 4\frac{|\xi|^2}{t} = 2\frac{|\xi|^2}{t}$$
$$= \frac{2|\xi|^2}{\big|x/|x| - |x|\xi\big|^2} = \frac{2|\xi|^2}{|x|^2\big|x/|x|^2 - \xi\big|^2}.$$

The last function is the Kelvin transform on x of the harmonic function $2|\xi|^2/\big|x-\xi\big|^2$ for $n = 4$, and since the Kelvin transform preserves the harmonicity [183], then this function is harmonic in $x \in S$, and therefore the function $E_4\big(x/|x|, |x|\xi\big)$ is biharmonic.

Let us check the boundary conditions (2.2.4). We start with the second one. It is not hard to calculate

$$-4\Lambda_x E_4(x,\xi) = \Lambda_x \ln|x-\xi| = \sum_{i=1}^{4} x_i\frac{x_i - \xi_i}{|x-\xi|^2} = \frac{|x|^2 - (x,\xi)}{|x-\xi|^2}$$

and

$$-4\Lambda_x E_4\Big(\frac{x}{|x|}, |x|\xi\Big) = \frac{1}{2}\Lambda_x \ln\big(1-2(x,\xi)+|x|^2|\xi|^2\big)$$
$$= \frac{1}{2}\sum_{i=1}^{4} x_i \frac{-2\xi_i + 2x_i|\xi|^2}{1-2(x,\xi)+|x|^2|\xi|^2} = \frac{|x|^2|\xi|^2-(x,\xi)}{\big|x/|x|-|x|\xi\big|^2},$$

and therefore for $x \in \partial S$

$$\Lambda_x\Big(E_4(x,\xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big)\Big) = -\frac{1}{4}\frac{|x|^2-|x|^2|\xi|^2}{\big|x/|x|-|x|\xi\big|^2} = \frac{|\xi|^2-1}{4}\frac{1}{\big|x/|x|-|x|\xi\big|^2}$$
$$= \frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big) = \Lambda_x\Big(\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big)\Big).$$

Hence, after moving the function from the right to the left side of the equality, we obtain that for $n = 4$ the function $G_4(x,\xi)$ from (2.2.20) for $x \in S$ satisfies the condition (2.2.22) $\frac{\partial}{\partial\nu}G_4(x,\xi)|_{\partial S} = 0$. The condition (2.2.21) $G_4(x,\xi)|_{\partial S} = 0$, where $x \in S$ is obviously also satisfied.

It is known [183, p. 25] that integrals of potential type

$$\int_S \frac{\rho(\xi)}{|x-\xi|^\alpha}\, d\xi$$

are functions from the class $C^p(\mathbb{R}^n)$ for a bounded integrable function $\rho(x)$, and differentiation is possible under the integral sign for any non-negative integer p such that $\alpha + p < n$. In our case $\alpha = n-4$, which means for the integral

$$u_1(x) = \frac{1}{\omega_n}\int_S E_4(x,\xi) f(\xi)\, d\xi$$

$p = 3$ and $u_1 \in C^3(\mathbb{R}^n)$. Therefore, taking into account Lemma 2.9, for $x \in S$ we obtain

$$\Delta u_1(x) = \frac{1}{\omega_n}\int_S \Delta E_4(x,\xi) f(\xi)\, d\xi = -\frac{1}{\omega_n}\int_S E(x,\xi) f(\xi)\, d\xi,$$

which means

$$\Delta^2 u_1(x) = \Delta\Big(-\frac{1}{\omega_n}\int_S E(x,\xi) f(\xi)\, d\xi\Big) = f(x), \quad x \in S.$$

The condition $f \in C^1(\bar{S})$ is necessary to satisfy the equality $\Delta(\Delta u_1(x)) = f(x)$ in S [28]. Further, by Lemma 2.12, the function

$$-E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big)$$

is biharmonic with respect to x in S for any $\xi \in S$ and can be differentiated with respect to x under the integral sign over ξ any number of times. Denoting the integral of this function multiplied by $1/\omega_n f(\xi)$ over $\xi \in S$ by $u_2(x)$, we find

$$\Delta^2 u_2(x) = -\frac{1}{\omega_n}\int_S \Delta_x^2\Big(E_4\Big(\frac{x}{|x|}, |x|\xi\Big) + \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big)\Big) f(\xi)\, d\xi = 0.$$

Therefore, taking into account (2.2.20), we get

$$\Delta^2 u(x) \equiv \Delta^2 u_1(x) + \Delta^2 u_2(x) = f(x).$$

Finally, due to the fact that $u \in C^3(\bar{S})$, from (2.2.21) and (2.2.22) we find

$$u(x)|_{\partial S} = \frac{1}{\omega_n}\int_S G_4(x,\xi)|_{x\in\partial S} f(\xi)\, d\xi = 0,$$
$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \frac{1}{\omega_n}\int_S \frac{\partial G_4(x,\xi)}{\partial \nu}\Big|_{x\in\partial S} f(\xi)\, d\xi = 0,$$

which means the conditions (2.2.4) for $u(x)$ are satisfied. The theorem is proved. □

The form of the Green's function obtained in Theorem 2.4 differs from that found in [56].

Remark 2.5. The Green's function $G_4(x,\xi)$, taking into account Lemma 2.9, can be written as

$$G_4(x,\xi) = E_4(x,\xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) + \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\Delta E_4\Big(\frac{x}{|x|}, |x|\xi\Big).$$

Theorem 2.5. *Let $n > 4$. The function $G_4(x,\xi)$ for $|\xi| < |x|$ can be written as*

$$G_4(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty}\Bigg(\frac{|x|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|x|^2}{2k+n-4} - \frac{|\xi|^2}{2k+n}\right) - \frac{1}{2k+n-2}$$
$$\times\left(\frac{1}{2k+n-4} - \frac{|x|^2|\xi|^2}{2k+n} + \frac{|x|^2-1}{2}\left(|\xi|^2-1\right)\right)\Bigg)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi). \quad (2.2.23)$$

For $|x| < |\xi|$, the representation for $G_4(x,\xi)$ can be obtained from (2.2.23) *by interchanging the variables x and ξ.*

Proof. In Lemma 2.13 for $|\xi| < |x|$ the representation is given

$$\hat{E}_4(x,\xi) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|x|^2}{2k+n-4} - \frac{|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi)$$
$$- \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2k+n-2}\left(\frac{1}{2k+n-4} - \frac{|x|^2|\xi|^2}{2k+n}\right)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi),$$

which for $|x| < |\xi|$ can be obtained from this representation by interchanging the variables x and ξ. Recalling the equality (2.2.6) from Lemma 2.8, the equality (2.2.15) for the function $\hat{E}_4(x,\xi)$ and the definition of the function $G_4(x,\xi)$ from (2.2.20) we get (2.2.23). Finally, the symmetry of the function $G_4(x,\xi)$ and the equality (2.2.23) with respect to x and ξ occurs due to Remark 2.3. The theorem is proved. □

Theorem 2.6. *Let $f(x) = |x|^{2l}H_m(x)$, where $H_m(x)$ is a homogeneous harmonic polynomial of degree $m \in \mathbb{N}_0$, $l \in \mathbb{R}_+$ and $n > 4$. Then*

$$\frac{1}{\omega_n}\int_{|\xi|<1} G_4(x,\xi)f(\xi)\,d\xi = \frac{|x|^{2l+4} - 1 - (l+2)(|x|^2-1)}{C_{l,m}}H_m(x), \qquad (2.2.24)$$

where $C_{l,m} = (2l+2)(2l+4)(2l+2m+n)(2l+2m+n+2)$.

Proof. Let first $f(x) = |x|^{2l}H_m^{(j)}(x)$, where $H_m^{(j)}(x)$ is some polynomial from the complete system $\{H_m^{(i)}(x) : i = 1,\dots,h_m, m \in \mathbb{N}_0\}$ of homogeneous degree $m \in \mathbb{N}_0$ orthogonal spherical harmonics and $l \in \mathbb{R}_+$. Let's denote the left side of the formula (2.2.24) by $u(x)$ and calculate it. Let $\varepsilon > 0$ be so small that $|x|+\varepsilon < 1$ and $|x|-\varepsilon > 0$, then due to the integrable singularity of the function $G_4(x,\xi)$ we have

$$u(x) = \frac{1}{\omega_n}\int_S G_4(x,\xi)f(\xi)\,d\xi = \int_0^1 \rho^{n-1}d\rho\frac{1}{\omega_n}\int_{|\xi|=1} G_4(x,\rho\xi)f(\rho\xi)\,ds_\xi$$
$$= \lim_{\varepsilon\to 0}\left(\int_0^{|x|-\varepsilon} + \int_0^{|x|+\varepsilon}\right)\rho^{n-1}d\rho\frac{1}{\omega_n}\int_{|\xi|=1} G_4(x,\rho\xi)f(\rho\xi)\,ds_\xi$$
$$\equiv \lim_{\varepsilon\to+0}(I_1^\varepsilon(x) + I_2^\varepsilon(x)).$$

Here the integral $I_1^\varepsilon(x)$ for $x = 0$ must be omitted. Let us calculate the first integral $I_1^\varepsilon(x)$. Since in this integral $|\xi| < |x| - \varepsilon \equiv a < |x|$, then by Lemma 2.7 the series from (2.2.23) representing the function $G_4(x,\xi)$, converges uniformly in ξ, which means integration and summation can be interchanged

$$
\begin{aligned}
I_1^\varepsilon(x) &= \frac{1}{2}\int_0^{|x|-\varepsilon} \rho^{2l+m+n-1}\Bigg(\sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\left(\frac{|x|^2}{2k+n-4}-\frac{\rho^2}{2k+n}\right)\\
&\quad -\frac{1}{2k+n-2}\left(\frac{1}{2k+n-4}-\rho^2\frac{|x|^2}{2k+n}+\frac{|x|^2-1}{2}\left(\rho^2-1\right)\right)\Bigg)\,d\rho\\
&\qquad\times\rho^k\sum_{i=1}^{h_k}H_k^{(i)}(x)\frac{1}{\omega_n}\int_{|\xi|=1}H_k^{(i)}(\xi)H_m^{(j)}(\xi)\,ds_\xi.
\end{aligned}
$$

Taking into account the orthogonality of the polynomials $\{H_k^{(i)}(x) : i=1,\dots,h_k,\ k\in\mathbb{N}_0\}$, we will have

$$
\begin{aligned}
I_1^\varepsilon(x) &= \frac{1}{2}\int_0^{|x|-\varepsilon} \rho^{2l+2m+n-1}\Bigg(\frac{|x|^{-(2m+n-2)}}{2m+n-2}\left(\frac{|x|^2}{2m+n-4}-\frac{\rho^2}{2m+n}\right)\\
&\quad -\frac{1}{2m+n-2}\left(\frac{1}{2m+n-4}-\frac{\rho^2|x|^2}{2m+n}+\frac{|x|^2-1}{2}\left(\rho^2-1\right)\right)\Bigg)d\rho H_m^{(j)}(x)\\
&= \frac{1}{2}\Bigg(\frac{|x|^{-(2m+n-2)}}{2m+n-2}\Big(\frac{|x|^2(|x|-\varepsilon)^{2l+2m+n}}{(2l+2m+n)(2m+n-4)}\\
&\quad -\frac{(|x|-\varepsilon)^{2l+2m+n+2}}{(2m+n)(2l+2m+n+2)}\Big)-\frac{1}{2m+n-2}\\
&\quad\times\Big(\frac{(|x|-\varepsilon)^{2l+2m+n}}{(2m+n-4)(2l+2m+n)}-\frac{|x|^2(|x|-\varepsilon)^{2l+2m+n+2}}{(2m+n)(2l+2m+n+2)}\\
&\quad +\frac{|x|^2-1}{2}\Big(\frac{(|x|-\varepsilon)^{2l+2m+n+2}}{2l+2m+n+2}-\frac{(|x|-\varepsilon)^{2l+2m+n}}{2l+2m+n}\Big)\Big)\Bigg)H_m^{(j)}(x).
\end{aligned}
$$

In the limit at $\varepsilon\to+0$ we have

$$
\begin{aligned}
\lim_{\varepsilon\to+0}I_1^\varepsilon(x) &= \frac{1}{2(2m+n-2)}\Bigg(\frac{|x|^{2l+4}}{(2l+2m+n)(2m+n-4)}\\
&\quad -\frac{|x|^{2l+4}}{(2m+n)(2l+2m+n+2)}-\frac{|x|^{2l+2m+n}}{(2m+n-4)(2l+2m+n)}\\
&\quad +\frac{|x|^{2l+2m+n+4}}{(2m+n)(2l+2m+n+2)}\\
&\quad -\frac{|x|^2-1}{2}\left(\frac{|x|^{2l+2m+n+2}}{2l+2m+n+2}-\frac{|x|^{2l+2m+n}}{2l+2m+n}\right)\Bigg)H_m^{(j)}(x).
\end{aligned}
$$

Similarly, using the symmetry $G_4(x,\xi)$, we find

$$I_2^\varepsilon(x) = \frac{1}{2}\int_{|x|+\varepsilon}^1 \rho^{2l+2m+n-1}\left(\frac{\rho^{-(2m+n-2)}}{2m+n-2}\left(\frac{\rho^2}{2m+n-4} - \frac{|x|^2}{2m+n}\right)\right.$$
$$\left. - \frac{1}{2m+n-2}\left(\frac{1}{2m+n-4} - \rho^2\frac{|x|^2}{2m+n} + \frac{|x|^2-1}{2}\left(\rho^2-1\right)\right)\right) d\rho H_m^{(j)}(x)$$
$$= \frac{1}{2(2m+n-2)}\left(\frac{1-(|x|+\varepsilon)^{2l+4}}{(2l+4)(2m+n-4)} - |x|^2\frac{1-(|x|+\varepsilon)^{2l+2}}{(2m+n)(2l+2)}\right.$$
$$- \frac{1-(|x|+\varepsilon)^{2l+2m+n}}{(2m+n-4)(2l+2m+n)} + |x|^2\frac{1-(|x|+\varepsilon)^{2l+2m+n+2}}{(2m+n)(2l+2m+n+2)}$$
$$\left. - \frac{|x|^2-1}{2}\left(\frac{1-(|x|+\varepsilon)^{2l+2m+n+2}}{2l+2m+n+2} - \frac{1-(|x|+\varepsilon)^{2l+2m+n}}{2l+2m+n}\right)\right) H_m^{(j)}(x).$$

In the limit at $\varepsilon \to +0$ we have

$$\lim_{\varepsilon\to+0} I_2^\varepsilon(x) = \frac{1}{2(2m+n-2)}\left(\frac{1-|x|^{2l+4}}{(2l+4)(2m+n-4)} - \frac{|x|^2-|x|^{2l+4}}{(2m+n)(2l+2)}\right.$$
$$- \frac{1-|x|^{2l+2m+n}}{(2m+n-4)(2l+2m+n)} + \frac{|x|^2-|x|^{2l+2m+n+4}}{(2m+n)(2l+2m+n+2)}$$
$$\left. - \frac{|x|^2-1}{2}\left(\frac{1-|x|^{2l+2m+n+2}}{2l+2m+n+2} - \frac{1-|x|^{2l+2m+n}}{2l+2m+n}\right)\right) H_m^{(j)}(x).$$

From here we get

$$u(x) = I_1^0(x) + I_2^0(x) = \frac{1}{2(2m+n-2)}\left(\frac{|x|^{2l+4}}{(2l+2m+n)(2m+n-4)}\right.$$
$$- \frac{|x|^{2l+4}}{(2m+n)(2l+2m+n+2)} + \frac{1-|x|^{2l+4}}{(2l+4)(2m+n-4)} - \frac{|x|^2-|x|^{2l+4}}{(2m+n)(2l+2)}$$
$$- \frac{1}{(2m+n-4)(2l+2m+n)} + \frac{|x|^2}{(2m+n)(2l+2m+n+2)}$$
$$\left. - \frac{|x|^2-1}{2}\left(\frac{1}{2l+2m+n+2} - \frac{1}{2l+2m+n}\right)\right) H_m^{(j)}(x). \quad (2.2.25)$$

Using "Mathematica", we calculate the coefficients for $|x|^{2l+4}H_m^{(j)}(x)$

$$\frac{1}{2(2m+n-2)}\Big(\frac{1}{(2l+2m+n)(2m+n-4)}-\frac{1}{(2m+n)(2+2l+2m+n)}$$
$$-\frac{1}{(2l+4)(2m+n-4)}+\frac{1}{(2l+2)(2m+n)}\Big)=\frac{1}{C_{l,m}},$$

then for $|x|^2H_m^{(j)}(x)$

$$\frac{1}{2(2m+n-2)}\Big(-\frac{1}{(2l+2)(2m+n)}\Big)+\frac{1}{(2l+2m+n+2)(2m+n)}$$
$$-\frac{1}{2(2l+2m+n+2)}+\frac{1}{2(2l+2m+n)}\Big)=-\frac{l+2}{C_{l,m}}$$

and finally for $H_m^{(j)}(x)$

$$\frac{1}{2(2m+n-2)}\Big(\frac{1}{(2l+4)(2m+n-4)}-\frac{1}{(2l+2m+n)(2m+n-4)}$$
$$+\frac{1}{2(2l+2m+n+2)}-\frac{1}{2(2l+2m+n))}\Big)=\frac{l+1}{C_{l,m}},$$

where $C_{l,m}=(2l+2)(2l+4)(2l+2m+n)(2l+2m+n+2)$. Substituting the found values of the coefficients into (2.2.25) we get

$$u(x)=\frac{|x|^{2l+4}-(l+2)|x|^2+l+1}{C_{l,m}}H_m^{(j)}(x),$$

which coincides with (2.2.24) for $H_m(x)=H_m^{(j)}(x)$.

Due to the completeness of the system $\{H_m^{(i)}(x):i=1,\ldots,h_m,m\in\mathbb{N}_0\}$ for the homogeneous harmonic polynomial $H_m(x)$ the representation $H_m(x)=\sum_{j=1}^{h_m}\alpha_jH_m^{(j)}(x)$ is true, which means for $f(x)=|x|^{2l}H_m(x)$

$$\frac{1}{\omega_n}\int_{|\xi|<1}G_4(x,\xi)f(\xi)\,d\xi=\sum_{j=1}^{h_m}\alpha_j\frac{1}{\omega_n}\int_{|\xi|<1}G_4(x,\xi)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi$$
$$=\frac{|x|^{2l+4}-1-(l+2)(|x|^2-1)}{C_{l,m}}\sum_{j=1}^{h_m}\alpha_jH_m^{(j)}$$
$$=\frac{|x|^{2l+4}-1-(l+2)(|x|^2-1)}{C_{l,m}}H_m(x).$$

The theorem is proved. □

The result of Theorem 2.6 coincides with the result obtained in [91] for $m \in \mathbb{N}_0$. The method for constructing these solutions is based on Almansi type expansions [70, 71]. Note that polynomial solutions of the Dirichlet problem for the 3-harmonic equation is constructed in [77].

Theorem 2.4 can be supplemented with the following statement.

Theorem 2.7. *Let $n = 2$. The next function*

$$G_4(x,\xi) = E_4(x,\xi) \\ - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\Big(E\Big(\frac{x}{|x|}, |x|\xi\Big) + \frac{1}{2}\Big) \quad (2.2.26)$$

is the Green's function of the Dirichlet problem (2.2.3), (2.2.4). *The Green's function $G_4(x,\xi)$ is biharmonic for $x, \xi \in S$ and $x \neq \xi$.*

Proof. Let us prove that the function $G_4(x,\xi)$ is biharmonic for $x \in S$ and $x \neq \xi$ and satisfies the homogeneous conditions (2.2.4). Biharmonicity of the function

$$E_4(x,\xi) = \frac{|x-\xi|^2}{4}\big(\ln|x-\xi| - 1\big)$$

for $x \neq \xi$ is established in Lemma 2.9. Let us study the function $E_4\big(x/|x|, |x|\xi\big)$. Similarly to the case $n = 4$ we denote

$$\ln\Big|\frac{x}{|x|} - |x|\xi\Big| = \frac{1}{2}\ln\Big|1 - 2(x,\xi) + |x|^2|\xi|^2\Big| \equiv \frac{1}{2}\ln t.$$

Then

$$\frac{\partial}{\partial x_i}\frac{1}{2}\ln t = \frac{-\xi_i + x_i|\xi|^2}{t}, \qquad \frac{\partial^2}{\partial x_i^2}\frac{1}{2}\ln t = -2\frac{(-\xi_i + x_i|\xi|^2)^2}{t^2} + \frac{|\xi|^2}{t}$$

and that means

$$\Delta_x\frac{1}{2}\ln t = -2\frac{|\xi|^2(1 - 2(x,\xi) + |x|^2|\xi|^2)}{t^2} + 2\frac{|\xi|^2}{t} = 0,$$

i.e. $\ln\big|x/|x| - |x|\xi\big|$ is a harmonic function in x. Since the factor before the logarithm is equal to $\big|x/|x| - |x|\xi\big|^2 = 1 - 2(x,\xi) + |x|^2|\xi|^2$, then the function $E_4\big(x/|x|, |x|\xi\big)$ for $n = 2$ is biharmonic in x for $x, \xi \in S$. Finally, the function

$$\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\Big(E\Big(\frac{x}{|x|}, |x|\xi\Big) + \frac{1}{2}\Big)$$

is biharmonic since the function $E\big(x/|x|, |x|\xi\big)$ is harmonic for $x \in S$. Let us check the boundary conditions (2.2.4). We start with the second condition. It is not hard to see that

$$\Lambda_x|x-\xi|^2 = 2(|x|^2-(x,\xi)), \quad \Lambda_x \ln|x-\xi| = \frac{|x|^2-(x,\xi)}{|x-\xi|^2},$$

and therefore

$$\begin{aligned}&4\Lambda_x E_4(x,\xi)\\ &= 2(|x|^2-(x,\xi))\big(\ln|x-\xi|-1\big)+|x|^2-(x,\xi) = \big(|x|^2-(x,\xi)\big)(2\ln|x-\xi|-1).\end{aligned}$$

Similar to the case $n=4$

$$\Lambda_x\big|x/|x|-|x|\xi\big|^2 = 2(|x|^2|\xi|^2-(x,\xi)),\ \Lambda_x \ln\big|x/|x|-|x|\xi\big| = \frac{|x|^2|\xi|^2-(x,\xi)}{\big|x/|x|-|x|\xi\big|^2},$$

and therefore, since the operator Λ is of the first order, we find

$$\begin{aligned}4\Lambda_x E_4\big(x/|x|,|x|\xi\big) &= 2(|x|^2|\xi|^2-(x,\xi))\big(\ln\big|x/|x|-|x|\xi\big|-1\big)\\ &+\big|x/|x|-|x|\xi\big|^2\frac{|x|^2|\xi|^2-(x,\xi)}{\big|x/|x|-|x|\xi\big|^2} = \big(|x|^2|\xi|^2-(x,\xi)\big)\big(2\ln\big|x/|x|-|x|\xi\big|-1\big).\end{aligned}$$

Thus, for $x \in \partial S$

$$\begin{aligned}4\Lambda_x\big(E_4(x,\xi)-E_4\big(x/|x|,|x|\xi\big)\big) &= \big(2\ln\big|x/|x|-|x|\xi\big|-1\big)(1-|\xi|^2)\\ &= 4\Big(-\ln\big|x/|x|-|x|\xi\big|+\frac{1}{2}\Big)\frac{|\xi|^2-1}{2} = 4\Big(E\big(x/|x|,|x|\xi\big)+\frac{1}{2}\Big)\frac{|\xi|^2-1}{2}\\ &= 4\Lambda_x\Big(\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\Big(E\big(x/|x|,|x|\xi\big)+\frac{1}{2}\Big)\Big).\end{aligned}$$

After elementary transformations we obtain the equality $\frac{\partial}{\partial\nu}G_4(x,\xi)\big|_{x\in\partial S} = 0$, where $\xi \in S$. Condition $G_4(x,\xi)|_{x\in\partial S} = 0$, where $\xi \in S$ is obviously also satisfied. The further proof repeats the end of the proof of Theorem 2.4, due to which differentiation and passage to the limit can be entered under the integral sign in

$$u(x) = \frac{1}{\omega_n}\int_S G_4(x,\xi)f(\xi)\,d\xi.$$

Therefore, the function $u(x)$ defines a solution to the problem (2.2.3)-(2.2.4) for $f \in C^1(\bar{S})$. The theorem is proved. □

2.3 Green's function of the Dirichlet problem for the polyharmonic equation

In this section, based on the results of Section 2.2, the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball is constructed. First, the fundamental solution of the m-harmonic equation $\mathcal{E}_{2m}(x,\xi)$ is determined and its properties are studied in Lemmas 2.15 and 2.16. The function $\mathcal{E}_{2m}(x,\xi)$ differs slightly from the elementary solution of the polyharmonic equation $E_{2m}(x,\xi)$ studied in section 4.2.1. Theorem 2.8 defines the Green's function $G_{2m}(x,\xi)$ of the Dirichlet problem, Theorem 2.9 finds a solution to the homogeneous Dirichlet problem for the polyharmonic equation in the unit ball, and in Theorem 2.10 a special case of the right-hand side of the equation in the form $f(x) = |x|^{2l}H_k(x)$ is considered. The main results of Section 2.3 are published in papers [77, 110, 123, 124, 127].

The explicit forms of Green's functions for various boundary value problems are presented in many works. Let us list just a few of them. For example, in the two-dimensional case, in the work [20], based on the well-known harmonic Green's function, the Green's functions of various biharmonic problems are presented. An explicit form of the Green's function for the Robin boundary value problem is found in the papers [22, 107, 156], and the Green's functions in a sector for the biharmonic and 3-harmonic equations are considered in [187, 188]. The papers [32, 56, 57] are devoted to the construction of the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball, and the paper [116] finds solutions to the Dirichlet and Neumann problems for the homogeneous polyharmonic equation. The explicit form of the Green's functions for the biharmonic equation is obtained in Section 2.2. In [77, 110] the explicit form of the Green's functions for the 3-harmonic equation in the unit ball is given. As the most general results on the generalized Neumann problem containing powers of normal derivatives in the boundary conditions, we note the paper [164].

In the equality (2.2.7) from the previous section, the elementary solution of the biharmonic equation is defined in the form

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n>4,\ n=3 \\ -\dfrac{1}{4}\ln|x-\xi|, & n=4 \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi|-1\big), & n=2 \end{cases}. \tag{2.3.1}$$

Theorem 2.4 proves that for $n \geq 3$ a function of the form

$$G_4(x,\xi) = E_4(x,\xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big), \tag{2.3.2}$$

where $E(x,\xi)$ is an elementary solution of the Laplace equation [28], is the Green's function of the Dirichlet problem for the biharmonic equation in the ball S, satisfying the equalities $G_4(x,\xi)|_{x\in\partial S} = \partial G_4(x,\xi)/\partial\nu|_{x\in\partial S} = 0$ for $\xi \in S$. In [110], the function

$$E_6(x,\xi) = \begin{cases} \dfrac{|x-\xi|^{6-n}}{2\cdot 4(n-2)(n-4)(n-6)}, & n \geq 3, n \neq 4,6, \\ -\dfrac{1}{64}\ln|x-\xi|, & n=6 \\ \dfrac{|x-\xi|^2}{32}\Big(\ln|x-\xi| - \dfrac{3}{4}\Big), & n=4 \\ -\dfrac{|x-\xi|^4}{64}\Big(\ln|x-\xi| - \dfrac{3}{2}\Big), & n=2 \end{cases}, \tag{2.3.3}$$

is introduced, which, by analogy with the function $E_4(x,\xi)$ from (2.3.1), is called the elementary solution of the 3-harmonic equation. For $x \neq \xi$ the equality $\Delta_x E_6(x,\xi) = -E_4(x,\xi)$ holds. The Green's function in this case for $n \geq 3$ and $n \neq 4$ is presented in the form

$$\begin{aligned} G_6(x,\xi) = E_6(x,\xi) - E_6\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2} \\ \times E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}E\Big(\frac{x}{|x|}, |x|\xi\Big). \end{aligned} \tag{2.3.4}$$

In the resulting equality, the singularity is contained only in the first term $E_6(x,\xi)$, similar to (2.3.2), and all other terms are 3-harmonic functions in S. In this section, we study the representation of solutions to the homogeneous Dirichlet problem for the m-harmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^m u(x) = f(x),\ x \in S, \tag{2.3.5}$$

$$u|_{\partial S} = 0,\ \frac{\partial u}{\partial\nu}|_{\partial S} = 0, \ldots, \frac{\partial^{m-1}u}{\partial\nu^{m-1}}|_{\partial S} = 0. \tag{2.3.6}$$

The work [32] shows that the Green's function $G_{2m}(x,\xi)$ of this problem has the form (Boggio formula)

$$G_{2m}(x,\xi) = k_m|x-\xi|^{2m-n}\int_1^{g(x,\xi)}(t^2-1)^{m-1}t^{1-n}\,dt,$$

where

$$g(x,\xi) = \frac{1}{|x-\xi|}\left|\frac{x}{|x|} - |x|\xi\right|, \quad k_m = \frac{1}{\omega_n((2m-2)!)^2},$$

and in the work [25] the function $G_{2m}(x,\xi)$ is constructed in the case of $n = 2$. In the work [57] an explicit representation of the function $G_{2m}(x,\xi)$ is found depending on the parity of n and the positivity of $2m-n$. In [47,52] some estimates of the Green's function $G_{2m}(x,\xi)$ are obtained using Boggio's formula. This section gives another representation of the Green's function $G_{2m}(x,\xi)$, which coincides, in particular, with the equalities (2.3.2) and (2.3.4).

2.3.1 Fundamental solution of the m-harmonic equation

Let $m \in \mathbb{N}$. Then the set $\mathbb{N} \setminus \{1\}$ can be divided into two disjoint sets $\mathbb{N}_m = \{n \in \mathbb{N} : n > 2m > 1\} \cup (2\mathbb{N}+1)$ and its complement $\mathbb{N}_m^c = \{2, 4, \ldots, 2m\}$. Since the set $\mathbb{N}_m^c$ is finite, then $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}_{m-1}^c \subset \mathbb{N}_m^c$, and therefore $\mathbb{N}_m \subset \mathbb{N}_{m-1}$.

Definition 2.1. The fundamental solution of the m-harmonic equation $\Delta^m u = 0$ is a function of the form

$$\mathcal{E}_{2m}(x,\xi) = \begin{cases} \dfrac{(-1)^m |x-\xi|^{2m-n}}{(2-n,2)_m (2,2)_{m-1}}, & n \in \mathbb{N}_m, \\ \dfrac{(-1)^m |x-\xi|^{2m-n}}{(2-n,2)_m^* (2,2)_{m-1}} \left(\ln|x-\xi| - \sum\limits_{k=1}^{m-n/2} \frac{1}{2k} \right), & n \in \mathbb{N}_m^c, \end{cases} \tag{2.3.7}$$

where $(a,b)_k = a(a+b)\ldots(a+kb-b)$ is a generalized Pochhammer symbol with the convention $(a,b)_0 = 1$, and the symbol $(a,b)_k^*$ means that if there is 0 among the factors $a, (a+b), \ldots (a+kb-b)$ included in $(a,b)_k$, then it should be replace with 1. For example, $(-2,2)_3^* = (-2)\cdot 1 \cdot 2 = -4$. In addition, if in the sum included in (2.3.7) the upper index becomes less than the lower index, then the sum is considered equal to zero.

It should be kept in mind that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and this notation is not related to $\mathcal{E}_{2m}(x,\xi)$.

Remark 2.6. From the equality (2.3.7) it follows that $\mathcal{E}_2(x,\xi)$ coincides with the elementary solution of the Laplace equation $E(x,\xi)$ [28] and, moreover, $\mathcal{E}_4(x,\xi) = E_4(x,\xi)$ for $n \geq 3$ (2.3.1) and $\mathcal{E}_6(x,\xi) = E_6(x, xi)$ for $n \geq 3, n \neq 4$ (2.3.3).

Remark 2.7. The fundamental solution $\mathcal{E}_{2m}(x,\xi)$ differs slightly from the fundamental solution of the polyharmonic equation $G_{m,n}(x)$ considered by S.L. Sobolev in [162]. For $n \in \mathbb{N}_m$ the difference is in the factor $(-1)^m$, and for $n \in \mathbb{N}_m^c$ the difference is more noticeable: instead of $\ln(C|x|)$, as from Sobolev, the expression $\ln|x| - \sum_{k=1}^{n/2} \frac{1}{2k}$ is taken.

Let us introduce the notation

$$\begin{gathered}\mathcal{E}_{2m}^*(x,\xi) = \mathcal{E}_{2m}\big(x/|x|, |x|\xi\big),\\ h(x,\xi) = |x-\xi|^2,\ h_*(x,\xi) = \big|x/|x| - |x|\xi\big|^2.\end{gathered} \tag{2.3.8}$$

It is clear that $h(x,\xi)$ and $h_*(x,\xi)$ are second-order polynomials. Some statements are necessary regarding the functions $\mathcal{E}_{2m}(x,\xi)$ and $\mathcal{E}_{2m}^*(x,\xi)$.

Lemma 2.15. *a). The symmetric function $\mathcal{E}_{2m}(x,\xi)$ is m-harmonic in $x \in S$ for $x \neq \xi$.*

b). The symmetric function $\mathcal{E}_{2m}^*(x,\xi)$ is m-harmonic in $x \in S$ for $\xi \in \bar{S}$.

Proof. a). The symmetry of the function $\mathcal{E}_{2m}(x,\xi)$ is clear from the definition (2.3.7). In Section 4.2.1 on page 344 we will study the function $E_{2m}(x,\xi)$, which is related by the following equality to the function $\mathcal{E}_{2m}(x,\xi)$

$$E_{2m}(x,\xi) = \begin{cases} \mathcal{E}_{2m}(x,\xi), & n \in \mathbb{N}_m \\ \mathcal{E}_{2m}(x,\xi) - \dfrac{(-1)^m|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}} \displaystyle\sum_{k=n/2}^{m-1} \frac{1}{2k}, & n \in \mathbb{N}_m^c \end{cases}. \tag{2.3.9}$$

Lemma 4.2 proves that the function $E_{2m}(x,\xi)$ is m-harmonic in $x \in S$ for $x \neq \xi$. Therefore, for $n \in \mathbb{N}_m$ the function $\mathcal{E}_{2m}(x,\xi)$ is also m-harmonic in $x \in S$ for $x \neq \xi$. Next, for $n \in \mathbb{N}_m^c$ we consider the difference $\mathcal{E}_{2m}(x,\xi) - E_{2m}(x,\xi)$. Since for $n \in \mathbb{N}_m^c = \{2, 4, \ldots, 2m\}$ the inclusion $2m - n \in 2\mathbb{N}_0$ is true, then the function $C|x-\xi|^{2m-n}$ is a polynomial of degree less than or equal to $2m-2$, which means it is an m-harmonic function everywhere. Here C is the corresponding numerical coefficient from (2.3.9). Therefore the function $\mathcal{E}_{2m}(x,\xi) = E_{2m}(x,\xi) + C|x-\xi|^{2m-n}$ is m-harmonic in $x \in S$ at $x \neq \xi$. Statement a) is proved.

b). It is easy to see that $\big|x/|x| - |x|\xi\big| \geq \big|x/|x|\big| - |x|\big|\xi\big| = 1 - |x||\xi| \geq 1 - |x| > 0$ for $x \in S$, $\xi \in \bar{S}$ and hence the function $\mathcal{E}_{2m}^*(x,\ xi)$ is defined and differentiable for given values of x and ξ. The symmetry of $\mathcal{E}_{2m}^*(x,\xi)$ follows from the equality $\big|x/|x| - |x|\xi\big|^2 = 1 - 2x\cdot\xi + |x|^2|\xi|^2 = \big|\xi/|\xi| - |\xi|x\big|^2$. Further, it is easy to see that

$$\frac{\partial}{\partial x_i} f(-x|\xi| + \xi/|\xi|) = -|\xi| f_i(-x|\xi| + \xi/|\xi|),$$

where $f_i(x)$ is the derivative of the function $f(x)$ with respect to the ith variable. For $f(x) = |x|^{2m-n}(c_1 \ln|x| + c_2)$, the derivative $f_i(x)$ exists if $|x| > 0$. In our case, the argument is $\big|x/|x| - |x|\xi\big| > 0$, which means there are derivatives of any order of the function $f(x)$. Therefore, by virtue of the first statement of the lemma, we have

$$\Delta^m \mathcal{E}_{2m}^*(x,\xi) = \Delta^m(\mathcal{E}_{2m}(-x|\xi| + \xi/|\xi|)) \\ = (-|\xi|)^{2m}(\Delta^{2m}\mathcal{E}_{2m})(-x|\xi| + \xi/|\xi|)) = 0.$$

The second statement of the lemma is proved. □

Remark 2.8. In Section 4.2 (see Definition 4.3 on page 344) an elementary solution of the polyharmonic equation $E_{2m}(x,\xi)$ is considered, which is related to the function $\mathcal{E}_{2m}(x,\ xi)$ by the equality (2.3.9). By Lemma 4.2 such a function $E_{2m}(x,\xi)$ satisfies the equality $\Delta_x E_{2m}(x,\xi) = -E_{2m-2}(x,\xi)$ for $x \neq \xi$, which is used when constructing the Green's functions of the Navier and Riquier-Neumann problems in Sections 4.4 and 4.5.

Let us study the product of polyharmonic functions. It is known that the product of harmonic functions can be either a harmonic function or a function that is not m-harmonic for any $m \in \mathbb{N}$. For example, the functions $u \cdot v$ and u^2 with $u(x_1,x_2) = e^{x_1}\cos x_2$ and $v(x_1,x_2) = e^{x_1}\sin x_2$. In addition, according to the well-known Almansi formula, the function $|x|^{2m-2}u(x)$ is an m-harmonic function if the function $u(x)$ is harmonic. Let us introduce the operator

$$\Lambda u(x) = \sum_{i=1}^n x_i u_{x_i}, \tag{2.3.10}$$

for which, as is easy to check, the equality $\Delta\Lambda u = (\Lambda + 2)\Delta u$ is true. From this equality it follows that $\Delta^m\Lambda u = (\Lambda + 2m)\Delta^m u$, and therefore, if the function $u(x)$ is m-harmonic, then the function Λu is also m-harmonic, although its representation contains terms of the form $x_i u_{x_i}$.

The following result requires a slight modification of the method of mathematical induction.

(*) *Let statements $A_{n,m}$ be given for $n,m \in \mathbb{N}$ and the following conditions are satisfied for them: 1) statements $A_{n,1}$ and $A_{1,m}$ are true for $n,m \in \mathbb{N}$; 2) from the truth of the statements $A_{n-1,m}$ and $A_{n,m-1}$, the index of which belongs to $\mathbb{N}^2$, the truth of $A_{n,m}$ follows. Then the statements $A_{n,m}$ are true for all $n,m \in \mathbb{N}$.*

Proof. Suppose that under conditions 1) and 2) there are statements $A_{n,m}$ that are not true. Let us choose among them A_{n^*,m^*}, for which the sum of indices n^*+m^* is the smallest and $(n^*,m^*)\in\mathbb{N}^2$. By condition 1) for A_{n^*,m^*} there can be neither $n^*=1$ nor $m^*=1$. Then, by condition 2) either the statement A_{n^*-1,m^*} or the statement A_{n^*,m^*-1} must not be satisfied. For these statements, the sum of the indices is equal to n^*+m^*-1, which cannot be due to the choice of the statement A_{n^*,m^*}. This means that all statements $A_{n,m}$ are true. □

Lemma 2.16. *Let $k_1,k_2\in\mathbb{N}_0$, $k_3\in\mathbb{N}$ and $k=k_1+k_2+k_3$. Then the function*

$$h^{k_1}(x,\xi)h_*^{k_2}(x,\xi)\mathcal{E}_{2k_3}^*(x,\xi)$$

is k-harmonic in $x\in S$ for $\xi\in\bar{S}$.

Proof. 1^0. Let us prove that the function

$$F_{k_2,k_3}(x,\xi)\equiv h_*^{k_2}(x,\xi)\mathcal{E}_{2k_3}^*(x,\xi)$$

for $k_2\in\mathbb{N}_0$, $k_3\in\mathbb{N}$ is (k_2+k_3)-harmonic in $x\in S$. Let first $n\in\mathbb{N}_{k_2+k_3}$. It is easy to see that in this case $n\in\mathbb{N}_{k_3}$ and therefore from (2.3.7) we obtain

$$F_{k_2,k_3}(x,\xi)=C_{k_3}h_*^{k_3-n/2+k_2}(x,\xi)=\frac{C_{k_3}}{C_{k_2+k_3}}\mathcal{E}_{2k_2+2k_3}^*(x,\xi),$$

where C_{k_3} is the numerical coefficient of $|x-\xi|^{2k_3-n}$ in $\mathcal{E}_{2k_3}$ when $n\in\mathbb{N}_{k_3}$. Therefore, by Lemma 2.15, the function $F_{k_2,k_3}(x,\xi)$ is (k_2+k_3)-harmonic in $x\in S$. If $n\in\mathbb{N}_{k_2+k_3}^c$, but $n\in\mathbb{N}_{k_3}$, then there is a positive integer s such that $k_3+1\le s\le k_2+k_3$ and $2s=n$. In this case

$$F_{k_2,k_3}(x,\xi)=C_{k_3}h_*^{k_3+k_2-s}(x,\xi).$$

Further it is easy to calculate that

$$\Delta\big|x/|x|-|x|\xi\big|^{\alpha}=\alpha(\alpha+n-2)|\xi|^2\big|x/|x|-|x|\xi\big|^{\alpha-2}\tag{2.3.11}$$

and that means since $1\le k_3+k_2-s+1\le k_2$, then the polynomial $F_{k_2,k_3}(x,\xi)$ is (k_2+1)-harmonic in $x\in S$. Since $k_2+1\le k_2+k_3$, then what is asserted in 1^0 is true. Let now $n\in\mathbb{N}_{k_3}^c$. Then there is a positive integer $s\le k_3$ such that $2s=n$, and therefore the polynomial $h_*^{k_3-s}(x,\xi)$, similar to the previous case, is (k_3-s+1)-harmonic. From the equality (2.3.7) we find

$$\mathcal{E}_{2k_3}^*(x,\xi)=C_{k_3}h_*^{k_3-s}(x,\xi)\ln\big|x/|x|-|x|\xi\big|-C_{k_3}L_{k_3}h_*^{k_3-s}(x,\xi),$$

where C_{k_3} is a numerical factor for $|x-\xi|^{2k_3-n}$, and L_{k_3} is a numerical term for a logarithm in $\mathcal{E}_{2k_3}$, when $n \in \mathbb{N}^c_{k_3}$. That is why

$$F_{k_2,k_3}(x,\xi) = C_{k_3} h_*^{k_3+k_2-s}(x,\xi) \ln \big| x/|x| - |x|\xi \big| - C_{k_3} L_{k_3} h_*^{k_3+k_2-s}(x,\xi).$$

Since $n \in \mathbb{N}^c_{k_2+k_3}$, then from (2.3.7) it follows

$$h_*^{k_2+k_3-s}(x,\xi) \ln \big| x/|x| - |x|\xi \big| = C^{-1}_{k_2+k_3} \mathcal{E}^*_{2k_2+2k_3}(x,\xi) + L_{k_2+k_3} h_*^{k_2+k_3-s}(x,\xi)$$

and that means

$$F_{k_2,k_3}(x,\xi) = C_{k_3} C^{-1}_{k_2+k_3} \mathcal{E}^*_{2k_2+2k_3}(x,\xi) + C_{k_3}(L_{k_2+k_3} - L_{k_3}) h_*^{k_2+k_3-s}(x,\xi).$$

The first term in the resulting equality, according to Lemma 2.15, is a (k_2+k_3)-harmonic function in $x \in S$, and the second term is (k_2+k_3-s+1)-harmonic polynomial. Since $s \geq 1$, then the statement 1^0 is proved.

2^0. Now we prove that the function $G_{k_1,k_2'}(x,\xi) = h^{k_1}(x,\xi) F_{k_2'}(x,\xi)$, where $F_{k_2'}(x,\xi)$ is an arbitrary k_2'-harmonic function over $x \in S$ and $k_1 \in \mathbb{N}_0$, $k_2' \in \mathbb{N}$ is (k_1+k_2')-harmonic in $x \in S$. We carry out the proof by induction (*) using two indices $k_1 \in \mathbb{N}_0$ and $k_2' \in \mathbb{N}$.

1) If $k_1 = 0$, then $G_{0,k_2'}(x,\xi) = F_{k_2'}(x,\xi)$ and therefore $G_{0,k_2'}(x,\xi)$ is k_2'-harmonic by condition. If $k_2' = 1$, then we have $G_{k_1,1}(x,\xi) = h^{k_1}(x,\xi) F_1(x,\xi)$ and since $h(x,\xi) = |x-\xi|^2$, then according to Almansi formula the function $G_{k_1,1}(x,\xi)$ is (k_1+1)-harmonic.

2) Suppose that the functions $h^{k_1-1}(x,\xi) F_{k_2'}(x,\xi)$ and $h^{k_1}(x,\xi) F_{k_2'-1}(x,\xi)$ are $(k_1+k_2'-1)$-harmonic in $x \in S$. Because by (2.3.8)

$$\frac{\partial}{\partial x_i} h^{k_1}(x,\xi) = 2k_1(x_i - \xi_i) h^{k_1-1}(x,\xi),$$

then using the operator Λ from (2.3.10) we get

$$\begin{aligned}
\Delta_x G_{k_1,k_2'}(x,\xi) &= F_{k_2'}(x,\xi) \Delta_x h^{k_1}(x,\xi) \\
&\quad + 4k_1 h^{k_1-1}(x,\xi) \sum_{i=1}^n (x_i - \xi_i) \frac{\partial F_{k_2'}(x,\xi)}{\partial x_i} + h^{k_1}(x,\xi) \Delta_x F_{k_2'}(x,\xi) \\
&= 2k_1(2k_1+n-2) h^{k_1-1}(x,\xi) F_{k_2'}(x,\xi) + 4k_1 h^{k_1-1}(x,\xi) \\
&\qquad \times \big(\Lambda_x F_{k_2'}(x,\xi) - \xi \cdot \nabla_x F_{k_2'}(x,\xi)\big) + h^{k_1}(x,\xi) \Delta_x F_{k_2'}(x,\xi). \quad (2.3.12)
\end{aligned}$$

In the last equality the following identity is used, similar to (2.3.11)

$$\Delta_x h^{k_1}(x,\xi) = 2k_1(2k_1+n-2)h^{k_1-1}(x,\xi).$$

By the induction hypothesis and by the property of the operator Λ, each term on the right-hand side of (2.3.12) is a $(k_1+k_2'-1)$-harmonic function in $x \in S$. Therefore, the function $G_{k_1,k_2'}(x,\xi)$ is (k_1+k_2')-harmonic in $x \in S$. Statement 2^0 is proved.

3^0. Now we choose $F_{k_2'}(x,\xi) = F_{k_2,k_3}(x,\xi)$, where $k_2' = k_2+k_3$. This is possible because, firstly, by Lemma 2.15 the function $F_1(x,\xi) = F_{0,1}(x,\xi) = E_2^*(x,\xi)$ is harmonic in $x \in S$, and secondly, according to the statement 1^0 of the lemma, the function $F_{k_2,k_3}(x,\xi)$ is k_2'-harmonic in $x \in S$, when $\xi \in \bar{S}$. Therefore, in accordance with 2^0, the statement of the lemma is true. The lemma is completely proved. □

2.3.2 Green's function $G_{2m}(x,\xi)$

Now we are able to obtain the Green's function representation for the Dirichlet problem (2.3.5)-(2.3.6).

Theorem 2.8. *The Green's function $G_{2m}(x,\xi)$ of the Dirichlet problem* (2.3.5)-(2.3.6) *for $n \geq 2$ and $m \in \mathbb{N}$ can be written as*

$$G_{2m}(x,\xi) = \mathcal{E}_{2m}(x,\xi) - \sum_{k=0}^{m-1} \frac{(|x|^2-1)^k(|\xi|^2-1)^k}{(2m-2,-2)_k(2,2)_k}\mathcal{E}_{2m-2k}\Big(\frac{x}{|x|}, |x|\xi\Big). \tag{2.3.13}$$

The symmetric function $G_{2m}(x,\xi)$ is m-harmonic in $x \in S$, $x \neq \xi \in \bar{S}$ and satisfies the equalities for $\xi \in S$

$$G_{2m}(x,\xi)|_{x\in\partial S} = 0,\ \frac{\partial G_{2m}(x,\xi)}{\partial\nu}\Big|_{x\in\partial S} = 0, \ldots, \frac{\partial^{m-1}G_{2m}(x,\xi)}{\partial\nu^{m-1}}\Big|_{x\in\partial S} = 0. \tag{2.3.14}$$

Proof. 1. First, let us check the correspondence of the function $G_{2m}(x,\xi)$, defined in (2.3.13), with the previously found functions (2.3.2) and (2.3.4). For $m = 1$ from (2.3.13), taking into account the notation (2.3.8), the condition $(a,b)_0 = 1$ and the equality $E(x,\xi) = \mathcal{E}_2(x,\xi)$, we obtain the well-known Green's function for the Laplace equation. For $m = 2$ from (2.3.13) we get

$$G_4(x,\xi) = \mathcal{E}_4(x,\xi) - \mathcal{E}_4^*(x,\xi) - \frac{(|x|^2-1)(|\xi|^2-1)}{2\cdot 2}\mathcal{E}_2^*(x,\xi),$$

which coincides with (2.3.2) for $n \geq 3$, since in this case, according to Remark 2.6, $\mathcal{E}_4(x,\xi) = E_4(x,\xi)$. If $m = 3$ and $n \geq 3$, $n \neq 4$, then by Remark 2.6 $\mathcal{E}_6(x,\xi) = E_6(x,\xi)$ and therefore, taking into account the previous equalities, (2.3.13) gives the equality

$$G_6(x,\xi) = \mathcal{E}_6(x,\xi) \\ - \mathcal{E}_6^*(x,\xi) - \frac{(|x|^2-1)(|\xi|^2-1)}{4\cdot 2}\mathcal{E}_4^*(x,\xi) - \frac{(|x|^2-1)^2(|\xi|^2-1)^2}{8\cdot 8}\mathcal{E}_2^*(x,\xi),$$

which corresponds to (2.3.4).

2. Let us check whether the function $G_{2m}(x,\xi)$ from (2.3.13) is m-harmonic with respect to $x \in S$ for $x \neq \xi \in \bar{S}$. To do this, note that in accordance with the notation (2.3.8) the equalities are true

$$h_*(x,\xi) - h(x,\xi) = 1 - 2x\cdot\xi + |x|^2|\xi|^2 - |x|^2 + 2x\cdot\xi - |\xi|^2 = (|x|^2-1)(|\xi|^2-1),$$

and therefore the equality (2.3.13) can be rewritten in the form

$$G_{2m}(x,\xi) = \mathcal{E}_{2m}(x,\xi) - \sum_{k=0}^{m-1}\frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k}\mathcal{E}^*_{2m-2k}(x,\xi). \tag{2.3.15}$$

By virtue of Lemma 2.15, it is sufficient to check the m-harmonicity of functions under the sum sign in (2.3.15). But this immediately follows from Lemma 2.16 since the common term of the sum in (2.3.15) is a $(k+m-k)$-harmonic function in $x \in S$ for $\xi \in \bar{S}$.

3. Now let us check the fulfillment of the boundary conditions (2.3.14). The outer unit normal derivative of the function $G_{2m}(x,\xi)$ to a sphere of radius $|x| < 1$ is equal

$$\frac{\partial G_{2m}(x,\xi)}{\partial \nu_x} = \sum_{i=1}^{n}\frac{x_i}{|x|}\frac{\partial G_{2m}(x,\xi)}{\partial x_i} = \frac{1}{|x|}\Lambda_x G_{2m}(x,\xi).$$

Therefore

$$\frac{\partial^2 G_{2m}(x,\xi)}{\partial \nu_x^2} = \Big(\frac{1}{|x|}\Lambda_x\frac{1}{|x|}\Lambda_x\Big)G_{2m}(x,\xi) = \frac{1}{|x|^2}\Lambda_x(-1+\Lambda_x)G_{2m}(x,\xi)$$

and therefore, using the notation $\Lambda^{[k]} = \Lambda(\Lambda-1)\ldots(\Lambda-k+1)$, we can write

$$\frac{\partial^k G_{2m}(x,\xi)}{\partial \nu_x^k} = \Big(\frac{1}{|x|}\Lambda_x\Big)^k G_{2m}(x,\xi)$$

$$= \frac{1}{|x|^k}\Lambda_x(\Lambda_x - 1)\ldots(\Lambda_x - k + 1)G_{2m}(x,\xi) = \frac{1}{|x|^k}\Lambda_x^{[k]}G_{2m}(x,\xi). \quad (2.3.16)$$

Thus, to satisfy the boundary conditions (2.3.14) we must prove that the equalities hold true

$$G_{2m}(x,\xi)|_{x\in\partial S} = 0,\ \Lambda_x G_{2m}(x,\xi)|_{x\in\partial S} = 0, \ldots, \Lambda_x^{[m-1]}G_{2m}(x,\xi)|_{x\in\partial S} = 0.$$

It is easy to see that these equalities are equivalent to the following equalities

$$G_{2m}(x,\xi)|_{x\in\partial S} = 0,\ \Lambda_x G_{2m}(x,\xi)|_{x\in\partial S} = 0, \ldots, \Lambda_x^{m-1}G_{2m}(x,\xi)|_{x\in\partial S} = 0. \quad (2.3.17)$$

Let us prove that they are true, which means that the boundary conditions (2.3.14) are also satisfied. If we put $x \in \partial S$ in (2.3.15), i.e. $|x| = 1$, then for $\xi \in S$, taking into account that $\mathcal{E}_{2m}(x,\xi) = \mathcal{E}^*_{2m}(x,\xi)$ and $h(x,\xi) = h_*(x,\xi)$ on ∂S, we get

$$G_{2m}(x,\xi)|_{x\in\partial S} = \mathcal{E}_{2m}(x,\xi)|_{x\in\partial S} - \frac{\mathcal{E}_{2m}(x,\xi)}{(2m-2,-2)_0(2,2)_0}\Big|_{x\in\partial S} = 0 \qquad (2.3.18)$$

for any $m \geq 1$.

Let us explore other boundary conditions from (2.3.17). It is easy to verify the equality

$$\Lambda_x G_{2m}(x,\xi) = \Lambda_x\mathcal{E}_{2m}(x,\xi) - \sum_{k=0}^{m-1}\frac{(h_*(x,\xi) - h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k}\Lambda_x\mathcal{E}^*_{2m-2k}(x,\xi)$$
$$- \sum_{k=1}^{m-1} k\Lambda_x(h_*(x,\xi) - h(x,\xi))\frac{(h_*(x,\xi) - h(x,\xi))^{k-1}}{(2m-2,-2)_k(2,2)_k}\mathcal{E}^*_{2m-2k}(x,\xi). \quad (2.3.19)$$

Consider the last term in the first sum on the right in (2.3.19), which has the form

$$\varepsilon_{m-2}(x) \equiv \frac{(h_*(x,\xi) - h(x,\xi))^{m-1}}{(2m-2,-2)_{m-1}(2,2)_{m-1}}\Lambda_x\mathcal{E}^*_2(x,\xi).$$

By Lemma 2.16 this function is m-harmonic in S. Since $h_*(x,\xi) = h(x,\xi)$ for $x \in \partial S$, then after applying operators Λ_x^k to the function $\varepsilon_{m-2}(x)$ for $k = 0, \ldots, m-2$ the limits for $x \to \partial S$ of all functions obtained in this case is turn to 0. The subscript of $\varepsilon_{m-2}(x)$ indicates the maximum degree of the operator Λ_x^k for which this property holds. Therefore, in the following presentation, all functions that have this property are denoted by one symbol $\varepsilon_{m-2}(x)$. These functions do not affect the fulfillment of boundary conditions (2.3.17).

Let $n \in \mathbb{N}_m$. Then, in accordance with (2.3.7), the equalities are true

$$\Lambda_x \mathcal{E}_{2m}(x,\xi) = \Lambda_x \frac{(-1)^m |x-\xi|^{2m-n}}{(2-n,2)_m (2,2)_{m-1}} = \frac{(-1)^m}{(2-n,2)_m(2,2)_{m-1}} \Lambda_x h^{m-n/2}(x,\xi)$$
$$= (m-n/2)\frac{-\mathcal{E}_{2m-2}(x,\xi)}{(2m-n)(2m-2)}\Lambda_x h(x,\xi) = -\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\mathcal{E}_{2m-2}(x,\xi). \quad (2.3.20)$$

Let us study the case $n \in \mathbb{N}_m^c$. Let first $n = 2m$ and that means $n \in \mathbb{N}_{m-1}$. Then, in accordance with (2.3.7), we have

$$\Lambda_x \mathcal{E}_{2m}(x,\xi) = \Lambda_x \frac{(-1)^m \ln|x-\xi|}{(2-2m,2)_m^* (2,2)_{m-1}} = \frac{(-1)^m}{(-1)^{m-1}(2,2)_{m-1}(2,2)_{m-1}} \frac{1}{2}\Lambda_x \ln h(x,\xi)$$
$$= \frac{-\mathcal{E}_{2m-2}(x,\xi)}{2(2m-2)}\Lambda_x h(x,\xi) = -\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\mathcal{E}_{2m-2}(x,\xi).$$

Let $n \in \mathbb{N}_m^c$, but $n \neq 2m$. Then again, in accordance with (2.3.7), we have

$$\Lambda_x \mathcal{E}_{2m}(x,\xi) = \Lambda_x \frac{(-1)^m |x-\xi|^{2m-n}}{(2-n,2)_m^* (2,2)_{m-1}}\left(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k}\right)$$
$$= -(m-n/2)\frac{(-1)^{m-1}|x-\xi|^{2m-2-n}\Lambda_x h(x,\xi)}{(2m-n)(2m-2)(2-n,2)_{m-1}^*(2,2)_{m-2}} \times \left(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k}\right)$$
$$+ \frac{(-1)^m|x-\xi|^{2m-n}}{(2m-n)(2m-2)(2-n,2)_{m-1}^*(2,2)_{m-2}}\frac{1}{2}\Lambda_x \ln h(x,\xi)$$
$$= -\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\frac{(-1)^{m-1}|x-\xi|^{2m-2-n}}{(2-n,2)_{m-1}^*(2,2)_{m-2}}\left(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} + \frac{1}{2m-n}\right)$$
$$= -\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\mathcal{E}_{2m-2}(x,\xi).$$

Thus, the formula (2.3.20) is valid for all $n \geq 2$. Similar to the calculations done above, it is easy to obtain for $n \geq 2$ the equality

$$\Lambda_x \mathcal{E}^*_{2m}(x,\xi) = -\frac{\Lambda_x h_*(x,\xi)}{2(2m-2)} \mathcal{E}^*_{2m-2}(x,\xi).$$

In accordance with (2.3.20) we rewrite the equality (2.3.19) in the form

$$\begin{aligned}\Lambda_x G_{2m}(x,\xi) &= -\frac{\Lambda_x h(x,\xi)}{2(2m-2)} \mathcal{E}_{2m-2}(x,\xi) \\ &+ \sum_{k=0}^{m-2} \frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k} \frac{\Lambda_x h_*(x,\xi)}{2(2m-2k-2)} \mathcal{E}^*_{2m-2k-2}(x,\xi) + \varepsilon_{m-2}(x) \\ &- \sum_{k=0}^{m-2} (k+1)\Lambda_x(h_*(x,\xi)-h(x,\xi)) \frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-2,-2)_{k+1}(2,2)_{k+1}} \mathcal{E}^*_{2m-2k-2}(x,\xi).\end{aligned} \tag{2.3.21}$$

Note that in the last sum the substitution of the index $k \to k+1$ is made. Let us combine the last two sums, which have the same upper and lower indices, into one sum

$$\begin{aligned}&\sum_{k=0}^{m-2} \frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k} \mathcal{E}^*_{2m-2k-2}(x,\xi) \\ &\qquad \times \left(\frac{\Lambda_x h_*(x,\xi)}{2(2m-2k-2)} - \frac{(k+1)\Lambda_x(h_*(x,\xi)-h(x,\xi))}{(2m-2k-2)(2k+2)}\right) \\ &\qquad\qquad = \sum_{k=0}^{m-2} \frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k} \frac{\Lambda_x h(x,\xi)}{2(2m-2k-2)} \mathcal{E}^*_{2m-2k-2}(x,\xi),\end{aligned}$$

and then, using the equality

$$(2m-2,-2)_{k+1} = (2m-2,-2)_k(2m-2k-2) = (2m-2)(2m-4,-2)_k,$$

we can write

$$\frac{\Lambda_x h(x,\xi)}{2(2m-2)} \sum_{k=0}^{m-2} \frac{(h_*(x,\xi)-h(x,\xi))^k}{(2m-4,-2)_k(2,2)_k} \mathcal{E}^*_{2m-2k-2}(x,\xi).$$

Thus (2.3.21) is converted to the form

$$\Lambda_x G_{2m}(x,\xi) = -\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\mathcal{E}_{2m-2}(x,\xi)$$
$$+\frac{\Lambda_x h(x,\xi)}{2(2m-2)}\sum_{k=0}^{(m-1)-1}\frac{(h_*(x,\xi)-h(x,\xi))^k}{(2(m-1)-2,-2)_k(2,2)_k}\mathcal{E}^*_{2(m-1)-2k}(x,\xi)+\varepsilon_{m-2}(x),$$

from which we get

$$\Lambda_x G_{2m}(x,\xi) = -\frac{\Lambda_x h(x,\xi)}{4(m-1)}G_{2m-2}(x,\xi)+\varepsilon_{m-2}(x). \tag{2.3.22}$$

It immediately follows that

$$\Lambda_x^2 G_{2m}(x,\xi) = -\frac{\Lambda_x^2 h(x,\xi)}{4(m-1)}G_{2m-2}(x,\xi)-\frac{\Lambda_x h(x,\xi)}{4(m-1)}\Lambda_x G_{2m-2}(x,\xi)+\Lambda_x\varepsilon_{m-2}(x).$$

Due to the equality (2.3.22), taken for $m = m-1$, and taking into account the equalities

$$\Lambda_x\varepsilon_{m-2}(x)=\varepsilon_{m-3}(x), \quad -\frac{\Lambda_x h(x,\xi)}{4(m-1)}\varepsilon_{m-3}(x)=\varepsilon_{m-3}(x),$$

and also since $\varepsilon_{m-3}(x)+\varepsilon_{m-3}(x)=\varepsilon_{m-3}(x)$ we can write

$$\Lambda_x^2 G_{2m}(x,\xi) = -\frac{\Lambda_x^2 h(x,\xi)}{4(m-1)}G_{2m-2}(x,\xi)$$
$$+\frac{(\Lambda_x h(x,\xi))^2}{4^2(m-1)(m-2)}G_{2m-4}(x,\xi)+\varepsilon_{m-3}(x). \tag{2.3.23}$$

Let us apply induction on k. Based on the equality (2.3.23) and taking into account (2.3.22), where $k=1$, we assume that for some $1<k<m-1$ the equality is true

$$\Lambda_x^{k-1}G_{2m}(x,\xi)=\sum_{i=1}^{k-1}g_{2i}^{(k-1)}(x,\xi)G_{2m-2i}(x,\xi)+\varepsilon_{m-k}(x), \tag{2.3.24}$$

where $g_{2i}^{(k-1)}(x,\xi)$ are some polynomials of degree $2i$ in x. For example, from (2.3.22) and (2.3.23), where $k=1$ and $k=2$, respectively, we find

$$g_2^{(1)}(x,\xi)=-\frac{\Lambda_x h(x,\xi)}{4(m-1)}, \quad g_2^{(2)}(x,\xi)=-\frac{\Lambda_x^2 h(x,\xi)}{4(m-1)},$$
$$g_4^{(2)}(x,\xi)=\frac{(\Lambda_x h(x,\xi))^2}{16(m-1)(m-2)}.$$

Using (2.3.24) and (2.3.22) with $m = m - i$ we write

$$\Lambda_x^k G_{2m}(x,\xi) = \sum_{i=1}^{k-1} \Lambda_x g_{2i}^{(k-1)}(x,\xi) G_{2m-2i}(x,\xi) - \sum_{i=1}^{k-1} g_{2i}^{(k-1)}(x,\xi) \frac{\Lambda_x h(x,\xi)}{4(m-i-1)} \times G_{2m-2i-2}(x,\xi) + \sum_{i=1}^{k-1} g_{2i}^{(k-1)}(x,\xi)\varepsilon_{m-i-2}(x) + \Lambda_x \varepsilon_{m-k}(x).$$

Hence, since

$$\sum_{i=1}^{k-1} g_{2i}^{(k-1)}(x,\xi) \frac{\Lambda_x h(x,\xi)}{4(m-i-1)} G_{2m-2i-2}(x,\xi) = \sum_{i=2}^{k} g_{2i-2}^{(k-1)}(x,\xi) \frac{\Lambda_x h(x,\xi)}{4(m-i)} G_{2m-2i}(x,\xi),$$

we get the equality

$$\Lambda_x^k G_{2m}(x,\xi) = \sum_{i=1}^{k} \Big(\Lambda_x g_{2i}^{(k-1)}(x,\xi) - g_{2i-2}^{(k-1)}(x,\xi) \frac{\Lambda_x h(x,\xi)}{4(m-i)} \Big) G_{2m-2i}(x,\xi) + \varepsilon_{m-k-1}(x), \quad (2.3.25)$$

where we should assume that $g_{2i}^{(k-1)}(x,\xi) = 0$ for $i = 0$ or $i > k-1$. In addition, it is taken into account that

$$\sum_{i=1}^{k-1} g_{2i}^{(k-1)}(x,\xi)\varepsilon_{m-i-2}(x) + \Lambda_x \varepsilon_{m-k}(x) = \varepsilon_{m-k-1}(x),$$

since the smallest index of the functions $\varepsilon_{m-i-2}(x)$ under the sum sign is equal to $m-k-1$. If we designate

$$g_{2i}^{(k)}(x,\xi) = \Lambda_x g_{2i}^{(k-1)}(x,\xi) - g_{2i-2}^{(k-1)}(x,\xi) \frac{\Lambda_x h(x,\xi)}{4(m-i)},$$

then (2.3.25) coincides with (2.3.24) for $k = k-1$. It is clear that the function $g_{2i}^{(k)}(x,\xi)$ is a polynomial of degree $2i$ in x, since by the induction hypothesis $\deg g_{2i}^{(k-1)}(x,\xi) = 2i$. Therefore, the induction step is proved. Thus, for $1 \le k \le m-1$ the equality is valid

$$\Lambda_x^k G_{2m}(x,\xi) = \sum_{i=1}^{k} g_{2i}^{(k)}(x,\xi) G_{2m-2i}(x,\xi) + \varepsilon_{m-k-1}(x).$$

If we use the equality (2.3.18) here, then, taking into account the definition of the functions $\varepsilon_k(x)$, we get

$$\Lambda_x^k G_{2m}(x,\xi)|_{x\in\partial S} = \sum_{i=1}^{k} g_{2i}^{(k)}(x,\xi) G_{2m-2i}(x,\xi)|_{x\in\partial S} + \varepsilon_{m-k-1}(x)|_{x\in\partial S} = 0,$$

for $k = 0,\dots,m-1$. This means the fulfillment of the boundary conditions (2.3.17) and therefore the equalities (2.3.14). The theorem is proved. □

Remark 2.9. The previously obtained formula (2.3.2) for the case $m = 2$ is not correct for all $n \geq 2$. For $n = 2$, in accordance with Theorem 2.7 on page 108, the Green's function has the form

$$G_4(x,\xi) = E_4(x,\xi) \\ - E_4^*(x,\xi) - \frac{(|x|^2-1)(|\xi|^2-1)}{4}\Big(E_2^*(x,\xi) + \frac{1}{2}\Big). \quad (2.3.26)$$

Nevertheless, the equality (2.3.13) is correct in this case as well. Indeed, since $\mathbb{N}_2^c = \{2,4\}$, and $\mathbb{N}_1^c = \{2\}$, then by virtue of (2.3.9) we have

$$E_2(x,\xi) = \mathcal{E}_2(x,\xi),$$
$$E_4(x,\xi) = \mathcal{E}_4(x,\xi) - \frac{|x-\xi|^2}{(0,2)_2^*(2,2)_1}\sum_{k=1}^{1}\frac{1}{2k} = \mathcal{E}_4(x,\xi) - \frac{|x-\xi|^2}{8}.$$

Substituting these values into (2.3.26) and taking into account that $h_*(x,\xi) - h(x,\xi) = (|x|^2-1)(|\xi|^2-1)$ we get the equality (2.3.13)

$$G_4(x,\xi) = \mathcal{E}_4(x,\xi) - \frac{|x-\xi|^2}{8} - \mathcal{E}_4^*(x,\xi) \\ + \frac{|x/|x| - |x|\xi|^2}{8} - \frac{(|x|^2-1)(|\xi|^2-1)}{4}\Big(\mathcal{E}_2^*(x,\xi) + \frac{1}{2}\Big) \\ = \mathcal{E}_4(x,\xi) - \mathcal{E}_4^*(x,\xi) - \frac{(|x|^2-1)(|\xi|^2-1)}{4}\mathcal{E}_2^*(x,\xi).$$

For $m = 3$ and $n = 4$ we have $n \in \mathbb{N}_2^c$. In this case, in [110, Theorem 2] a representation of the Green's function is obtained, which also differs in form from (2.3.4), but coincides with (2.3.13).

2.3.3 Solution of the homogeneous Dirichlet problem

To construct a solution to the Dirichlet problem, we formulate the following statement following from Theorem 2.8.

Theorem 2.9. *The solution to the homogeneous Dirichlet boundary value problem* (2.3.5)-(2.3.6) *for the polyharmonic equation in the unit ball S for $f \in C^1(\bar{S})$ and $n \geq 2$ can be represented as*

$$u(x) = \frac{(-1)^m}{\omega_n} \int_S G_{2m}(x, \xi) f(\xi)\, d\xi, \tag{2.3.27}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Proof. Let us calculate the result of applying the operator Δ^{m-1} to the function $u(x)$ given by the equality (2.3.27). To do this, note that potential-type integrals $\int_S \frac{\rho(\xi)}{|x-\xi|^\alpha}\, d\xi$ are functions from the class $C^p(\mathbb{R}^n)$ for a bounded and integrable function $\rho(x)$ and, in this case, differentiation of order $p \in \mathbb{N}_0$ with respect to x can be entered under the integral sign for any p such that $\alpha + p < n$ [183, p. 25]. In our case, for the singular term of the function $G_{2m}(x, \xi)$ from (2.3.13) we have $\alpha = n - 2m + 1$, and therefore for the integral

$$u_1(x) = \frac{(-1)^m}{\omega_n} \int_S \mathcal{E}_{2m}(x, \xi) f(\xi)\, d\xi$$

we have $p = 2m - 2$ and therefore $u_1 \in C^{2m-2}(\mathbb{R}^n)$, which means the operator Δ^{m-1} can be entered under the integral sign.

Let $n \in \mathbb{N}_m$, then from (2.3.9) it follows that $\mathcal{E}_{2m}(x, \xi) = E_{2m}(x, \xi)$ and therefore since due to Lemma 4.2 $\Delta E_{2m}(x, \xi) = -E_{2m-2}(x, \xi)$ for $x \neq \xi$, then $\Delta \mathcal{E}_{2m}(x, \xi) = -\mathcal{E}_{2m-2}(x, \xi)$ for $x \neq \xi$. That is why

$$\Delta^{m-1} \mathcal{E}_{2m}(x, \xi) = (-1)^{m-1} \mathcal{E}_2(x, \xi) = (-1)^{m-1} E_2(x, \xi),$$

where $E_2(x, \xi)$ coincides with $E(x, \xi)$ from [28]. If $n \in \mathbb{N}_m^c$ and $n \neq 2$, then taking into account the equality $\Delta|x - \xi|^\alpha = \alpha(\alpha + n - 2)|x - \xi|^{\alpha-2}$ and in accordance with (2.3.9), we get

$$\begin{aligned}
\Delta^{m-1} \mathcal{E}_{2m}(x, \xi) &= \Delta^{m-1} E_{2m}(x, \xi) + \Delta^{m-1} \frac{(-1)^m |x - \xi|^{2m-n}}{(2 - n, 2)^*_m (2, 2)_{m-1}} \sum_{k=n/2}^{m-1} \frac{1}{2k} \\
&= (-1)^{m-1} E_2(x, \xi) + |x - \xi|^{2-n} \frac{(-1)^m (4 - n, 2)_{m-1} (2, 2)_{m-1}}{(2 - n, 2)^*_m (2, 2)_{m-1}}
\end{aligned}$$

$$\times \sum_{k=n/2}^{m-1} \frac{1}{2k} = (-1)^{m-1} E_2(x,\xi).$$

The fraction is turned to zero because among the factors in the expression $(4-n,2)_{m-1}$ from its numerator, due to the conditions $n \in \mathbb{N}_m^c$ and $n \neq 2$, there is one zero. In the case $n = 2$ we similarly find

$$\Delta^{m-1} \mathcal{E}_{2m}(x,\xi) = (-1)^{m-1}\big(E_2(x,\xi) - M\big),$$

where $M = \sum_{k=1}^{m-1} \frac{1}{2k}$. This means that for $n > 2$ we have

$$\Delta^{m-1} u_1(x) = \frac{(-1)^m}{\omega_n} \int_S \Delta_x^{m-1} \mathcal{E}_{2m}(x,\xi) f(\xi)\, d\xi = -\frac{1}{\omega_n} \int_S E_2(x,\xi) f(\xi)\, d\xi,$$

and for $n = 2$ we get accordingly

$$\Delta^{m-1} u_1(x) = -\frac{1}{\omega_n} \int_S (E_2(x,\xi) - M) f(\xi)\, d\xi.$$

As a result, for $n > 2$, by the property of the volume potential, we can write

$$\Delta^m u_1(x) = \Delta\Big(-\frac{1}{\omega_n} \int_S E_2(x,\xi) f(\xi)\, d\xi\Big) = f(x), \quad x \in S$$

and for $n = 2$ we have the same equality

$$\Delta^m u_1(x) = \Delta\Big(-\frac{1}{\omega_n} \int_S E_2(x,\xi) f(\xi)\, d\xi + \frac{M}{\omega_n} \int_S f(\xi)\, d\xi\Big) = f(x), \quad x \in S.$$

The condition $f \in C^1(\bar{S})$ on the function f is sufficient to satisfy the equality $\Delta(\Delta^{m-1} u_1)(x) = f(x) in$S) [28]. In Lemma 2.16, taking into account the notation $(|x|^2-1)(|\xi|^2-1) = h_*(x,\xi) - h(x,\xi)$ from Theorem 2.8, it is proved that a function of the form

$$\sum_{k=0}^{m-1} \frac{(|x|^2-1)^k(|\xi|^2-1)^k}{(2m-2,-2)_k(2,2)_k} \mathcal{E}^*_{2m-2k}(x,\xi) = \sum_{k=0}^{m-1} \frac{(h_*(x,\xi) - h(x,\xi))^k}{(2m-2,-2)_k(2,2)_k} \mathcal{E}^*_{2m-2k}(x,\xi)$$

is m-harmonic with respect to x in S for any $\xi \in \bar{S}$ and can be differentiated with respect to x under the integral sign over ξ any number of times. Let us denote the integral over $\xi \in S$ of this function multiplied by $(-1)^m/\omega_n f(\xi)$ by $u_2(x)$. Then we have

$$\Delta^m u_2(x) = \frac{(-1)^m}{\omega_n} \sum_{k=0}^{m-1} \int_S \Delta_x^m \frac{(|x|^2-1)^k(|\xi|^2-1)^k}{(2m-2,-2)_k(2,2)_k} \mathcal{E}^*_{2m-2k}(x,\xi) f(\xi)\, d\xi = 0.$$

Therefore, the function $u(x)$ from (2.3.27), taking into account the equality (2.3.13), satisfies the equation (2.3.5)

$$\Delta^m u(x) = \Delta^m u_1(x) - \Delta^m u_2(x) = f(x), \quad x \in S.$$

Further, due to the fact that for the function $u(x)$ from (2.3.27) we ultimately have the inclusion $u \in C^{2m-2}(\bar{S})$, then passing to the limit $x \to \partial S$ for functions $\Lambda_x^{[k]} u(x)$, $k = 0, \dots, m-1$ can be entered under the integral sign. Using (2.3.17) and (2.3.16) we find

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial S} = \frac{(-1)^m}{\omega_n} \int_S \Lambda_x^{[k]} G_{2m}(x,\xi)\big|_{x \in \partial S} f(\xi)\, d\xi = 0,$$

for $k = 0, \dots, m-1$, which means the function $u(x)$ from (2.3.27) satisfies all boundary conditions (2.3.6) . The theorem is proved. □

2.3.4 The simplest right-hand side of the equation

Let us consider the simplest special case of the polynomial right-hand side in the equation (2.3.5). We generalize the result of Theorem 2.6.

Theorem 2.10. *Let in the equation* (2.3.5) $f(x) = |x|^{2l} H_k(x)$, *where* $H_k(x)$ *is a homogeneous harmonic polynomial of degree* $k \in \mathbb{N}_0$, $l \in \mathbb{N}$ *and* $n \geq 2$. *Then the solution to the Dirichlet problem* (2.3.5)-(2.3.6) *has the form*

$$u(x) = \frac{(-1)^m}{\omega_n} \int_{|\xi|<1} G_{2m}(x,\xi)|x|^{2l} H_k(\xi)\, d\xi = \Bigg(|x|^{2l+2m} - \sum_{i=0}^{m-1} \binom{l+m}{i} \big(|x|^2 - 1\big)^i\Bigg) \frac{H_k(x)}{(2l+2,2)_m (2l+2k+n,2)_m}, \tag{2.3.28}$$

where $(a,b)_m$ *is the generalized Pochhammer symbol (see Definition 1.3).*

Proof. The case $m = 2$ is considered in Theorem 2.6. For $m = 3$ in [110, Corollary 2] it is established that the solution to the Dirichlet problem for the 3-harmonic equation for $f(x) = |x|^{2l} H_m(x)$ can be written as

$$u(x) = \frac{|x|^{2l+6} - 1 - (l+3)(|x|^2-1) - \frac{1}{2}(l+2)(l+3)(|x|^2-1)^2}{(2l+2,2)_3 (2l+2k+n,2)_3} H_k(x).$$

Due to the uniqueness of the solution to the Dirichlet problem (Theorem 2.16) and the equality (2.3.27) we easily obtain the equality (2.3.28) for $m = 3$. The equality

(2.3.28) generalizes this formula to arbitrary $m \in \mathbb{N}$. Let us check that the function $u(x)$ given by the right-hand side of (2.3.28) is a solution to the homogeneous Dirichlet problem with $f(x) = |x|^{2l} H_k(x)$. It is not difficult to see that since

$$\Delta|x|^{2l} H_k(x) = 2l(2l + 2k + n - 2)|x|^{2l-2} H_k(x),$$

then from (2.3.28), taking into account that the functions $|x|^{2k} H_k(x)$ for $k \leq m - 1$ are m-harmonic polynomials, we find

$$\begin{aligned}\Delta^m u(x) \\ &= \frac{(2l + 2m) \ldots (2l + 2) \cdot (2l + 2m + 2k + n - 2) \ldots (2l + 2k + n)}{(2l + 2, 2)_m (2l + 2k + n, 2)_m} \\ &\qquad \times |x|^{2l} H_k(x) = |x|^{2l} H_k(x),\end{aligned}$$

i.e. $u(x)$ satisfies the equation (2.3.5).

Let us check the boundary conditions (2.3.6). First, we select a constant factor from the function $u(x)$, i.e. consider the function $u^*(x) = (2l + 2, 2)_m (2l + 2k + n, 2)_m u(x)$. It is clear that if $u^*(x)$ satisfies the conditions (2.3.17), then the function $u(x)$ also satisfies them, which means it also satisfies the conditions (2.3.6). Next, let us represent the function $u^*(x)$ as a product of two polynomials $u^*(x) = G_m(x) H_k(x)$, where

$$G_m(x) = |x|^{2l+2m} - \sum_{i=0}^{m-1} \binom{l + m}{i} \left(|x|^2 - 1\right)^i.$$

It is easy to see that $G_m(x)|_{\partial S} = 0$ for $m \geq 1$, which means $u^*(x)|_{\partial S} = 0$. Let us calculate $\Lambda u^*(x)$. We have

$$\begin{aligned}\Lambda u^*(x) &= H_k(x) \Lambda G_m(x) + G_m(x) \Lambda H_k(x) = \Big((2l + 2m)|x|^{2l+2m} \\ &\quad - \sum_{i=1}^{m-1} \binom{l + m}{i} 2i|x|^2 \left(|x|^2 - 1\right)^{i-1}\Big) H_k(x) + k G_m(x) H_k(x) \\ &= (2l + 2m)\Big(|x|^{2l+2m-2} - \sum_{i=1}^{m-1} \binom{l + m - 1}{i - 1} \left(|x|^2 - 1\right)^{i-1}\Big)|x|^2 H_k(x) \\ &\quad + k G_m(x) H_k(x) = (2l + 2m) G_{m-1}(x) \mathcal{H}_{k+2}(x) + k G_m(x) H_k(x),\end{aligned}$$

whence it follows that $\Lambda u^*|_{\partial S} = 0$ for $m \geq 2$. Here we denoted $\mathcal{H}_{k+2}(x) = |x|^2 H_k(x)$. Further we assume that $\mathcal{H}_{k+2i+2}(x) = |x|^2 \mathcal{H}_{k+2i}(x)$ for $i \in \mathbb{N}$. It is easy to see that, similar to what was done above, the equality holds

$$\begin{aligned}\Lambda\big(G_{m-i}(x)\mathcal{H}_{k+2i}(x)\big)\\ &= (2l+2m-2i)G_{m-i-1}(x)\mathcal{H}_{k+2i+2}(x) + (k+2i)G_{m-i}(x)\mathcal{H}_{k+2i}(x),\end{aligned}$$

where $m-i-1 \geq 1$. If we now continue to apply the operator Λ to $\Lambda u^*(x)$, we can see that the polynomial $\Lambda^s u^*(x)$ is written as a sum of polynomials of the form $G_{m-i}(x)\mathcal{H}_{k+2i}(x)$, where $i = 0, \dots, s$ with some coefficients.

Next, separating separately from the polynomial $\Lambda^s u^*(x)$ the term with the lowest index i from the polynomial $G_i(x)$, and the sum of the remaining terms with a higher index i from $G_i(x)$ denoting as $O_i(x)$, we can write the equality

$$\Lambda^s u^*(x) = (2l+2m)\dots(2l+2m-2s+2)G_{m-s}(x)\mathcal{H}_{k+2s}(x) + O_{m-s}(x),$$

where $\mathcal{H}_{k+2s}(x) = |x|^{2s}H_k(x)$ and $m-1 \geq s$. Since $G_i(x)|_{\partial S} = 0$ for $i \geq 1$, which means $O_{m-s}(x)|_{\partial S} = 0$, for $s \leq m-1$, then from the resulting equality it follows that $\Lambda^s u^*(x)|_{\partial S} = 0$, for $s = 0, \dots, m-1$. Thus, the boundary conditions (2.3.6) for the function $u^*(x)$, and therefore for the function $u(x)$ from the right side of (2.3.28), are satisfied. From the uniqueness of the solution to the Dirichlet problem (Theorem 2.16) and the equality (2.3.27) it follows that the equality (2.3.28) is valid. The theorem is proved. □

2.4 Dirichlet problem for the homogeneous polyharmonic equation

In connection with the explicit representation of the solution to the Dirichlet problem with zero data on the boundary ∂S, obtained in Theorem 2.9, in order to find a solution to the complete Dirichlet problem for m-harmonic equation in a ball, there is a lack of a way to construct a solution to the Dirichlet problem for a homogeneous m-harmonic equation with nonzero functions $\varphi_i(x)$, $i = 0, \dots, m-1$ on ∂S. This section is devoted to that issue. For polynomial data, Theorem 2.15 provides a formula for finding a solution to the Dirichlet problem for the Poisson equation. The results of Section 2.4 are published in papers [86, 116, 120].

In the work [112] it is established that the solution to the Dirichlet problem for the homogeneous biharmonic equation in S can be transformed to the form

$$u(x) = u_0(x) + \frac{1-|x|^2}{2}\Lambda u_0(x) - \frac{1-|x|^2}{2}u_1(x), \tag{2.4.1}$$

where the harmonic functions $u_0(x)$ and $u_1(x)$ are solutions of the Dirichlet problems with values $\varphi_0(x)$ and $\varphi_1(x)$ on the boundary ∂S, respectively. Obviously, the functions $u_i(x)$ for $i = 0, 1$ can be represented in the form

$$u_i(x) = -\frac{1}{\omega_n}\int_{\partial S}\frac{\partial G_2(x,\xi)}{\partial\nu}\varphi_i(\xi)\,ds_\xi = \frac{1}{\omega_n}\int_{\partial S}\frac{1-|x|^2}{|x-\xi|^n}\varphi_i(\xi)\,ds_\xi,$$

where $G_2(x,\xi)$ is the Green's function of the Dirichlet problem for the Poisson equation.

The goal of this section is to obtain a formula similar to (2.4.1) to represent the solution to the Dirichlet problem for the homogeneous m-harmonic equation in the ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$, but without using the Green's function (see remark 2.14 on page 147).

Consider the complete Dirichlet problem

$$\Delta^m u(x) = f(x), \quad x \in S, \tag{2.4.2}$$

$$u|_{\partial\Omega} = \varphi_0(s), \left.\frac{\partial u}{\partial\nu}\right|_{\partial S} = \varphi_1(s), \ldots, \left.\frac{\partial^{m-1}u}{\partial\nu^{m-1}}\right|_{\partial S} = \varphi_{m-1}(s), s \in \partial S, \tag{2.4.3}$$

where ν is the outer normal to the unit sphere ∂S. The required smoothness of the functions $f(x)$ and $\varphi_k(s)$, $k = 0, \ldots, m-1$ will be indicated below.

2.4.1 Preliminary transformations

Similarly to the representation of the solution to the Dirichlet problem in the biharmonic case (2.4.1), we take an m-harmonic function in S satisfying the boundary value conditions (2.4.3) in the form

$$u(x) = \sum_{k=0}^{m-1} p_k(x)(|x|^2-1)^k, \tag{2.4.4}$$

where $p_k(x) \in C^{m-1}(\bar{S})$, $k = 0, \ldots, m-1$ is a set of harmonic functions in S. The above required smoothness of the functions $p_k(x)$ is necessary, since the function $u(x)$ must satisfy the boundary conditions (2.4.3).

Let $P_k(t) = \sum_{s=0}^{k} c_s t^s$ be some polynomial. We use the factorial polynomial corresponding to the polynomial $P_k(t)$ by the equality $P_{[k]}(t) = \sum_{s=0}^{k} c_s t^{[s]}$, where the factorial monomial $t^{[k]}$ is defined by the equality $t^{[k]} = t(t-1)\ldots(t-k+1)$. Recall that the operator $\Lambda u = \sum\limits_{k=1}^{n} x_k u_{x_k}$ has an important boundary property (see Lemma 2.1)

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial S} = \Lambda^{[k]} u|_{\partial S}, \tag{2.4.5}$$

and if the function $p(x)$ is harmonic, then the function $\Lambda p(x)$ is also harmonic.

To apply the operator Λ to a product of functions, the following lemma is needed (see Lemma 3.33 on page 288).

Lemma 2.17. *Let $p, q \in C^k(S)$, then the operator Λ satisfies the equality*

$$\Lambda^{[k]}(pq) = \sum_{i=0}^{k} \binom{k}{i} \Lambda^{[i]} p \Lambda^{[k-i]} q. \tag{2.4.6}$$

Let us prove the following main relation.

Theorem 2.11. *Let, for the given harmonic in S functions $q_i(x) \in C^{k_i}(\bar{S})$, $i = 0, \ldots, m-1$, where $k_i \geq 0$, there exist harmonic functions $p_i(x) \in C^{m-1}(\bar{S})$, $i = 0, \ldots, m-1$, such that the equalities hold:*

$$\begin{gathered} r_s(x) = \sum_{k=s}^{m-1} (-1)^{k-s} \binom{k}{s} p_k(x), \quad s_j(x) = \sum_{s=0}^{m-1} (2s)^{[j]} r_s(x), \\ q_i(x) = \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} s_j(x), \end{gathered} \tag{2.4.7}$$

where $s, j, i = 0, \ldots, m-1$. Then, for such $p_0(x), \ldots, p_{m-1}(x)$, the m-harmonic function $u(x) \in C^{m-1}(\bar{S})$, determined in (2.4.4)*, satisfies the Dirichlet boundary value conditions*

$$\frac{\partial^i u}{\partial \nu^i}\Big|_{\partial S} = q_i(x)|_{\partial S}, \quad i = 0, \ldots, m-1. \tag{2.4.8}$$

Proof. It is easy to see that due to (2.4.6), we have

$$\begin{aligned} \Lambda^{[i]} u(x) &= \sum_{k=0}^{m-1} \Lambda^{[i]} (p_k(x)(|x|^2-1)^k) = \sum_{k=0}^{m-1} \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} p_k(x) \Lambda^{[j]} (|x|^2-1)^k \\ &= \sum_{k=0}^{m-1} \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} p_k(x) \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} (2s) \cdots (2s-j+1) |x|^{2s}, \end{aligned}$$

where $i = 0, \dots, m-1$, and therefore, in accordance with the formulas (2.4.5) and (2.4.7) and the required smoothness of functions $p_i(x)$ and $q_i(x)$, we obtain

$$\begin{aligned}\frac{\partial^i u}{\partial \nu^i}\Big|_{\partial S} = \Lambda^{[i]} u(x)_{|\partial S} &= \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} \sum_{k=0}^{m-1} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} (2s)^{[j]} p_k(x)_{|\partial S} \\ &= \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} \sum_{s=0}^{m-1} (2s)^{[j]} \sum_{k=s}^{m-1} (-1)^{k-s} \binom{k}{s} p_k(x)_{|\partial S} \\ = \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} \sum_{s=0}^{m-1} (2s)^{[j]} r_s(x)(x)_{|\partial S} &= \sum_{j=0}^{i} \binom{i}{j} \Lambda^{[i-j]} s_j(x)_{|\partial S} = q_i(x)_{|\partial S},\end{aligned}$$

where $i = 0, \dots, m-1$. The theorem is proved. □

Theorem 2.11 implies that if by the given harmonic in S functions $\{q_i(x) \in C^{k_i}(\bar{S}), i = 0, \dots, m-1\}$ we are able to find functions $\{s_j(x), j = 0, \dots, m-1\}$, and then $\{r_s(x), s = 0, \dots, m-1\}$, and finally, $\{p_k(x) \in C^{m-1}(\bar{S}), k = 0, \dots, m-1\}$ such that the equalities (2.4.7) hold, then the m-harmonic function $u(x) \in C^{m-1}(\bar{S})$ satisfies the boundary conditions (2.4.8). Therefore, we need to reverse the transformations (2.4.7). This requires auxiliary constructions.

Let a polynomial $P(\lambda)$ be written in factorial monomials

$$P(\lambda) = \sum_{k=0}^{m} p_k \lambda^{[k]}. \tag{2.4.9}$$

Consider the following operation of "factorial" differentiation of polynomials

$$P^{(k)}(\lambda) \equiv \left(P^{(k-1)}(\lambda)\right)^{(1)} = P^{(k-1)}(\lambda+1) - P^{(k-1)}(\lambda), \tag{2.4.10}$$

where $P^{(0)}(\lambda) = P(\lambda)$. It is easy to see that this differentiation reduces the degree of a polynomial by 1 and

$$\left(\lambda^{[k]}\right)^{(1)} = (\lambda+1)^{[k]} - \lambda^{[k]} = \lambda^{[k-1]}(\lambda + 1 - (\lambda - k + 1)) = k\lambda^{[k-1]},$$

which means that

$$(\lambda^{[k]})^{(j)} = \begin{cases} k^{[j]}\lambda^{[k-j]}, & k \ge j \\ 0, & k < j \end{cases}.$$

Therefore, (2.4.9) implies

$$P^{(k)}(0) = \sum_{i=k}^{m} p_i i^{[k]} \lambda^{[i-k]}|_{\lambda=0} = p_k k!$$

, and hence, we have the equality

$$P(\lambda) = \sum_{k=0}^{m} \frac{P^{(k)}(0)}{k!} \lambda^{[k]}. \tag{2.4.11}$$

Moreover, the definition (2.4.10) of the derivative implies the following equality

$$P^{(k)}(\lambda) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(\lambda + i), \tag{2.4.12}$$

and therefore, coefficients of the polynomial $P(\lambda)$ from (2.4.9) can be calculated by the formula

$$p_k = \frac{P^{(k)}(0)}{k!} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(i).$$

The equality (2.4.12) can be easily proved by the induction. For $k = 1$, it coincides with the definition of the derivative. If it is true for the $(k-1)$th derivative, then

$$\begin{aligned}
P^{(k)}(\lambda) &= P^{(k-1)}(\lambda + 1) - P^{(k-1)}(\lambda) \\
&= \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} P(\lambda + 1 + i) + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k-1}{i} P(\lambda + i) \\
&= P(\lambda + k) + \sum_{i=1}^{k-1} (-1)^{k-i} \left(\binom{k-1}{i-1} + \binom{k-1}{i} \right) P(\lambda + i) + P(\lambda) \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(\lambda + i),
\end{aligned}$$

and hence the induction step is proved.

Now let us look at the last transformation from (2.4.7).

Lemma 2.18. *The last transformation from* (2.4.7) *is invertible in the form*

$$s_i(x) = \sum_{j=0}^{i} \binom{i}{j} (-\Lambda)^{[i-j]} q_j(x), \quad i = 0, \dots, m-1.$$

Proof. Let the operator matrix $B(\Lambda)$ be of the form $B(\Lambda) = \left\{\binom{i}{j}\Lambda^{[i-j]}\right\}_{i,j=\overline{0,m-1}}$, where $\binom{i}{j} = 0$ if $i < j$. We prove that

$$(B(\Lambda))^{-1} = A(\Lambda) \equiv \left\{\binom{i}{j}(-\Lambda)^{[i-j]}\right\}_{i,j=\overline{0,m-1}}.$$

Let us calculate the product $B(\Lambda)A(\Lambda) = (c_{i,j}(\Lambda))_{i,j=\overline{0,m-1}}$, where

$$c_{i,j}(\Lambda) = \sum_{k=0}^{m-1}\binom{i}{k}\Lambda^{[i-k]}\binom{k}{j}(-\Lambda)^{[k-j]}.$$

Let $i > j$. Since $\binom{i}{k}\binom{k}{j} = \binom{i}{j}\binom{i-j}{k-j}$, then

$$c_{i,j}(\Lambda) = \binom{i}{j}\sum_{k=j}^{i}\binom{i-j}{k-j}\Lambda^{[i-k]}(-\Lambda)^{[k-j]}$$

$$= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\Lambda^{[i-j-k]}(-\Lambda)^{[k]}. \quad (2.4.13)$$

By virtue of the binomial theorem, we have

$$0 = \big(\Lambda + (-\Lambda)\big)^{[m]} = \sum_{k=0}^{m}\binom{m}{k}\Lambda^{[m-k]}(-\Lambda)^{[k]},$$

where $m \in \mathbb{N}$. Hence, it follows that in the case $i > j$, we have $c_{i,j} = 0$. For $i = j$ from (2.4.13), we find

$$c_{i,i} = \binom{i}{i}\sum_{k=0}^{0}\binom{0}{k}\Lambda^{[-k]}(-\Lambda)^{[k]} = 1.$$

If $i < j$, then since $\binom{i}{k}\binom{k}{j} = 0$, it follows that $c_{i,j} = 0$. Therefore, $B(\Lambda)A(\Lambda) = I$, and hence, $(B(\Lambda))^{-1} = A(\Lambda)$. Thus, keeping in mind that $Q(x) = (q_0(x), \dots, q_{m-1}(x))^T$, and $S(x) = (s_0(x), \dots, s_{m-1}(x))^T$, we have $Q(x) = B(\Lambda)S(x) \Rightarrow S(x) = A(\Lambda)Q(x)$. The lemma is proved. □

Consider the following polynomials in λ

$$h_m(\lambda) = \prod_{k=0}^{m-1}(\lambda - 2k), \quad h_{m,i}(\lambda) = \frac{h_m(\lambda)}{(\lambda - 2i)h'_m(2i)}, \quad (2.4.14)$$

where $i = 0, \ldots, m-1$ and

$$h'_m(2i) = \prod_{k=0, k\neq i}^{m-1} (2i - 2k) = (-1)^{m-i-1}(2i)!!(2m-2i-2)!! \neq 0.$$

It is clear that $\deg h_{m,i}(\lambda) = m - 1$. Using the polynomials $h_{m,i}(\lambda)$, in accordance with the formula (2.4.9), we can define the coefficients $h_{m,i}^{(j)}$ by the following equalities

$$h_{m,i}(\lambda) = \sum_{j=0}^{m-1} h_{m,i}^{(j)} \lambda^{[j]}, \quad i = 0, \ldots, m-1. \tag{2.4.15}$$

By virtue of (2.4.11), we have $h_{m,i}^{(j)} = h_{m,i}^{(j)}(0)/j!$. Consider the matrix $\mathbb{H}_m = (h_{m,i}^{(j)})_{i,j=0,\ldots,m-1}$.

Lemma 2.19. *The second transformation from* (2.4.7) *can be reversed in the form*

$$r_j(x) = \sum_{i=0}^{m-1} h_{m,j}^{(i)} s_i(x), \quad j = 0, \ldots, m-1.$$

Proof. Denote

$$C = ((2j)^{[i]})_{i,j=\overline{0,m-1}} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 2^{[1]} & 4^{[1]} & \ldots & (2m-2)^{[1]} \\ & & & \ldots & \\ 0 & 2^{[m-1]} & 4^{[m-1]} & \ldots & (2m-2)^{[m-1]} \end{pmatrix},$$

and prove that

$$C^{-1} = \mathbb{H}_m \equiv \left(h_{m,i}^{(j)}\right)_{i,j=\overline{0,m-1}}. \tag{2.4.16}$$

It is easy to see that for $i = 1, \ldots, m-1$

$$\left(0, 2^{[i]}, \ldots, (2m-2)^{[i]}\right) \cdot \mathbb{H}_m = \left(\sum_{k=0}^{m-1} (2k)^{[i]} h_{m,k}^{(0)}, \ldots, \sum_{k=0}^{m-1} (2k)^{[i]} h_{m,k}^{(m-1)}\right). \tag{2.4.17}$$

Consider the following polynomials of degree $m-1$

$$g_i(\lambda) = \sum_{k=0}^{m-1} (2k)^{[i]} h_{m,k}(\lambda) = \sum_{k=0}^{m-1} \frac{(2k)^{[i]}}{h'_m(2k)} \frac{h_m(\lambda)}{(\lambda - 2k)}.$$

Using the notation of the formula (2.4.15), we write

$$g_i(\lambda) = \sum_{k=0}^{m-1} (2k)^{[i]} \sum_{j=0}^{m-1} h_{m,k}^{(j)} \lambda^{[j]} = \sum_{j=0}^{m-1} \lambda^{[j]} \sum_{k=0}^{m-1} (2k)^{[i]} h_{m,k}^{(j)} \equiv \sum_{j=0}^{m-1} g_i^{(j)} \lambda^{[j]},$$

and therefore, (2.4.17) can be written as

$$\left(0, 2^{[i]}, \ldots, (2m-2)^{[i]}\right) \cdot \mathbb{H}_m = \left(g_i^{(0)}, g_i^{(1)}, \ldots, g_i^{(m-1)}\right). \tag{2.4.18}$$

It is not hard to see that for $j = 0, \ldots, m-1$, the equalities

$$g_i(2j) = \sum_{k=0}^{m-1} \frac{(2k)^{[i]}}{h_m'(2k)} \frac{h_m(2j)}{(2j-2k)} = \frac{(2j)^{[i]}}{h_m'(2j)} h_m'(2j) = (2j)^{[i]}$$

hold. Since $g_i(\lambda)$ is a polynomial of degree $m-1$ and the equalities $(g_i(\lambda) - \lambda^{[i]})_{|\lambda=2j} = 0$ are true at m points $\lambda = 2j$, where $j = 0, \ldots, m-1$, then $g_i(\lambda) \equiv \lambda^{[i]}$. Therefore, from (2.4.18) it follows

$$\left(0, 2^{[i]}, \ldots, (2m-2)^{[i]}\right) \cdot \mathbb{H}_m = e_i$$

which means $C\mathbb{H}_m = I$. Thus, if $S(x) = (s_0(x), \ldots, s_{m-1}(x))^T$ and $R(x) = (r_0(x), \ldots, r_{m-1}(x))^T$ then $S(x) = CR(x) \Rightarrow R(x) = \mathbb{H}_m S(x)$. The lemma is proved. □

Remark 2.10. In Theorem 3.4 on page 174, it is established that the following connection between the kth ($0 \le k \le m-1$) row of the matrix $\mathbb{H}_m$ and the coefficients in the representation of the value $\Delta^k u(0)$ (function $u(x)$ is m-harmonic in S) through the values of its normal derivatives in the form

$$\Delta^k u(0) = \frac{(2,2)_k (n,2)_k}{\omega_n} \int_{\partial S} \sum_{j=0}^{m-1} h_{m,k}^{(j)} \frac{\partial^j u}{\partial \nu^j} \, ds_\xi, \quad k = 0, \ldots, m-1, \tag{2.4.19}$$

where $(a,b)_k = a(a+b)\ldots(a+(k-1)b)$ is the generalized Pochhammer symbol with convention $(a,b)_0 = 1$.

Remark 2.11. The last row of the matrix $\mathbb{H}_m$ gives the necessary solvability condition of the Neumann boundary value problem for the $(m-1)$-harmonic equation with boundary conditions: $\frac{\partial^j u}{\partial \nu^j}\big|_{\partial S} = \varphi_j$, $j = 1, \ldots, m-1$ (see also Corollary 3.1 on page 173) in the form

$$\int_{\partial S} \sum_{j=1}^{m-1} h_{m,m-1}^{(j)} \varphi_j \, ds_\xi = 0, \tag{2.4.20}$$

because in this case $\Delta^{m-1} u(x) = 0$ in S.

Example 2.4. Let us find the matrix $\mathbb{H}_4$. Since $h_4(\lambda) = \prod_{k=0}^{3}(\lambda-2k) = \lambda(\lambda-2)(\lambda-4)(\lambda-6)$, then

$$h_4'(2i) = (-1)^{3-i}\frac{(6)!!}{C_3^i}. \tag{2.4.21}$$

Therefore, according to (2.4.14),

$$h_{4,i}(\lambda) = \frac{h_4(\lambda)}{(\lambda-2i)h_4'(2i)} = (-1)^{3-i}\frac{C_3^i}{(6)!!}\prod_{k=0,k\neq i}^{3}(\lambda-2k).$$

From here, we can find $h_{4,0}(\lambda) = -\frac{1}{6!!}(\lambda-2)(\lambda-4)(\lambda-6)$. Let us use (2.4.12)

$$h_{4,0}^{(k)}(0) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}h_{4,0}(i).$$

Then, we obtain

$$\begin{aligned}
h_{4,0}^{(0)}(0) &= -\frac{1}{6!!}(-2)(-4)(-6) = 1,\\
h_{4,0}^{(1)}(0) &= -\frac{1}{6!!}((-1)(-3)(-5) - (-2)(-4)(-6)) = -\frac{33}{6!!},\\
h_{4,0}^{(2)}(0) &= -\frac{1}{6!!}(0 - 2\cdot(-1)(-3)(-5) + (-2)(-4)(-6)) = \frac{18}{6!!},\\
h_{4,0}^{(3)}(0) &= -\frac{1}{6!!}(1(-1)(-3) - 3\cdot 0 + 3\cdot(-1)(-3)(-5)\\
&\quad -(-2)(-4)(-6)) = -\frac{6}{6!!},
\end{aligned}$$

whence by virtue of (2.4.11)

$$h_{4,0}(\lambda) = \sum_{k=0}^{3}\frac{h_{4,0}^{(k)}(0)}{k!}\lambda^{[k]} = 1 - \frac{33}{48}\lambda^{[1]} + \frac{9}{48}\lambda^{[2]} - \frac{1}{48}\lambda^{[3]},$$

and therefore, the first row of the matrix $\mathbb{H}_4$ has the form $(1, -\frac{11}{16}, \frac{3}{16}, -\frac{1}{48})$. To find the expansion of the polynomial $h_{4,1}(\lambda) = \frac{3}{6!!}\lambda(\lambda-4)(\lambda-6)$ we can also use the Stirling triangle of the second kind (1.1.3) from which we get $\lambda^2 = \lambda + \lambda^{[2]}$ and $\lambda^3 = \lambda + 3\lambda^{[2]} + \lambda^{[3]}$, which means

$$h_{4,1}(\lambda) = \frac{1}{16}(24\lambda - 10\lambda^2 + \lambda^3) = \frac{1}{16}(15\lambda - 7\lambda^{[2]} + \lambda^{[3]}).$$

This means that the second row of the matrix $\mathbb{H}_4$ has the form $(0, \frac{15}{16}, -\frac{7}{16}, \frac{1}{16})$. Similarly, we can find polynomial expansions $h_{4,2}(\lambda) = -\frac{3}{6!!}\lambda(\lambda-2)(\lambda-6)$, $h_{4,3}(\lambda) = \frac{1}{6!!}\lambda(\lambda-2)(\lambda-4)$, whence we get

$$\mathbb{H}_4 = \begin{pmatrix} 1 & -\dfrac{11}{16} & \dfrac{3}{16} & -\dfrac{1}{48} \\ 0 & \dfrac{15}{16} & -\dfrac{7}{16} & \dfrac{1}{16} \\ 0 & -\dfrac{5}{16} & \dfrac{5}{16} & -\dfrac{1}{16} \\ 0 & \dfrac{1}{16} & -\dfrac{1}{16} & \dfrac{1}{48} \end{pmatrix}. \tag{2.4.22}$$

According to (2.4.19), for the four-harmonic function $u(x)$, the first row of the matrix $\mathbb{H}_4$ gives the equality

$$u(0) = \frac{1}{\omega_n}\int_{\partial S}\Big(u - \frac{11}{16}\frac{\partial u}{\partial \nu} + \frac{3}{16}\frac{\partial^2 u}{\partial \nu^2} - \frac{1}{48}\frac{\partial^3 u}{\partial \nu^3}\Big)\,ds_\xi.$$

Moreover, according to (2.4.20), the fourth row of the matrix $\mathbb{H}_4$ gives the solvability condition of the Neumann boundary value problem for the 3-harmonic equation (see also triangle B on page 187)

$$\int_{\partial S}\big(3\varphi_1 - 3\varphi_2 + \varphi_3\big)\,ds_\xi = 0.$$

Lemma 2.20. *The first transformation from* (2.4.7) *can be reversed in the form*

$$p_k(x) = \sum_{j=k}^{m-1}\binom{j}{k} r_j(x), \quad k = 0,\dots,m-1.$$

Proof. Let us prove that if

$$C = \Big\{(-1)^{j-i}\binom{j}{i}\Big\}_{i,j=\overline{0,l-1}},$$

then

$$C^{-1} = \Big\{\binom{j}{i}\Big\}_{i,j=\overline{0,l-1}}.$$

To do this, let us make sure that the matrix $CC^{-1} \equiv (c_{i,j})_{i,j=\overline{0,l-1}}$ is the identity matrix. Indeed, it is not difficult to see that

$$c_{i,j}=\sum_{k=0}^{l-1}(-1)^{k-i}\binom{k}{i}\binom{j}{k}=\sum_{k=i}^{j}(-1)^{k-i}\binom{k}{i}\binom{j}{k}$$

$$=\binom{j}{i}\sum_{k=0}^{j-i}(-1)^{k}\binom{j-i}{k}=0$$

and therefore for $j\neq i$ we have $c_{i,j}=0$, but $c_{i,i}=1$. So $CC^{-1}=E$.

Thus, if $P(x)=(p_0(x),\dots,p_{m-1}(x))^T$ and $R(x)=(r_0(x),\dots,r_{m-1}(x))^T$, then $R(x)=CP(x)$ and therefore $P(x)=C^{-1}R(x)$. This proves the lemma. □

2.4.2 Inversion of the basic relation

From the proved Lemmas 2.18-2.20 and Theorem 2.11 the following statement follows.

Theorem 2.12. *Let $q_i(x)\in C^{2m-2-i}(\bar S)$, $i=0,\dots,m-1$ be some system of a harmonic in S functions and a harmonic in S functions $p_s(x)$, defined by the equalities*

$$p_s(x)=\sum_{j=0}^{s}\frac{1}{j!}H_s^{(j)}(-\Lambda)q_j(x),\qquad s=0,\dots,m-1,\tag{2.4.23}$$

where

$$H_s(\lambda)=\frac{1}{(2s)!!}\lambda(\lambda-2)\cdots(\lambda-2s+2),\qquad s\in\mathbb{N},\tag{2.4.24}$$

with convention $H_0(\lambda)=1$, and the jth-order derivative $H_s^{(j)}(\lambda)$ of the polynomial $H_s(\lambda)$ is taken in the sense of definition (2.4.10). *Then the m-harmonic function*

$$u(x)=\sum_{k=0}^{m-1}p_k(x)(|x|^2-1)^k\tag{2.4.25}$$

satisfies the boundary conditions of the Dirichlet boundary value problem (2.4.8) *and $u(x)\in C^{m-1}(\bar S)$.*

Proof. Let us use Theorem 2.11 and consider the equalities (2.4.7). If we consistently apply Lemmas 2.18–2.20, then the equalities (2.4.7) can be inverted in the form

$$p_s(x)=\sum_{k=s}^{m-1}\binom{k}{s}\sum_{i=0}^{m-1}h_{m,k}^{(i)}\sum_{j=0}^{i}\binom{i}{j}(-\Lambda)^{[i-j]}q_j(x),\tag{2.4.26}$$

where $s = 0, \dots, m-1$ and the numbers $h^{(i)}_{m,k}$ are the elements of the matrix $\mathbb{H}_m$ from (2.4.16). Therefore, by Theorem 2.11, for such polynomials $p_s(x)$, the function $u(x)$ from (2.4.25) satisfies the Dirichlet boundary conditions (2.4.8).

Simplify the equalities (2.4.26). It is not hard to see that

$$p_s(x) = \sum_{k=s}^{m-1} \binom{k}{s} \sum_{i=0}^{m-1} h^{(i)}_{m,k}\psi(i) = \sum_{i=0}^{m-1} \psi(i) \sum_{k=s}^{m-1} \binom{k}{s} h^{(i)}_{m,k}, \tag{2.4.27}$$

where $\psi(i)$ denotes the internal sum over j in (2.4.26). Consider the following polynomial

$$H_s(\lambda) = \sum_{k=s}^{m-1} \binom{k}{s} h_{m,k}(\lambda),$$

where the polynomials $h_{m,k}(\lambda)$ are defined in (2.4.14). Using (2.4.21), the polynomial $h_{m,k}(\lambda)$ can be written in the form

$$\begin{aligned} h_{m,k}(\lambda) = \frac{h_m(\lambda)}{(\lambda - 2k)h'(2k)} &= \frac{(-1)^{m-k-1}}{(2m-2)!!}\binom{m-1}{k}\frac{h_m(\lambda)}{(\lambda - 2k)} \\ &= \frac{(-1)^{m-k-1}}{(2k)!!(2m-2-2k)!!}\frac{h_m(\lambda)}{(\lambda - 2k)}. \end{aligned}$$

Here, as in (2.4.14), it is denoted that $h_m(\lambda) = \lambda(\lambda-2)\cdots(\lambda-2m+2)$. Therefore, we have

$$\begin{aligned} H_s(\lambda) = \sum_{k=s}^{m-1} \binom{k}{s} h_{m,k}(\lambda) &= \frac{h_m(\lambda)}{(2s)!!} \sum_{k=s}^{m-1} \frac{(-1)^{m-k-1}}{(2k-2s)!!(2m-2-2k)!!}\frac{1}{\lambda - 2k} \\ &= \frac{h_m(\lambda)}{(2s)!!(2m-2-2s)!!} \sum_{k=s}^{m-1} \binom{m-1-s}{k-s} \frac{(-1)^{m-k-1}}{\lambda - 2k} \\ &= \frac{h_m(\lambda)}{(2s)!!(2m-2-2s)!!} \sum_{k=0}^{m-1-s} \binom{m-1-s}{k} \frac{(-1)^{m-k-s-1}}{\lambda - 2k - 2s}. \end{aligned} \tag{2.4.28}$$

Next, we use the simple equality

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{\mu - 2k} = \frac{(2m)!!}{\mu(\mu-2)\cdots(\mu-2m)},$$

which is obvious for $m = 0$ and can be easily proved by the induction on $m > 0$:

$$\begin{aligned}
\frac{(2m)!!}{\mu(\mu-2)\cdots(\mu-2m)} &= \frac{(2m-2)!!}{(\mu-2)\cdots(\mu-2m+2)}\Big(\frac{1}{\mu-2m}-\frac{1}{\mu}\Big)\\
&= \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{(-1)^{m-1-k}}{\mu-2k-2} - \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{(-1)^{m-1-k}}{\mu-2k}\\
&= \sum_{k=1}^{m}\binom{m-1}{k-1}\frac{(-1)^{m-k}}{\mu-2k} + \sum_{k=0}^{m}\binom{m-1}{k}\frac{(-1)^{m-k}}{\mu-2k}\\
&= \sum_{k=0}^{m}\left(\binom{m-1}{k-1}+\binom{m-1}{k}\right)\frac{(-1)^{m-k}}{\mu-2k} = \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^{m-k}}{\mu-2k}.
\end{aligned}$$

Thus, from (2.4.28) for $m' = m-1-s$ and $\mu = \lambda - 2s$, we get

$$H_s(\lambda) = \sum_{k=s}^{m-1}\binom{k}{s}h_{m,k}(\lambda) = \frac{h_m(\lambda)}{(2s)!!(\lambda-2s)\cdots(\lambda-2m+2)}.$$

Hence, for $s = 0$, we find $H_0(\lambda) = 1$. If $s = 1,\dots,m-1$, then using the value of $h_{m,k}(\lambda)$ from (2.4.14) we obtain $H_s(\lambda) = \frac{1}{(2s)!!}\lambda(\lambda-2)\cdots(\lambda-2s+2)$, which is the same as the polynomial defined in (2.4.24). Further, we expand the polynomial $H_s(\lambda)$ in monomial powers $\lambda^{[i]}$ as in (2.4.11)

$$H_s(\lambda) = \sum_{i=0}^{s}\frac{H_s^{(i)}(0)}{i!}\lambda^{[i]}.$$

By the definition of $H_s(\lambda)$ and using (2.4.15), we get

$$H_s(\lambda) = \sum_{k=s}^{m-1}\binom{k}{s}h_{m,k}(\lambda) = \sum_{k=s}^{m-1}\binom{k}{s}\sum_{i=0}^{m-1}h_{m,k}^{(i)}\lambda^{[i]} = \sum_{i=0}^{m-1}\lambda^{[i]}\sum_{k=s}^{m-1}\binom{k}{s}h_{m,k}^{(i)},$$

whence it follows that

$$\sum_{k=s}^{m-1}\binom{k}{s}h_{m,k}^{(i)} = \frac{H_s^{(i)}(0)}{i!}.$$

Substituting the found value of the sum into the equality (2.4.27) and using the definition of $\psi(i)$, we get

$$p_s(x) = \sum_{i=0}^{s}\frac{H_s^{(i)}(0)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-\Lambda)^{[i-j]}q_j(x), \quad s = 0,\dots,m-1.$$

Reversing the order of summations and using the expansion of the polynomial $H_s^{(j)}(\lambda)$ in the form (2.4.11), we have

$$p_s(x) = \sum_{j=0}^{s} \frac{1}{j!}\Big(\sum_{i=j}^{s} \frac{(H_s^{(j)})^{(i-j)}(0)}{(i-j)!}(-\Lambda)^{[i-j]}\Big) q_j(x) = \sum_{j=0}^{s} \frac{1}{j!} H_s^{(j)}(-\Lambda) q_j(x),$$

where $s = 0, \dots, m-1$, which is the same as (2.4.23).

Let us check the smoothness of the functions $p_s(x)$. In Theorem 2.11, it is required that $p_0(x), \dots, p_{m-1}(x) \in C^{m-1}(\bar{S})$. Since $\deg H_s^{(j)}(\lambda) = s - j$, then for such smoothness from the formula (2.4.23), for example for $s = m-1$, it follows that $q_j(x) \in C^{2m-2-j}(\bar{S})$. For $s < m-1$, the smoothness conditions for $q_j(x)$ are weaker than those indicated. Hence, in Theorem 2.11, we can take $k_j = 2m-2-j$, $j = 0, \dots, m-1$. The theorem is proved. □

Remark 2.12. The formula (2.4.25) is similar to the well-known Almansi formula [5, 71], but in (2.4.25), the harmonic components $p_k(x)$ are known. In (3.1.5) the equality for finding $p_k(x)$ from a known $u(x)$ is given.

2.4.3 Solution of the Dirichlet problem

The assertions proved above allow us to formulate the main result on the Dirichlet boundary value problem.

Theorem 2.13. *Solution of the Dirichlet boundary value problem for the polyharmonic equation*

$$\Delta^m u(x) = f(x), \quad x \in S,$$
$$u|_{\partial S} = \varphi_0(x), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_1(s), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\partial S} = \varphi_{m-1}(s), \quad s \in \partial S,$$

for $f \in C^1(\bar{S})$, $\varphi_k \in C^{2m-2-k+\varepsilon}(\partial S)$, $k = 0, \dots, m-1$, $\varepsilon > 0$, *having smoothness* $u \in C^{m-1}(\bar{S})$ *can be written as*

$$u(x) = \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(-\Lambda; |x|^2 - 1) q_j(x) + \int_S G_m(x,\xi) f(\xi)\, d\xi, \tag{2.4.29}$$

where

$$K_{m-1}(\lambda; |x|^2 - 1) = \sum_{k=0}^{m-1} (|x|^2 - 1)^k H_k(\lambda), \tag{2.4.30}$$

the polynomials $H_k(\lambda)$ *are determined from* (2.4.24), $K^{(j)}_{m-1}(\lambda; |x|^2-1)$ *is the derivative with respect to* λ *of order* j*, which is taken in the sense of the definition* (2.4.10), *and the harmonic functions* $q_k(x)$*,* $k=0,\dots,m-1$ *are solutions of the Dirichlet problems*

$$\Delta q_k(x)=0,\ x\in S;\quad q_k|_{\partial S}=\varphi_k(s),\ s\in\partial S, \tag{2.4.31}$$

and $G_m(x,\xi)$ *is the Green's function of the Dirichlet boundary value problem in the unit ball for the* m*-harmonic equation.*

Proof. Let $q_i(x)\in C^{2m-2-i}(\bar S)$, $i=0,\dots,m-1$ be a system of harmonic functions in S that are solutions to problems (2.4.31). Then, by Theorem 2.12 and using the properties of Green's function $G_m(x,\xi)$, the function

$$u(x)=\sum_{k=0}^{m-1}(|x|^2-1)^k p_k(x)+\int_S G_m(x,\xi)f(\xi)\,d\xi, \tag{2.4.32}$$

where the harmonic functions $p_k(x)$ for $k=0,\dots,m-1$ are found from the equalities

$$p_k(x)=\sum_{j=0}^{k}\frac{1}{j!}H_k^{(j)}(-\Lambda)q_j(x),\quad k=0,\dots,m-1,$$

the polynomial $H_k(\lambda)$ has the form $H_k(\lambda)=\frac{1}{(2k)!!}\lambda(\lambda-2)\cdots(\lambda-2k+2)$, $k\in\mathbb{N}$, and the jth-order derivative $H_k^{(j)}(\lambda)$ is taken in the sense of the definition (2.4.10), satisfies the equation (2.4.2) and the Dirichlet boundary conditions (2.4.3)

$$\frac{\partial^i u}{\partial\nu^i}\Big|_{\partial S}=q_i(x)|_{\partial S}+\int_S\frac{\partial^i G_m(x,\xi)}{\partial\nu_x^i}f(\xi)\,d\xi|_{x\in\partial S}=\varphi_i(s),\ s\in\partial S,$$

for $i=0,\dots,m-1$. We transform the first term in the resulting solution (2.4.32). It is easy to see that using the notation (2.4.30), we can write

$$\sum_{k=0}^{m-1}(|x|^2-1)^k p_k(x)=\sum_{k=0}^{m-1}(|x|^2-1)^k\sum_{j=0}^{k}\frac{1}{j!}H_k^{(j)}(-\Lambda)q_j(x)=\sum_{j=0}^{m-1}\frac{1}{j!}$$
$$\times\Big(\sum_{k=j}^{m-1}(|x|^2-1)^k H_k(\mu)\Big)^{(j)}_{\mu}\Big|_{\mu=-\Lambda}q_j(x)=\sum_{j=0}^{m-1}\frac{1}{j!}K^{(j)}_{m-1}(-\Lambda;|x|^2-1)q_j(x),$$

where it is taken into account that the jth-order derivative of $H_k(\mu)$ for $k<j$ is equal to zero. Finally, by virtue of paper [4, Lemma 2.7], in order for the harmonic

in S functions $q_i(x)$ to have the smoothness $q_i(x) \in C^{2m-2-i}(\bar{S})$, it is sufficient to require that $\varphi_k \in C^{2m-2-k+\varepsilon}(\partial S)$, $k = 0, \dots, m-1$ for some $\varepsilon > 0$. Therefore, the m-harmonic function $u(x)$ from (2.4.29) is a solution to the Dirichlet boundary value problem (2.4.2) and (2.4.3). The theorem is proved. □

Remark 2.13. The existence of a solution to the Dirichlet problem is also considered in Lemma 3.21 on page 211.

Corollary 2.2. In Theorem 2.13 the smoothness of the boundary functions φ_k, $k = 0, \dots, m-1$ can be reduced if instead of the condition $u \in C^{m-1}(\bar{S})$ we require only $\Lambda^k u|_{\partial S} = \varphi_k$ for $k = 0, \dots, m-1$. In this case, Theorem 2.13 remains true for $\varphi_k \in C^{m-1-k+\varepsilon}(\partial S)$, $k = 0, \dots, m-1$ and the solution has the same form (2.4.29).

Proof. Let $f(x) = 0$, since we are studying the smoothness of boundary functions. Let us rewrite the solution (2.4.29) of the Dirichlet problem again in the form (2.4.4)

$$u(x) = \sum_{k=0}^{m-1} p_k(x)(|x|^2 - 1)^k,$$

where the harmonic functions $p_k(x)$ have the form

$$p_k(x) = \sum_{j=k}^{m-1} \frac{1}{j!} H_k^{(j)}(-\Lambda) q_j(x).$$

Since $\varphi_j \in C^{m-1-j+\varepsilon}(\partial S)$, then by [4, Lemma 2.7] $q_j \in C^{m-1-j+\varepsilon}(\bar{S})$ for $j = 0, \dots, m-1$, which means $H_k^{(j)}(-\Lambda) q_j \in C^{m-1-k+\varepsilon}(\bar{S})$ and therefore $p_k \in C^{m-1-k+\varepsilon}(\bar{S})$, whence $u \in C^{\varepsilon}(\bar{S})$. Let $l = 0, \dots, m-1$. Let us split the function $u(x)$ into two terms

$$u(x) = \sum_{k=0}^{m-1-l} p_k(x)(|x|^2 - 1)^k + \sum_{k=m-l}^{m-1} p_k(x)(|x|^2 - 1)^k \equiv u_0(x) + u_l(x).$$

Let us study the limit of the function $\Lambda^l u(x)$ at $x \to \partial S$ for the m-harmonic function $u(x)$. If the smoothness of the functions φ_j is the same as in Theorem 2.13, then the limit at $x \to \partial S$ of the function $\Lambda^l u(x)$ exists as in Theorem 2.13 it is proved that $u \in C^{m-1}(\bar{S})$. It is easy to see that $\Lambda^l u_0 \in C^{\varepsilon}(\bar{S})$ and therefore the limit of the function $\Lambda^l u_0(x)$ exists, which cannot be said for the function $\Lambda^l u_l(x)$. Let us prove that for the terms included in $\Lambda^l u_l(x)$, for which, due to the reduced smoothness of the functions φ_j, the existence of a limit at $x \to \partial S$ cannot be guaranteed, this limit will still be equal to 0. It is clear that the limit of these same terms with greater smoothness of the functions φ_j as in Theorem 2.13 will also be equal to 0, which means that the limit of $\Lambda^l u(x)$ with decreasing smoothness of boundary functions will not change.

For this purpose we will use [4, Lemma 2.2]. By virtue of this lemma, for the harmonic functions $p_k \in C^\varepsilon(\bar{S})$ for $l > 0$ the following inequality is true:

$$\left|(1-|x|)^l \Lambda^l p_k(x)\right| \le C(1-|x|)^\varepsilon, \tag{2.4.33}$$

where $x \in S$ and $C > 0$ is some constant. We will examine the terms in the sum representing $\Lambda^l u_l(x)$ for which the existence of a limit cannot be guaranteed. Let us first estimate the last term in the sum representing $\Lambda^l u_l(x)$. Since $p_{m-1} \in C^\varepsilon(\bar{S})$ and $\Lambda^l(|x|^2-1)^{m-1} = P(x)(|x|^2-1)^{m-1-l}$, where $P(x)$ is some polynomial whose value does not interest us, then in accordance with (2.4.33) for $l > 0$ we get

$$\begin{aligned}
|\Lambda^l p_{m-1}(x)(|x|^2-1)^{m-1}| &\le \sum_{l_1=0}^{l}\binom{l}{l_1}|\Lambda^{l_1} p_{m-1}(x)\Lambda^{l-l_1}(1-|x|^2)^{m-1}| \\
&\le C_1|P(x)|(1-|x|)^{m-1-l}\sum_{l_1=0}^{l}|\Lambda^{l_1} p_{m-1}(x)(1-|x|)^{l_1}| \le C_2(1-|x|)^{m-1-l+\varepsilon},
\end{aligned}$$

where $C_1 > 0, C_2 > 0$ are some constants. For $l = 0$, the considered term is included in $u_0(x)$. It follows that

$$\lim_{x\to\partial S}\Lambda^l(p_{m-1}(x)(|x|^2-1)^{m-1}) = 0.$$

Let $0 \le i < l$. Consider the term $\Lambda^l(p_{m-1-i}(x)(|x|^2-1)^{m-1-i})$ included in the sum representing $\Lambda^l u_l(x)$. Since $p_{m-1-i} \in C^{i+\varepsilon}(\bar{S})$, then when applying the operator Λ^l to a product of functions, we consider only terms of the form $\Lambda^{l_1} p_{m-1-i}(x)\Lambda^{l_2}(|x|^2-1)^{m-1-i}$ for $l_1 > i$, $l_1 + l_2 = l$, since the limits for other terms are defined. Denoting $g_i(x) = \Lambda^i p_{m-1-i}(x) \in C^\varepsilon(\bar{S})$, similarly to the previous case, we have

$$\begin{aligned}
|\Lambda^{l_1} p_{m-1-i}(x)\Lambda^{l_2}(|x|^2-1)^{m-1-i}| &= |\Lambda^{l_1-i} g_i(x)P(x)(1-|x|^2)^{m-1-i-l_2}| \\
&= |P(x)(1-|x|^2)^{m-1-l}||\Lambda^{l_1-i} g_i(x)(1-|x|^2)^{l_1-i}| \le C_3(1-|x|)^{m-1-l+\varepsilon},
\end{aligned}$$

whence it follows that

$$\lim_{x\to\partial S}\Lambda^{l_1} p_{m-1-i}(x)\Lambda^{l_2}(|x|^2-1)^{m-1-i} = 0.$$

Thus, the limits at $x \to \partial S$ of all terms included in $\Lambda^l u_l(x)$, for which the existing smoothness of the boundary functions does not guarantee the existence of the limit, are equal to 0. If we consider the same function $\Lambda^l u_l(x)$ if the data is smooth as in theorem 2.13, then the limits of the same terms from $\Lambda^l u_l(x)$ exist and must be equal to 0. This means that the limit $\Lambda^l u_l(x)$ will not change as the smoothness of the boundary functions decreases. Since the limit $\Lambda^l u_0(x)$ exists, then $\Lambda^{[l]} u|_{\partial S} = \varphi_l$. $\square$

Example 2.5. Let us find the solution to the Dirichlet boundary value problem for the three-harmonic equation in the unit ball. In this case $m = 3$. First, find the polynomials $H_0(\lambda)$, $H_1(\lambda)$ and $H_2(\lambda)$ by the formula (2.4.24). We have

$$H_0(\lambda) = 1, \quad H_1(\lambda) = \frac{1}{2}\lambda^{[1]}, \quad H_2(\lambda) = \frac{1}{4!!}\lambda(\lambda-2) = \frac{1}{4!!}(\lambda^{[2]} - \lambda^{[1]}),$$

which means

$$\begin{aligned} &K_2(\lambda; |x|^2-1) \\ &\quad = \sum_{k=0}^{2}(|x|^2-1)^k H_k(\lambda) = 1 + \frac{1}{2}(|x|^2-1)\lambda^{[1]} + \frac{1}{4!!}(|x|^2-1)^2(\lambda^{[2]} - \lambda^{[1]}). \end{aligned}$$

If we recall the equality $\left(\lambda^{[k]}\right)^{(m)} = k^{[m]}\lambda^{[k-m]}$, then it is not hard to find

$$\begin{aligned} K_2^{(1)}(\lambda; |x|^2-1) &= \frac{1}{2}(|x|^2-1) + \frac{1}{4!!}(|x|^2-1)^2(2\lambda^{[1]}-1), \\ K_2^{(2)}(\lambda; |x|^2-1) &= \frac{2}{4!!}(|x|^2-1)^2. \end{aligned}$$

Hence, by the formulas (2.4.29) and (2.3.27) from Section 2.3, we can write

$$\begin{aligned} u(x) &= \sum_{j=0}^{2}\frac{1}{j!}K_2^{(j)}(-\Lambda; |x|^2-1)q_j(x) - \frac{1}{\omega_n}\int_S G_6(x,\xi)f(\xi)\,d\xi \\ &= \Big(1 - \frac{1}{2}(|x|^2-1)\Lambda + \frac{1}{4!!}(|x|^2-1)^2(\Lambda^2 + 2\Lambda)\Big)q_0(x) \\ &+ \Big(\frac{1}{2}(|x|^2-1) - \frac{1}{4!!}(|x|^2-1)^2(2\Lambda+1)\Big)q_1(x) + \frac{1}{4!!}(|x|^2-1)^2 q_2(x) \\ &- \frac{1}{\omega_n}\int_S G_6(x,\xi)f(\xi)\,d\xi, \quad (2.4.34) \end{aligned}$$

where $G_6(x,\xi)$ is the Green's function of the Dirichlet boundary value problem for $m = 3$.

Remark 2.14. Note that the equality (2.4.34) for $f(x) = 0$ implies a representation of the solution to the biharmonic Dirichlet problem (2.4.1), if all terms containing $(|x|^2 - 1)^2$ are removed from it. This is so because, in accordance with (2.4.30), it is precisely these terms that distinguish the functions $K_2(\lambda; |x|^2-1)$ and $K_1(\lambda; |x|^2-1)$.

The possibility of recursive construction of the solution (2.4.29) to the Dirichlet boundary value problem (2.4.2) and (2.4.3) for the homogeneous polyharmonic equation is given in the following theorem.

Theorem 2.14. *Let $u_m(x)$ be a solution of the Dirichlet boundary value problem* (2.4.2) *and* (2.4.3) *with $f(x) = 0$, written as* (2.4.29)*. Then, the $(m+1)$-harmonic function $u_{m+1}(x)$ is a solution to the Dirichlet boundary value problem* (2.4.2) *and* (2.4.3) *with the same boundary functions $\varphi_k \in C^{2m-2-k+\varepsilon}(\partial S)$, $k = 0, \dots, m-1$ and $\varphi_m \in C^{m+\varepsilon}(\partial S)$ can be written as*

$$u_{m+1}(x) = u_m(x) + (|x|^2 - 1)^m \sum_{j=0}^{m} \frac{1}{j!} H_m^{(j)}(-\Lambda) q_j(x), \tag{2.4.35}$$

where the polynomial $H_m(\lambda)$ is defined in (2.4.24)*, the derivative $H_m^{(j)}(\lambda)$ is taken in the sense of* (2.4.10)*, and the harmonic functions $q_k(x)$, $k = 0, \dots, m-1$ are solutions of the corresponding Dirichlet boundary value problems, as in Theorem 2.13.*

Proof. By Theorem 2.13, solutions of the Dirichlet boundary value problems $u_{m+1}(x)$ and $u_m(x)$ can be written as (2.4.29). Transform the difference of these solutions

$$\begin{aligned} u_{m+1}(x) - u_m(x) &= \sum_{j=0}^{m-1} \frac{1}{j!} \big(K_m^{(j)}(-\Lambda; |x|^2-1) - K_{m-1}^{(j)}(-\Lambda; |x|^2-1)\big) q_j(x) \\ &+ \frac{1}{m!} K_m^{(m)}(-\Lambda; |x|^2-1) q_m(x) = (|x|^2-1)^m \sum_{j=0}^{m-1} \frac{1}{j!} H_m^{(j)}(-\Lambda) q_j(x) \\ &+ \frac{1}{m!} (|x|^2-1)^m H_m^{(m)}(-\Lambda) q_m(x) = (|x|^2-1)^m \sum_{j=0}^{m} \frac{1}{j!} H_m^{(j)}(-\Lambda) q_j(x). \end{aligned}$$

This equality proves the theorem. □

Example 2.6. Let us use the formula (2.4.35) to construct a solution to the Dirichlet boundary value problem for the homogeneous four-harmonic equation using the solution (2.4.34) of the Dirichlet boundary value problem for the homogeneous three-harmonic equation obtained in Example 2.5. To do this, we need to use the "factorial" representation of the polynomial $H_3(\lambda)$ given in Example 2.4. Its coefficients are located in the last row of the matrix $\mathbb{H}_4$ from (2.4.22):

$$H_3(\lambda) = \frac{1}{6!!}\lambda(\lambda-2)(\lambda-4) = \frac{1}{6!!}\Big(3\lambda^{[1]} - 3\lambda^{[2]} + \lambda^{[3]}\Big).$$

Using the equality $\big(\lambda^{[k]}\big)^{(m)} = k^{[m]}\lambda^{[k-m]}$, we get

$$H_3^{(1)}(\lambda) = \frac{1}{6!!}\Big(3 - 6\lambda^{[1]} + 3\lambda^{[2]}\Big),\ H_3^{(2)}(\lambda) = \frac{1}{6!!}\Big(-6 + 6\lambda^{[1]}\Big),\ H_3^{(3)}(\lambda) = 1,$$

and hence, by (2.4.35) for $m = 3$, we obtain

$$\begin{aligned} & u_4(x) - u_3(x) \\ & = \frac{(|x|^2-1)^3}{6!!}\Big(-(\Lambda^3 + 6\Lambda^2 + 8\Lambda)q_0 + (3\Lambda^2 + 9\Lambda + 3)q_1 - (3\Lambda + 3)q_2 + q_3\Big). \end{aligned}$$

From here, using (2.4.34), we find

$$\begin{aligned} u_4(x) = \Big(1 &- \frac{(|x|^2-1)}{2}\Lambda + \frac{(|x|^2-1)^2}{4!!}(\Lambda^2 + 2\Lambda) \\ &- \frac{(|x|^2-1)^3}{6!!}(\Lambda^3 + 6\Lambda^2 + 8\Lambda)\Big)q_0(x) + \Big(\frac{(|x|^2-1)}{2} \\ &- \frac{(|x|^2-1)^2}{4!!}(2\Lambda + 1) + \frac{(|x|^2-1)^3}{6!!}(3\Lambda^2 + 9\Lambda + 3)\Big)q_1(x) \\ &+ \Big(\frac{(|x|^2-1)^2}{4!!} - \frac{(|x|^2-1)^3}{6!!}(3\Lambda + 3)\Big)q_2(x) + \frac{(|x|^2-1)^3}{6!!}q_3(x). \end{aligned}$$

When solving the Dirichlet problem for polyharmonic equation with polynomial boundary data, the harmonic functions $q_j(x)$ also turn out to be polynomials. Therefore, when applying Theorem 2.14 to such a problem, the following result will be useful.

Consider the Dirichlet problem for the Laplace equation in the ball S

$$\Delta v(x) = Q(x),\ x \in S; \quad v|_{\partial S} = P(x)|_{\partial S}, \tag{2.4.36}$$

with polynomial boundary function $P(x)$ and right-hand side $Q(x)$ and for $n > 2$. Let us formulate a statement that complements the statement of Theorem 2.2 on page 84.

Theorem 2.15. *The solution to the Dirichlet problem* (2.4.36) *for the Poisson equation can be written in the form*

$$v(x) = P(x) + \frac{|x|^2-1}{2}\int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^s}{(2s+2)!!(2s)!!}\Delta^s(Q-\Delta P)(\alpha x)\,\alpha^{n/2-1}\,d\alpha. \quad (2.4.37)$$

Proof. Using theorem 2.2 we find a solution to the following Dirichlet problem

$$\Delta u(x) = \Delta P(x),\ x\in S;\quad u|_{\partial S} = 0.$$

From the equality (2.1.24) for $m = 1$ it follows

$$u(x) = \frac{|x|^2-1}{2}\int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^s}{(2s+2)!!(2s)!!}\Delta^{s+1}P(\alpha x)\,\alpha^{n/2-1}\,d\alpha.$$

Then a polynomial of the form

$$v(x) = P(x) - u(x) = P(x) - \frac{|x|^2-1}{2}\int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^s}{(2s+2)!!(2s)!!}\Delta^{s+1}P(\alpha x)\,\alpha^{n/2-1}\,d\alpha$$

is harmonic since $\Delta v(x) = \Delta P(x) - \Delta u(x) = 0$ and satisfies the boundary condition $v(x)|_{\partial S} = P(x)|_{\partial S}$. If we add the solution from (2.1.24) at $m = 1$ to $v(x)$, then we obtain the desired solution to problem (2.4.36) in the form (2.4.37). □

Remark 2.15. A similar formula for a polynomial solution to the Dirichlet problem for the biharmonic equation is given in Example 3.7.

Note that in the paper [90] a similar method is used to solve the Dunkle-Poisson equation and related equations in superspace.

Example 2.7. Let us find a solution to the Dirichlet problem for the 3-harmonic equation

$$\Delta^3 u(x) = 0,\quad x\in S,$$
$$u|_{\partial S} = q_0,\quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0,\quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = x_i^2|_{\partial S},$$

where $q_0 = \text{const}$.

In accordance with the equality (2.4.34) it is necessary to find the harmonic functions $q_0(x)$, $q_1(x)$ and $q_2(x)$. It is easy to see that $q_0(x) = q_0$, $q_1(x) = 0$. To find $q_2(x)$ we solve the problem (2.4.36) with $P(x) = x_i^2$

$$\Delta v(x) = 0,\ x \in S; \quad v|_{\partial S} = x_i^2|_{\partial S}.$$

The sum from the equality (2.4.37) have only one term at $s = 0$. Therefore we have

$$v(x) = x_i^2 - \frac{|x|^2 - 1}{2} \int_0^1 \frac{1}{2} \cdot 2\alpha^{n/2-1}\, d\alpha = x_i^2 + \frac{1}{n}(1 - |x|^2)$$

and that means $q_2(x) = x_i^2 + \frac{1}{n}(1 - |x|^2)$. Substituting the found values of the polynomials $q_0(x)$, $q_1(x)$ and $q_2(x)$ into (2.4.34) we obtain the desired solution

$$u(x) = q_0(x) + \frac{1}{4!!}(|x|^2 - 1)^2 q_2(x) = q_0 + \frac{x_i^2}{8}(|x|^2 - 1)^2 - \frac{1}{8n}(|x|^2 - 1)^3.$$

Let us prove the uniqueness theorem for the solution of the Dirichlet problem in the class of m-harmonic functions in which the existence of a solution was proven.

Theorem 2.16. *The solution to the Dirichlet problem* (2.4.2), (2.4.3) *is unique in the class of m-harmonic functions of the form* $\sum_{k=0}^{m-1}(|x|^2 - 1)^k u_k(x)$, *in which the harmonic functions* $u_k(x)$ *have the smoothness* $u_k \in C^{m-1-k+\varepsilon}(\bar{S})$.

Proof. By virtue of the Almansi formula on the representation of the m-harmonic function $u(x)$ in the starlike domains [162] we can write

$$u(x) = \sum_{k=0}^{m-1}(|x|^2 - 1)^k u_k(x).$$

Let in addition $u_k \in C^{m-1-k+\varepsilon}(\bar{S})$. Let us prove that if in this case the equalities $u|_{\partial S} = \ldots = \Lambda^{m-1}u|_{\partial S} = 0$ are satisfied, which are equivalent to the homogeneous Dirichlet conditions $u|_{\partial S} = \ldots = \partial^{m-1}u/\partial\nu^{m-1}|_{\partial S} = 0$, then $u(x) = 0$ in S. Similar to the reasoning from Corollary 2.2 on page 144, we are convinced that the limits for $x \to \partial S$ of all terms of the form $\Lambda^l(|x|^2 - 1)^k u_k(x)$ for $l = 0, \ldots, m-1$ exist. Let us write $u(x)$ in the form

$$u(x) = u_0(x) + \sum_{k=1}^{m-1}(|x|^2 - 1)^k u_k(x),$$

whence, taking into account that $u_0 \in C^{m-1+\varepsilon}(\bar{S})$, for $x \to \partial S$ we obtain $0 = u|_{\partial S} = u_0|_{\partial S}$, which means $u_0(x) = 0$. Let us rewrite the previous equality in the form

$$u(x) = (|x|^2 - 1)u_1(x) + \sum_{k=2}^{m-1}(|x|^2 - 1)^k u_k(x)$$

and apply the operator Λ to both sides of this equality

$$\Lambda u(x) = (|x|^2 - 1)\Lambda u_1(x) + 2|x|^2 u_1(x) + \Lambda \sum_{k=2}^{m-1}(|x|^2 - 1)^k u_k(x).$$

Hence, since $\Lambda u_1 \in C^{m-2+\varepsilon}(\bar{S})$, then for $x \to \partial S$ we have $0 = \Lambda u|_{\partial S} = 2u_1|_{\partial S}$, which means $u_1(x) = 0$. Proceeding in a similar way we get

$$u(x) = (|x|^2 - 1)^{m-1} u_{m-1}(x).$$

From here we can find

$$\Lambda^{m-1}u(x) = \sum_{k=0}^{m-1}\binom{m-1}{k}\Lambda^k(|x|^2 - 1)^{m-1} \cdot \Lambda^{m-1-k}u_{m-1}(x). \tag{2.4.38}$$

Since $u_{m-1} \in C^{\varepsilon}(\bar{S})$, then using [4, Lemma 2.2] we write the inequality

$$\left|(1 - |x|)^l \Lambda^l u_{m-1}(x)\right| \leq C(1 - |x|)^{\varepsilon},$$

where $l > 0$. Therefore, for all terms of the sum in (2.4.38) except the last one, i.e. for $k < m - 1$, we can take $l = m - 1 - k$ and then we have

$$|\Lambda^k(|x|^2 - 1)^{m-1} \cdot \Lambda^{m-1-k}u_{m-1}(x)| \leq C(1 - |x|)^{\varepsilon}.$$

The last term in the sum (2.4.38), i.e. for $k = m - 1$ is equal to

$$\Lambda^{m-1}(|x|^2 - 1)^{m-1} \cdot u_{m-1}(x).$$

If, taking into account the last inequality and equality, we pass to the limit in (2.4.38) at $x \to \partial S$, then we obtain $0 = \Lambda^{m-1}u(x)|_{\partial S} = C_1 u_{m-1}|_{\partial S}$, where $C_1 = \Lambda^{m-1}(|x|^2 - 1)^{m-1}|_{\partial S} \neq 0$. Therefore $u_{m-1}(x) = 0$ and consequently $u(x) = 0$ in S. The theorem is proved. □

Chapter 3

Neumann type problems for the polyharmonic equations

One of the classical high-order elliptic partial differential equations is the polyharmonic equation, and the classical problems for this equation are the Dirichlet (see Chapter 2) and Neumann boundary value problems. By Neumann problem we mean the problem (3.2.1), the formulation of which is proposed by A.V. Bitsadze [29], although in the literature there is no established definition of which problem among the Neumann type problems (see Section 3.4) should be considered as Neumann problem. Conditions for the solvability of problems of this type are studied in the classical theory of boundary value problems for elliptic equations and systems of equations satisfying the so-called complementarity condition. It has been established that all problems of this type are Fredholm and therefore their solvability for homogeneous boundary conditions is guaranteed by the orthogonality of the right-hand sides to all solutions of the homogeneous adjoint equation. In Section 1.1 (see also [60]) a more general boundary value problem for a polyharmonic equation is considered, containing high-order polynomials of normal derivatives in the boundary conditions. Theorem 1.2 on the solvability of this boundary value problem and on the representation of the solution is proved. The solvability conditions had the form of orthogonality of some vectors depending on the problem data. To find out under what data a particular problem can be solved, it is necessary to perform some difficult calculations. Since the Neumann problem is a classical problem, it would be nice for it to have solvability conditions in an easily verifiable form. In addition, it is also better to choose a classical domain of the problem, say the unit ball, and give a way to construct solutions to the problem in the case of the simplest boundary functions, for example, polynomial ones. The conditions obtained in this case can serve as a test example for studying more general problems for elliptic equations.

This chapter is devoted to the study of the solvability of Neumann-type problems for the polyharmonic equation in the unit ball. First, as preliminary results, in Section 3.1 the property of the mean value (3.1.4) over the unit sphere ∂S of normal derivatives of polyharmonic functions is studied and the number triangle H (3.1.24) is defined. Using the rows of this triangle in Theorems 3.2 and 3.4, a connection is established between the values of the powers of the Laplacian of the polyharmonic function at the center of the ball with the values of the linear combinations of some normal derivatives of that function on the boundary. Examples 3.1-3.3 are given to illustrate the results obtained.

In Section 3.2 another integer triangle B (3.2.18) is introduced, related to the solvability conditions of the Neumann problem for a homogeneous polyharmonic equation. Using Lemmas 3.9-3.15 it is possible to obtain recurrence equations for the elements of this triangle b_k^s, and then solve the resulting recurrence equation explicitly. In Theorem 3.7, based on matrices $\mathbb{H}_m$ of the form (2.4.16) associated with Theorem 3.4, necessary and sufficient conditions for the solvability of the Neumann problem are given and an explicit representation of its solution is found. Example 3.4 is given.

In Section 3.3, in addition to Section 3.2, the Neumann problem for an inhomogeneous polyharmonic equation is studied. First, in Theorem 3.10, based on Theorem 3.8 and Lemmas 3.18-3.20 the case of polynomial right-hand side $f(x)$ is studied. Then, in Theorem 3.11, this result is extended to the case of $f \in C^1(\bar{S})$. Theorem 3.12 proposes a method for finding a solution to the Neumann problem (3.3.1)-(3.3.2). In the case of polynomial data, an effective way to construct a solution to the Neumann problem is given. The main results are illustrated by Examples 3.5-3.7.

In Section 3.4 necessary and sufficient conditions for the solvability of Neumann type problems $\mathcal{N}_k$ (3.4.1)-(3.4.2) are found. Lemma 3.21 investigates the smoothness of the harmonic components of the solution to the Dirichlet problem $\mathcal{N}_0$. Further, in Lemmas 3.22 and 3.23 sufficient conditions for the solvability of the problem $\mathcal{N}_k$ are found in terms of the function $v(x)$ – a solution to the auxiliary Dirichlet problem and depending on the integer triangle B. Theorem 3.15 from Section 3.4.3 provides necessary conditions for the solvability of the problem $\mathcal{N}_k$ in terms of the boundary functions of the problem. In Lemma 3.29 and Theorem 3.17 from Section 3.4.4 the conditions from Theorem 3.15 in the case of even $k - l$ are expressed through the auxiliary function $v(x)$, in the case of odd $k - l$ this is done in Theorem 3.18. Finally, Theorem 3.19 from Section 3.4.5 proves that for $k \leq m$ the necessary conditions of Theorem 3.15 are also sufficient conditions for the solvability of the problem $\mathcal{N}_k$. Example 3.9 gives the dynamics of the emergence of solvability conditions in problems $\mathcal{N}_k$ depending on k for the 7-harmonic equation. For the inhomogeneous equation in Example 3.10 the problem $\mathcal{N}_3$ is also considered for the 7-harmonic equation.

In Section 3.5 the solvability of the boundary value problem (3.5.1)-(3.5.2) with boundary conditions of general form for the biharmonic equation in the unit ball, which we called the Dirichlet-Riquier problem, is investigated. Special cases of the problem under consideration are the classical problems of Dirichlet, Navier, Robin, listed above, the Steklov spectral problem, as well as many other boundary value problems generated by these boundary conditions. Theorem 3.23 formulates conditions for the uniqueness of a solution to the problem under study. Then, in Theorem 3.24, conditions for the unconditional solvability of the problem for the homogeneous biharmonic equation are found, which coincided with the uniqueness conditions from Theorem 3.23. Theorem 3.25 considers the case when the conditions of Theorem 3.24 are not satisfied, but a solution to the problem under consideration still exists. Finally, Section 3.5.4 considers five special cases of the general problem.

In Section 3.6 we studied the Neumann type problem (3.6.1)-(3.6.2) for the biharmonic equation in the unit ball, which does not fit into the class of Neumann type problems $\mathcal{N}_k$ from Sections 3.4 and 3.5, but is included in the general problems studied in Section 1.1. In Theorem 3.26, the condition for the existence of a solution to the problem under consideration (3.6.1)-(3.6.2) for a homogeneous equation is formulated and the explicit form of this solution (3.6.15) is given. Then, in Theorem 3.27, the uniqueness of the solution to the problem is established up to an arbitrary constant. Finally, in Theorem 3.28, with the help of Lemmas 3.34-3.35, necessary and sufficient conditions for the solvability of the problem under consideration for an inhomogeneous equation are found. In Example 3.11 solvability conditions for the problem with a simple right-hand side of the equation in the form $f(x) = |x|^m h(x)$ are given.

Finally, in Section 3.7 we construct the Green's function of the Neumann and Robin problems for the Laplace equation in the unit ball, which is needed in Chapter 4. In Lemma 3.36 a polynomial solution to the Neumann problem is found under zero boundary conditions and a polynomial right-hand side of the equation. Then in Theorem 3.29 the resulting solution is written through an integral over S with the kernel in the form of the function $\mathcal{N}(x,\xi)$ defined in (3.7.7). Theorem 3.30 shows that the function $\mathcal{N}(x,\xi)$ is the Green's function of the Neumann problem and gives an explicit representation of the solution to the Neumann problem. Then the Robin problem is studied. In Lemma 3.37 the case of a polynomial right-hand side is studied. Further, based on this, the case of the right-hand side of the equation of a general type is studied. Equality (3.7.17) from Theorem 3.31 gives the Green's function $\mathcal{N}_\lambda(x,\xi)$ of the Robin problem and gives an explicit representation of the solution to this problem in integral form. The found Green's functions of the Neumann and Robin problems complement each other.

3.1 Boundary properties of normal derivatives of polyharmonic functions

Section 3.1 is organized as follows. Theorem 3.1 from section 3.1.1 proves the property of the mean value (3.1.4) over ∂S for normal derivatives of a polyharmonic function. Sections 3.1.2-3.1.4 examine the number triangle H. This triangle appears in the equality, which specifies the value of the k-harmonic function at the center of the unit ball $u(0)$ through the integral of normal derivatives of this function along the boundary of the ball S. The properties of the numbers h_k^s are studied in Lemmas 3.1-3.7. In Theorem 3.2 of Section 3.1.5 the equality (3.1.7) is proved, and then in Theorem 3.4 it is generalized for values of $\Delta^m u(0)$ for $m < k$. The main results of Section 3.1 is published in papers [81, 85].

Let the function $u(x)$ be harmonic in the domain $\Omega \subset \mathbb{R}^n$ and $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. The property of the mean value for harmonic functions [165] is well known: if $x \in \Omega$ and $\overline{B_r(x)} \subset \Omega$, then for every function harmonic in Ω the equality holds

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, ds_y.$$

This property of the mean value is generalized by Pizzetti (see [42, 147]) for the k-harmonic function $u(x)$ in Ω in the following form

$$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, ds_y = \Gamma(n/2) \sum_{i=0}^{k-1} \frac{r^{2i} \Delta^i u(x_0)}{4^i i! \Gamma(i + n/2)},$$

where $\Gamma(\alpha)$ is the Euler gamma function. This property can be easily rewritten for the k-harmonic function $u \in C^{k-1}(\bar{S})$ in the unit ball $S \subset \mathbb{R}^n$ in the form

$$\frac{1}{\omega_n} \int_{\partial S} u(x)\, ds_x = \sum_{i=0}^{k-1} \frac{\Delta^i u(0)}{(2,2)_i (n,2)_i}, \tag{3.1.1}$$

where ω_n is the area of the unit sphere ∂S, and $(a, b)_k$ is the generalized Pochhammer symbol. Theorem 1.10 on page 52 proved a similar formula for calculating the integral of the homogeneous polynomial $Q_m(x)$ over the unit sphere

$$\int_{|x|=1} Q_m(x)\, ds_x = \begin{cases} 0, & m \in 2\mathbb{N}-1 \\ \dfrac{\Delta^{m/2} Q_m(x)}{m!!\, n \cdots (n+m-2)}\, \omega_n, & m \in 2\mathbb{N} \end{cases}.$$

Let us look again at the operator

$$\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}. \tag{3.1.2}$$

This operator plays an important role in subsequent studies because, as noted earlier (see Section 1.1.1), the following equality holds on ∂S

$$\frac{\partial^k u}{\partial \nu^k} = \Lambda^{[k]} u, \tag{3.1.3}$$

where ν is the outer normal to the unit sphere ∂S, and $t^{[k]} = t(t-1)\ldots(t-k+1)$ is the factorial power of t of order k. In addition, if u is a harmonic function, then the function $P(\Lambda)u$, where $P(\lambda)$ is a polynomial, is also a harmonic function.

Let us extend the equality (3.1.1) to the normal derivatives of the polyharmonic function over the sphere ∂S. Let $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.

Theorem 3.1. *For any $m \in \mathbb{N}_0$ and any polyharmonic function $u \in C^m(\bar{S})$ in the unit ball S the following equality holds*

$$\frac{1}{\omega_n} \int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = \sum_{k=0}^{\infty} \frac{(2k)^{[m]}}{(2,2)_k (n,2)_k} \Delta^k u(0), \tag{3.1.4}$$

where ν is the outer normal to the unit sphere ∂S.

Proof. In [89, Theorem 7.3] it is proved that for any polyharmonic function $u(x)$ in S the Almansi representation is valid

$$u(x) = v_0(x) + \sum_{k=1}^{\infty} \frac{1}{4^k} \frac{|x|^{2k}}{k!} \int_0^1 \frac{(1-\alpha)^{k-1}}{(k-1)!} \alpha^{n/2-1} v_k(\alpha x)\, d\alpha, \tag{3.1.5}$$

in which the harmonic functions $v_0(x), \ldots, v_k(x), \ldots$ in S are given by the formula

$v_k(x) = \Delta^k u(x)$

$$+\sum_{s=1}^{\infty}\frac{(-1)^s}{4^s}\frac{|x|^{2s}}{s!}\int_0^1\frac{(1-\alpha)^{s-1}\alpha^{s-1}}{(s-1)!}\alpha^{n/2-1}\Delta^{k+s}u(\alpha x)\,d\alpha. \quad (3.1.6)$$

The limits of summation in the sums above are infinite, then since the function $u(x)$ is polyharmonic in S, then in essence the summation is finite and $v_k(x) = 0$ starting from some k_0. It is easy to see that the equality $\Lambda\big(|x|^{2k}u\big) = |x|^{2k}(2k+\Lambda)u$ is true and therefore

$$\Lambda^{[2]}\big(|x|^{2k}u\big) = (\Lambda-1)\big(|x|^{2k}(2k+\Lambda)u\big) = |x|^{2k}(2k-1+\Lambda)(2k+\Lambda)u,$$

which means

$$\Lambda^{[m]}\big(|x|^{2k}u\big) = |x|^{2k}(2k-m+1+\Lambda)\cdots$$
$$\cdots(2k-1+\Lambda)(2k+\Lambda)u = (2k)^{[m]}u + Q_m(\Lambda)u,$$

where $Q_m(\lambda)$ is some polynomial such that $Q_m(0) = 0$. Therefore, in S the equalities are true

$$\Lambda^{[m]}u(x) = \Lambda^{[m]}v_0(x) + \sum_{k=1}^{\infty}\frac{(2k)^{[m]}|x|^{2k}}{4^k k!}\int_0^1\frac{(1-\alpha)^{k-1}}{(k-1)!}\alpha^{n/2-1}v_k(\alpha x)\,d\alpha$$
$$+\sum_{k=1}^{\infty}\frac{|x|^{2k}}{4^k k!}\int_0^1\frac{(1-\alpha)^{k-1}}{(k-1)!}\alpha^{n/2-1}Q_m(\Lambda)v_k(\alpha x)\,d\alpha.$$

Using the mean value property for harmonic functions and the harmonicity of Λv_k, it is easy to obtain the equality $\int_{\partial S}Q_m(\Lambda)v_k(\alpha x)\,ds_x = 0$. Therefore, using the equality (3.1.6) we find

$$\frac{1}{\omega_n}\int_{\partial S}\Lambda^{[m]}u(x)\,ds_x = \sum_{k=1}^{\infty}\frac{(2k)^{[m]}}{4^k k!(k-1)!}\int_0^1(1-\alpha)^{k-1}\alpha^{n/2-1}\,d\alpha\,v_k(0)$$
$$= \sum_{k=1}^{\infty}\frac{(2k)^{[m]}v_k(0)}{4^k k!(k-1)!}\frac{\Gamma(k)\Gamma(n/2)}{\Gamma(k+n/2)} = \sum_{k=1}^{\infty}\frac{(2k)^{[m]}v_k(0)}{(2,2)_k(n,2)_k} = \sum_{k=0}^{\infty}\frac{(2k)^{[m]}\Delta^k u(0)}{(2,2)_k(n,2)_k}.$$

Hence, by virtue of (3.1.3), we obtain what is stated in the theorem for $m > 0$. If $m = 0$, then by virtue of the equality (3.1.1) the formula (3.1.4) is true in this case as well. The theorem is proved. □

Example 3.1. 1^0. Let the function $u(x)$ be harmonic in S and $u \in C^m(\bar S)$, then from Theorem 3.1 it follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = 0, \quad m \geq 1.$$

A more general statement for the polyharmonic function can be found in Corollary 3.2 on page 223.

2^0. For a biharmonic function $u \in C^m(\bar S)$ in S, from theorem 3.1 it follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = \omega_n \frac{2^{[m]}}{2n} \Delta u(0) = 0, \quad m \geq 3$$

since $2^{[m]} = 0$ for $m \geq 3$ (see also example 3.3).

3^0. In general, if the function $u(x)$ is k-harmonic in S and $u \in C^m(\bar S)$, then from Theorem 3.1 it follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = \omega_n \sum_{i=0}^{k-1} \frac{(2i)^{[m]} \Delta^i u(0)}{(2,2)_i (n,2)_i} = 0, \quad m \geq 2k-1$$

since $(2k-2)^{[m]} = 0$ for $m \geq 2k-1$.

3.1.1 Number triangle H

Below (see Theorem 3.2) we prove that for every k-harmonic function $u \in C^{k-1}(\bar S)$ the equality

$$u(0) = \frac{1}{\omega_n} \int_{\partial S} \left(h_k^0 u + h_k^1 \frac{\partial u}{\partial \nu} + \cdots + h_k^{k-1} \frac{\partial^{k-1} u}{\partial \nu^{k-1}} \right) ds_x, \tag{3.1.7}$$

holds true, where ν is the outer normal to the sphere S, and h_k^i are some numbers. It is easy to see that $h_k^0 = 1$, since for a harmonic function $u \in C(\bar S)$ ($k = 1$) in S we have

$$u(0) = \frac{h_k^0}{\omega_n} \int_{\partial S} u(x)\, ds_x = h_k^0 u(0).$$

If we choose the function $u(x)$ biharmonic in S ($k = 2$) and $u \in C^1(\bar S)$, then by the Almansi formula we have $u(x) = u_0(x) + |x|^2 u_1(x)$ and therefore

$$\Lambda u(x) = \Lambda u_0(x) + \Lambda(|x|^2 u_1(x)) = \Lambda u_0(x) + 2|x|^2 u_1(x) + |x|^2 \Lambda u_1(x).$$

Therefore, integrating the previous equalities over ∂S we obtain

$$\frac{1}{\omega_n}\int_{\partial S} u(x)\,ds_x = \frac{1}{\omega_n}\int_{\partial S}(u_0(x)+u_1(x))\,ds_x = u_0(0)+u_1(0),$$
$$\frac{1}{\omega_n}\int_{\partial S}\frac{\partial u}{\partial \nu}\,ds_x = \frac{1}{\omega_n}\int_{\partial S}\Lambda u(x)\,ds_x$$
$$= \frac{1}{\omega_n}\int_{\partial S}(\Lambda u_0(x)+2u_1(x)+\Lambda u_1(x))\,ds_x = 2u_1(0)$$

and therefore we can easily find

$$u(0) = u_0(0) = \frac{1}{\omega_n}\int_{\partial S}\left(u(x)-\frac{1}{2}\frac{\partial u}{\partial \nu}\right)ds_x.$$

The study of the numbers h_k^s for $s=\overline{0,k-1}$ and $k\in\mathbb{N}$ is the subject of research in Sections 3.1.2-3.1.4. For example, we made sure that for $k=1,2$ we have $h_1^0=1$ and $h_2^0=1$, $h_2^1=-1/2$. Let us arrange the numbers h_k^s in the form of a triangle H like Pascal's triangle. Rows are numbered with a subscript $k\in\mathbb{N}_0$, and slanted columns parallel to the left side are numbered with a superscript $s\in\mathbb{N}_0$, but $s<k$

$$H = \begin{array}{c} 1 \\ 1 \quad -1/2 \\ 1 \quad h_3^1 \quad h_3^2 \\ 1 \quad h_4^1 \quad h_4^2 \quad h_4^3 \\ h_5^1 \quad h_5^2 \quad h_5^3 \quad h_5^4 \quad h_5^5 \\ \cdots \end{array}. \tag{3.1.8}$$

Let us put in the equality (3.1.7) $u(x)=|x|^{2i}v(x)$, where $v(x)$ is a harmonic function $v\in C^{k-1}(\bar S)$ such that $v(0)\neq 0$ and $0\le i\le k-1$. Then

$$\Lambda u(x) = v(x)\Lambda|x|^{2i} + |x|^{2i}\Lambda v(x) = v(x)\Lambda|x|^{2i} + v_1(x),$$

where $v_1(x)=|x|^{2i}\Lambda v(x)$ and therefore

$$\int_{\partial S} v_1(x)\,ds_x = \int_{\partial S}\Lambda v(x)\,ds_x = \omega_n\Lambda v(0) = 0.$$

From here

$$\Lambda^{[2]}u(x) = \Lambda(\Lambda-1)u(x) = v(x)\Lambda(\Lambda-1)|x|^{2i} + \Lambda v(x)\Lambda|x|^{2i}$$

$$+ (\Lambda - 1)v_1(x) = v(x)\Lambda(\Lambda - 1)|x|^{2i} + v_2(x) = v(x)\Lambda^{[2]}|x|^{2i} + v_2(x),$$

where $v_2(x) = \Lambda v(x)\Lambda|x|^{2i} + (\Lambda - 1)v_1(x)$ and therefore

$$\int_{\partial S} v_2(x)\,ds_x = \int_{\partial S} (2i\Lambda v(x) + (\Lambda - 1)v_1(x))\,ds_x$$
$$= \omega_n(2i\Lambda v(0) + (\Lambda - 1)v_1(0)) = 0.$$

Consequently we have

$$\Lambda^{[s]}u(x) = v(x)\Lambda^{[s]}|x|^{2i} + v_s(x),$$

where the functions $v_s(x)$ are such that $\int_{\partial S} v_s(x)\,ds_x = 0$. So, according to (3.1.7) and (3.1.3)

$$0 = \frac{1}{\omega_n}\int_{\partial S}\Big((2i)^{[0]}h_k^0 v(x) + (2i)^{[1]}h_k^1 v(x) + \cdots + (2i)^{[k-1]}h_k^{k-1}v(x)\Big)\,ds_x$$
$$= v(0)\Big((2i)^{[0]}h_k^0 + (2i)^{[1]}h_k^1 + \cdots + (2i)^{[k-1]}h_k^{k-1}\Big) \quad (3.1.9)$$

and therefore for $1 \le i \le k-1$ we have the equalities

$$(2i)^{[0]}h_k^0 + (2i)^{[1]}h_k^1 + \cdots + (2i)^{[k-1]}h_k^{k-1} = 0.$$

This means that the vector $h_k = (h_k^0, h_k^1, \ldots, h_k^{k-1})$ must satisfy the following system of equations

$$\begin{array}{c} 2^{[0]}h_k^0 + \cdots + 2^{[k-2]}h_k^{k-2} = -2^{[k-1]}h_k^{k-1} \\ \vdots \\ (2k-2)^{[0]}h_k^0 + \cdots + (2k-2)^{[k-2]}h_k^{k-2} = -(2k-2)^{[k-1]}h_k^{k-1} \end{array}. \quad (3.1.10)$$

Since the solution to this homogeneous system is unique up to a constant factor (the determinant of the matrix on the left is different from 0), it is necessary to choose a solution for which $h_k^0 = 1$. Then, with such a vector $h_k = (1, h_k^1, \ldots, h_k^{k-1})$ the equality (3.1.7) will be true. The opposite is also true, i.e. if $h_k = (1, h_k^1, \ldots, h_k^{k-1})$ satisfies the system (3.1.10), then the equalities (3.1.9) are true, and hence the equality (3.1.7) is also true for a k-harmonic function of the form $u(x) = |x|^{2i}v(x)$, where $0 \le i < k$ and $v(x)$ is harmonic in S function $v \in C^{k-1}(\bar{S})$. By virtue of the Almansi formula, what is stated in (3.1.7) is also true for an arbitrary k-harmonic function $u \in C^{k-1}(\bar{S})$.

We denote the determinant of the matrix in (3.1.10) on the left by Δ_k, and the determinant of the matrix obtained from this matrix by replacing the sth column at $0 \le s \le k-2$ with the column $(2^{[k-1]}, \dots, (2k-2)^{[k-1]})^T$ we denote by Δ_k^s, i.e. (similar formulas are discussed in Section 3.2.1 on page 177)

$$\Delta_k^s \begin{vmatrix} 2^{[0]} & \dots & 2^{[s-1]} & 2^{[k-1]} & 2^{[s+1]} & \dots & 2^{[k-2]} \\ 4^{[0]} & \dots & 4^{[s-1]} & 4^{[k-1]} & 4^{[s+1]} & \dots & 4^{[k-2]} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (2k-2)^{[0]} & \dots & (2k-2)^{[s-1]} & (2k-2)^{[k-1]} & (2k-2)^{[s+1]} & \dots & (2k-2)^{[k-2]} \end{vmatrix}$$

It is easy to see that

$$\Delta_k = \begin{vmatrix} 2^{[0]} & \dots & 2^{[k-2]} \\ 4^{[0]} & \dots & 4^{[k-2]} \\ \vdots & \ddots & \vdots \\ (2k-2)^{[0]} & \cdots & (2k-2)^{[k-2]} \end{vmatrix} = W[2, 4, \dots, 2k-2] = (2k-4)!! \cdots 2!!,$$

where $W[\lambda_0, \dots, \lambda_{k-2}]$ denotes the Vandermonde determinant of order $k-1$. Therefore, the system (3.1.10) is uniquely solvable for any h_k^{k-1}. Let us choose h_k^{k-1} so that $h_k^0 = 1$.

Let us calculate the determinants of Δ_k^s. We consider a more general case. Let $\lambda_0, \dots, \lambda_{k-2} \in \mathbb{R}$. Let us denote

$$\Delta_k^s(\lambda) \equiv \Delta_k^s(\lambda_0, \dots, \lambda_{k-2})$$
$$= \begin{vmatrix} \lambda_0^{[0]} & \dots & \lambda_{k-2}^{[0]} \\ \vdots & \ddots & \vdots \\ \lambda_0^{[s-1]} & \dots & \lambda_{k-2}^{[s-1]} \\ \lambda_0^{[k-1]} & \dots & \lambda_{k-2}^{[k-1]} \\ \lambda_0^{[s+1]} & \dots & \lambda_{k-2}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_0^{[k-2]} & \dots & \lambda_{k-2}^{[k-2]} \end{vmatrix} = (-1)^{k-s-2} \begin{vmatrix} \lambda_0^{[0]} & \dots & \lambda_{k-2}^{[0]} \\ \vdots & \ddots & \vdots \\ \lambda_0^{[s-1]} & \dots & \lambda_{k-2}^{[s-1]} \\ \lambda_0^{[s+1]} & \dots & \lambda_{k-2}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_0^{[k-2]} & \dots & \lambda_{k-2}^{[k-2]} \\ \lambda_0^{[k-1]} & \dots & \lambda_{k-2}^{[k-1]} \end{vmatrix} \quad (3.1.11)$$

and $\Delta_k(\lambda) = \det\left(\lambda_j^{[i]}\right)_{i,j=\overline{0,k-2}}$. It is obvious that $\Delta_k(\lambda) = \Delta_k^{k-1}(\lambda)$.

Lemma 3.1. *For an integer s such that* $0 \le s \le k-1$ *the equality holds true*

$$s^{[0]}\Delta_k^0(\lambda) + \dots + s^{[s]}\Delta_k^s(\lambda) = -\Delta_k(\lambda) \prod_{i=0}^{k-2} (s - \lambda_i). \quad (3.1.12)$$

Proof. Let s be an integer and such that $0 \le s \le k-1$. Consider a determinant of the form

$$V(s;\lambda) = \begin{vmatrix} s^{[0]} & \lambda_0^{[0]} & \dots & \lambda_{k-2}^{[0]} \\ \vdots & & \ddots & \vdots \\ s^{[s]} & \lambda_0^{[s]} & \dots & \lambda_{k-2}^{[s]} \\ 0 & \lambda_0^{[s+1]} & \dots & \lambda_{k-2}^{[s+1]} \\ \vdots & & \ddots & \vdots \\ 0 & \lambda_0^{[k-1]} & \dots & \lambda_{k-2}^{[k-1]} \end{vmatrix}.$$

Let us expand it along the first column. In accordance with (3.1.11) and taking into account the equality $s^{[i]} = 0$ true for $i > s$ we have

$$V(s;\lambda) = (-1)^{k-2}s^{[0]}\Delta_k^0(\lambda) - (-1)^{k-3}s^{[1]}\Delta_k^1(\lambda) + \cdots + \\ + (-1)^s(-1)^{k-s-2}s^{[s]}\Delta_k^s(\lambda) = (-1)^{k-2}\left(s^{[0]}\Delta_k^0(\lambda) + \cdots + s^{[s]}\Delta_k^s(\lambda)\right).$$

On the other hand, it is easy to see that the determinant $V(s;\lambda)$ is reduced to the Vandermonde determinant $W[s,\lambda_0,\dots,\lambda_{k-2}]$ using linear operations on rows. That is why

$$V(s;\lambda) = W[s,\lambda_0,\dots,\lambda_{k-2}] = \prod_{i=0}^{k-2}(\lambda_i - s)W[\lambda_0,\dots,\lambda_{k-2}] \\ = \prod_{i=0}^{k-2}(\lambda_i - s)\Delta_k(\lambda) = (-1)^{k-1}\prod_{i=0}^{k-2}(s-\lambda_i)\Delta_k(\lambda).$$

Therefore the equality is true

$$(-1)^{k-1}\prod_{i=0}^{k-2}(s-\lambda_i)\Delta_k(\lambda) = (-1)^{k-2}\left(s^{[0]}\Delta_k^0(\lambda) + \cdots + s^{[s]}\Delta_k^s(\lambda)\right).$$

This immediately implies the equality (3.1.12). □

3.1.2 Auxiliary equalities for numbers h_k^s

Let us use Lemma 3.1 to determine the numbers h_k^s.

Lemma 3.2. *For any integers $k > 1$ and s such that $0 \le s \le k-2$ the equality holds true*

$$s^{[0]} h_k^0 + \cdots + s^{[s]} h_k^s = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{s}{2} - 1\right)^{[k-1]}. \qquad (3.1.13)$$

Proof. By virtue of the notation made above, $\Delta_k^s(2, \ldots, 2k-2) = \Delta_k^s$ and therefore, by Lemma 3.1 with $\lambda_i = 2i + 2$ we have

$$s^{[0]} \Delta_k^0 + \cdots + s^{[s]} \Delta_k^s = -\Delta_k \prod_{i=0}^{k-2} (s - 2i - 2) = -2^{k-1} \left(\frac{s}{2} - 1\right)^{[k-1]} \Delta_k. \qquad (3.1.14)$$

Hence, for $s = 0$ and $k > 1$ we get

$$\Delta_k^0 = -\Delta_k 2^{k-1} (-1)^{[k-1]} = (-1)^k (2k-2)!! \Delta_k.$$

Remembering the system (3.1.10) we find $h_k^s = -h_k^{k-1} \dfrac{\Delta_k^s}{\Delta_k}$. For $s = 0$ we have

$$h_k^0 = -h_k^{k-1} \frac{\Delta_k^0}{\Delta_k} = (-1)^{k-1} h_k^{k-1} (2k-2)!!$$

and therefore for $h_k^{k-1} = (-1)^{k-1}/(2k-2)!!$ we have $h_k^0 = 1$. Thus

$$h_k^s = (-1)^k \frac{\Delta_k^s}{(2k-2)!! \Delta_k}.$$

Therefore, dividing both sides of the equality (3.1.14) by $(2k-2)!!\Delta_k \ne 0$ and multiplying by $(-1)^k$ we get (3.1.13). The lemma is proved. □

From the proof of Lemma 3.2 it follows that for $k \ge 1$

$$h_k^{k-1} = \frac{(-1)^{k-1}}{(2k-2)!!}. \qquad (3.1.15)$$

Thus, we have defined the left and right sides of the numerical triangle (3.1.8), consisting of the numbers h_k^s.

3.1.3 Formulas to determine h_k^s

Lemma 3.3. *For $k \in \mathbb{N}$ and an integer s such that $0 \le s \le k-2$ the numbers h_k^s are defined by the equality*

$$h_k^s = \frac{(-1)^{k+s}}{s!(k-1)!} \sum_{j=0}^{s} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2} - 1\right)^{[k-1]}. \tag{3.1.16}$$

Proof. Let us denote $\tilde{h}_k^s = s! h_k^s$ and assume that $\binom{a}{b} = 0$ for $b > a$. Then (3.1.13) can be rewritten as

$$\binom{s}{1} \tilde{h}_k^0 + \cdots + \binom{s}{s} \tilde{h}_k^s + \cdots + \binom{s}{k-2} \tilde{h}_k^{k-2} = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{s}{2} - 1\right)^{[k-1]}$$

and therefore the vector $\tilde{h}_k = (\tilde{h}_k^0, \dots, \tilde{h}_k^{k-2})$ is a solution to the algebraic system

$$C\tilde{h}_k = \frac{(-1)^{k-1}}{(k-1)!} \left(\left(\frac{j}{2} - 1\right)^{[k-1]} \right)_{j=\overline{0,k-2}}$$

with the matrix $C = \left(\binom{i}{j}\right)_{i,j=\overline{0,k-2}}$. It is easy to verify that the triangular matrix C is invertible and $C^{-1} = \left((-1)^{i+j}\binom{i}{j}\right)_{i,j= \ overline 0,k-2}$. This follows from the following equality

$$\sum_{s=0}^{k-2} (-1)^{s+j} \binom{i}{s} \binom{s}{j} = \delta_{i,j}.$$

Therefore we have

$$\tilde{h}_k = \frac{(-1)^{k-1}}{(k-1)!} \left(\sum_{j=0}^{k-2} (-1)^{i+j} \binom{i}{j} \left(\frac{j}{2} - 1\right)^{[k-1]} \right)_{i=\overline{0,k-2}}$$

and therefore, taking into account that $\tilde{h}_k^s = s! h_k^s$ we get

$$h_k^s = \frac{(-1)^{k+s}}{s!(k-1)!} \sum_{j=0}^{k-2} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2} - 1\right)^{[k-1]}.$$

This immediately follows (3.1.16). □

Let us check the resulting formula (3.1.16) for $s = 0$. It is easy to see that $h_k^0 = (-1)^{k+1}(-1)^{[k-1]}/(k-1)! = 1$, which coincides with what is found earlier. Next, let us calculate the unknown numbers h_k^1. For $k > 1$ from (3.1.16) we find

$$h_k^1 = \frac{(-1)^{k+2}}{(k-1)!}\Big((-1)^{[k-1]} - \Big(-\frac{1}{2}\Big)^{[k-1]}\Big) = -\Big(1 - \frac{(2k-3)!!}{(2k-2)!!}\Big). \tag{3.1.17}$$

So, we found the elements of the triangle (3.1.8) on the first diagonal parallel to the left side of the triangle. For $k = 2$ we have the value calculated above $h_2^1 = -1/2$. Let us try to transform the equality (3.1.16) to a more compact form.

Lemma 3.4. *For $k \in \mathbb{N}$ and an integer s such that $0 \le s \le k-2$ the equality holds*

$$h_k^s = \frac{(-1)^{k-1}}{s!(k-1)!}\left(\frac{1}{t}(\sqrt{t}-1)^s\right)^{(k-1)}\Big|_{t=1}. \tag{3.1.18}$$

Proof. It is easy to see that

$$\left(\frac{j}{2}-1\right)^{[k-1]} = \left(\frac{j}{2}-1\right)\dots\left(\frac{j}{2}-k+1\right) = \left(t^{\frac{j}{2}-1}\right)^{(k-1)}\Big|_{t=1}.$$

Therefore, for integer $k > 1$ and s such that $0 \le s \le k-2$, in accordance with (3.1.16), the equalities hold

$$\begin{aligned} h_k^s &= \frac{(-1)^{k+s}}{s!(k-1)!}\sum_{j=0}^{s}(-1)^{j+1}\binom{s}{j}\left(\frac{j}{2}-1\right)^{[k-1]} \\ &= \frac{(-1)^{k+1}}{s!(k-1)!}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}\left((\sqrt{t})^{j-2}\right)^{(k-1)}\Big|_{t=1} \\ &= \frac{(-1)^{k+1}}{s!(k-1)!}\left(\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(\sqrt{t})^{j-2}\right)^{(k-1)}\Big|_{t=1} \\ &= \frac{(-1)^{k-1}}{s!(k-1)!}\left(\frac{1}{t}(\sqrt{t}-1)^s\right)^{(k-1)}\Big|_{t=1}, \end{aligned}$$

that is the formula (3.1.18) is correct for the specified k and s. □

3.1.4 Recurrence equation for numbers h_k^s

Let us transform the equality for writing the numbers h_k^s a little more. Let us prove an auxiliary lemma. Let us remind that

$$\binom{k}{i_1\dots i_s}=\frac{k!}{i_1!\dots i_s!}.$$

Lemma 3.5. *Let $f_i\in C^k(a,b)$ for $i=1,\dots,s$ then*

$$(f_1(t)\cdots f_s(t))^{(k)}=\sum_{i_1+\ldots+i_s=k}\binom{k}{i_1\dots i_s}f_1^{(i_1)}(t)\cdots f_s^{(i_s)}(t) \tag{3.1.19}$$

at $t\in(a,b)$, where $i_1,\dots,i_s\in\mathbb{N}_0$.

Proof. For $k=0$ the equality is obvious. For $k=1$ we have the equality

$$(f_1(t)\cdots f_s(t))'=f_1'(t)f_2(t)\cdots f_s(t)+f_1(t)f_2'(t)\cdots f_s(t)+f_1(t)f_2(t)\cdots f_s'(t),$$

and using the equality (3.1.19) we get

$$\sum_{i_1+\ldots+i_s=1}\binom{1}{i_1\dots i_s}f^{(i_1)}(t)\cdots f^{(i_s)}(t)$$
$$=1\cdot f_1'(t)f_2(t)\cdots f_s(t)+1\cdot f_1(t)f_2'(t)\cdots f_s(t)+1\cdot f_1(t)f_2(t)\cdots f_s'(t),$$

and therefore the equality (3.1.19) is correct for $k=1$. Let now (3.1.19) be true for some $k\in\mathbb{N}$. Let us prove its validity for $k+1$. We have

$$\begin{aligned}
&(f_1(t)\cdots f_s(t))^{(k+1)}\\
&\quad=\left((f_1(t)\cdots f_s(t))^{(k)}\right)'=\left(\sum_{i_1+\cdots+i_s=k}\binom{k}{i_1\dots i_s}f_1^{(i_1)}(t)\cdots f_s^{(i_s)}(t)\right)'\\
&\quad=\sum_{i_1+\cdots+i_s=k}\binom{k}{i_1\dots i_s}\left(f_1^{(i_1+1)}(t)\cdots f_s^{(i_s)}(t)+\cdots+f_1^{(i_1)}(t)\cdots f_s^{(i_s+1)}(t)\right)\\
&\quad=\sum_{i_1+\cdots+i_s-1=k}\left(\binom{k}{i_1-1\dots i_s}+\cdots+\binom{k}{i_1\dots i_s-1}\right)f_1^{(i_1)}(t)\cdots f_s^{(i_s)}(t)\\
&\quad=\sum_{i_1+\cdots+i_s=k+1}\binom{k+1}{i_1\dots i_s}f_1^{(i_1)}(t)\cdots f_s^{(i_s)}(t),
\end{aligned}$$

where we should assume that if $i_j<0$, then $\binom{k}{i_1\dots i_s}=0$. That is what needed to be proved. □

Remark 3.1. The equality (3.1.18) is also true for $s = k - 1$. Indeed, from (3.1.18) for $s = k - 1$ using the Lemma 3.5 we find

$$h_k^{k-1} = \frac{(-1)^{k-1}}{(k-1)!(k-1)!}\left(\frac{1}{t}(\sqrt{t}-1)^{k-1}\right)^{(k-1)}\Big|_{t=1}$$
$$= \frac{(-1)^{k-1}}{(k-1)!(k-1)!}\binom{k-1}{01\cdots 1}\frac{1}{t}\Big(\frac{1}{2\sqrt{t}}\Big)^{k-1}\Big|_{t=1}$$
$$= \frac{(-1)^{k-1}}{(k-1)!(k-1)!}\frac{(k-1)!}{2^{k-1}} = \frac{(-1)^{k-1}}{(2k-2)!!},$$

which is the same as (3.1.15).

Using Lemma 3.5, the equality (3.1.18) allows one to find the numbers h_k^s for large values of s, which is quite difficult to do using the equality (3.1.16). For example, similar to Remark 3.1

$$h_k^{k-2} = \frac{(-1)^{k-1}}{(k-2)!(k-1)!}\left(\frac{1}{t}(\sqrt{t}-1)^{k-2}\right)^{(k-1)}\Big|_{t=1} = \frac{(-1)^{k}}{2^{k-2}(k-2)!(k-1)!}$$
$$\times\left(\binom{k-1}{1\ldots 1} + \frac{k-2}{2}\binom{k-1}{0\,2\ldots 1}\right) = \frac{(-1)^{k-2}}{4(2k-4)!!}(4+k-2) = \frac{k+2}{4}\frac{(-1)^{k-2}}{(2k-4)!!}.$$

Lemma 3.6. *For the numbers h_k^s the recurrent equality*

$$h_{k+1}^s = \left(1 - \frac{s}{2k}\right)h_k^s + \frac{1}{2k}h_k^{s-1}, \tag{3.1.20}$$

holds true, moreover, $h_k^s = 0$ for $s \geq k$ and $h_k^0 = 1$, $k \in \mathbb{N}$.

Proof. For $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$ we introduce the notation

$$a_k^s = \left(\frac{1}{t}(\sqrt{t}-1)^s\right)^{(k-1)}\Big|_{t=1}. \tag{3.1.21}$$

First, let us prove that the recurrence equality is true for the numbers a_k^s

$$a_{k+1}^s = \left(\frac{s}{2} - k\right)a_k^s + \frac{s}{2}a_k^{s-1}, \tag{3.1.22}$$

where $a_k^s = 0$ for $s \geq k$ and $a_k^0 = (-1)^{k-1}(k-1)!$, $k \in \mathbb{N}$.

Let us examine the triangle A, similar to the triangle (3.1.8), but consisting of the numbers a_k^s for $k \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $s \leq k-1$. If $s = 0$ or $k = 1$, then the value of

a_k^s can also be calculated $a_k^0 = (-1)^{k-1}(k-1)!$ and $a_1^s = 0$. In addition, by (3.1.21) $a_2^1 = \frac{1}{2}$ and $a_1^0 = 1$.

It is easy to see that if the degree of s is higher than or equal to k, then from (3.1.21) it follows that $a_k^s = 0$. The value of a_k^0 is calculated using the equality (3.1.21) above. Let us use the equality (3.1.19) from Lemma 3.5 for $s = s+1$ and $f_0(t) = 1/t$, $f_i(t) = \sqrt{t} - 1$. Since

$$(\sqrt{t}-1)^{(m)} = \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-m+1\right)t^{\frac{1}{2}-m} = \frac{(-1)^{m-1}}{2^m}(2m-3)!!\,t^{\frac{1}{2}-m},$$

then we can write

$$\begin{aligned}
&\left(\frac{1}{t}(\sqrt{t}-1)^s\right)^{(k-1)} \\
&\quad = \sum_{\substack{i_0+i_1+\cdots+i_s=k-1\\ i_0\ge 0, i_1>0,\ldots,i_s>0}} \binom{k-1}{i_0\, i_1\, \ldots\, i_s}(-1)^{k-s-1}\frac{(2i_0)!!(2i_1-3)!!\cdots(2i_s-3)!!}{2^{k-1}}t^{\frac{s}{2}-k} \\
&\quad + s\sum_{\substack{i_0+i_1+\cdots+i_{s-1}=k-1\\ i_0\ge 0, i_1>0,\ldots,i_{s-1}>0}} \binom{k-1}{i_0\, i_1\, \ldots\, i_{s-1}}(-1)^{k-s}\frac{(2i_0)!!(2i_1-3)!!\cdots(2i_{s-1}-3)!!}{2^{k-1}} \\
&\qquad \times t^{\frac{s-1}{2}-k}(\sqrt{t}-1) + s(s-1)\sum_{\substack{i_0+i_1+\cdots+i_{s-2}=k-1\\ i_0\ge 0, i_1>0,\ldots,i_{s-2}>0}} \binom{k-1}{i_0\, i_1\, \ldots\, i_{s-2}}(-1)^{k-s+1} \\
&\qquad \times \frac{(2i_0)!!(2i_1-3)!!\cdots(2i_{s-2}-3)!!}{2^{k-1}}t^{\frac{s-2}{2}-k}(\sqrt{t}-1)^2 + \cdots = a_k^s t^{\frac{s}{2}-k} \\
&\qquad\qquad + s a_k^{s-1} t^{\frac{s-1}{2}-k}(\sqrt{t}-1) + s(s-1)a_k^{s-2}t^{\frac{s-2}{2}-k}(\sqrt{t}-1)^2 + \cdots. \qquad (3.1.23)
\end{aligned}$$

Therefore

$$\left((\sqrt{t}-1)^s\right)^{(k)} = a_{k+1}^s t^{\frac{s}{2}-k-1} + s a_{k+1}^{s-1} t^{\frac{s-1}{2}-k-1}(\sqrt{t}-1) + \cdots.$$

On the other hand, differentiating (3.1.23) we get

$$\left((\sqrt{t}-1)^s\right)^{(k)} = \left(\frac{s}{2}-k\right)a_k^s t^{\frac{s}{2}-k-1} + \frac{s}{2}a_k^{s-1}t^{\frac{s-2}{2}-k} + (\sqrt{t}-1)F(t),$$

where $F(t)$ is some function. Assuming $t = 1$ in the previous equalities we get

$$a_{k+1}^s = \left(\frac{s}{2}-k\right)a_k^s + \frac{s}{2}a_k^{s-1}.$$

Remembering that $a_k^s = h_k^s(-1)^{k-1}s!(k-1)!$ we find

$$h_{k+1}^s(-1)^k s!k! = \left(\frac{s}{2}-k\right)h_k^s(-1)^{k-1}s!(k-1)! \\ + \frac{s}{2}h_k^{s-1}(-1)^{k-1}(s-1)!(k-1)!.$$

Dividing both sides of this equality by $(-1)^k s!k!$ we arrive at (3.1.20). For $s=0$ we have the equality $h_k^0 = a_k^0/((-1)^{k-1}(k-1)!) = 1$ and, in addition, $h_k^s = 0$ for $s \ge k$. Which is what needed to be proven. □

Remark 3.2. Note that the equation

$$a_{k+1}^s = (k-2s+1)a_k^s + \frac{1}{2}a_k^{s-1},$$

similar to (3.1.22) is obtained in (1.3.24) and is used in the construction of special polynomials. Moreover, the equality

$$b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1}$$

(3.2.17) is used when examining triangle B from (3.2.18), and the equality

$$a_{k+1}^s = \left(\frac{s}{2}-k\right)a_k^s + \frac{s}{2}a_k^{s-1}$$

is solved in Lemma 3.14.

An equation of the form (3.1.20) defines an arithmetic triangle like Pascal's, Euler's or Stirling's triangles (see (1.1.2)), but its elements are rational fractions. Calculating h_k^s using the equality (3.1.20) we expand the number triangle H from (3.1.8) to five rows

$$H = \begin{array}{c} 1 \\ 1 \quad -\frac{1}{2} \\ 1 \quad -\frac{5}{8} \quad \frac{1}{8} \\ 1 \quad -\frac{11}{16} \quad \frac{3}{16} \quad -\frac{1}{48} \\ 1 \quad -\frac{93}{128} \quad \frac{29}{128} \quad -\frac{7}{192} \quad \frac{1}{384} \\ \cdots \\ \cdots \quad h_{k+1}^s = (1-s/(2k))h_k^s - 1/(2k)h_k^{s-1} \quad \cdots \end{array}. \tag{3.1.24}$$

Note that the 4th row of the triangle H coincides with the first row of the matrix $\mathbb{H}_4$ from Example 2.4 on page 137.

Another look at the numbers h_k^s is given by the following lemma.

Lemma 3.7. *The rows of triangle H have the property: the numbers h_k^s for $0 \leq s \leq k-1$ and $k \geq 2$ are coefficients of the expansion of the polynomial of $(k-1)$th degree*

$$H_k(\lambda) = \frac{(-1)^{k-1}}{(2k-2)!!}(\lambda-2)\cdots(\lambda-2k+2)$$

by factorial powers $\lambda^{[s]}$

$$H_k(\lambda) = h_k^{k-1}\lambda^{[k-1]} + h_k^{k-2}\lambda^{[k-2]} + \cdots + h_k^1\lambda^{[1]} + h_k^0.$$

Proof. Consider the following polynomial of $(k-1)$th degree in λ

$$\hat{H}_k(\lambda) = h_k^{k-1}\lambda^{[k-1]} + h_k^{k-2}\lambda^{[k-2]} + \cdots + h_k^1\lambda^{[1]} + h_k^0.$$

From the equality (3.1.13) of Lemma 3.2 it follows that for $0 \leq s \leq k-2$ and $k \geq 2$

$$\hat{H}_k(s) = h_k^{k-1}s^{[k-1]} + \cdots + h_k^1 s^{[1]} + h_k^0$$
$$= h_k^s s^{[s]} + \cdots + h_k^0 s^{[0]} = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{s}{2}-1\right)^{[k-1]}.$$

Because if the values of two polynomials of degree $k-1$, having the same free terms equal to 1, coincide at $k-1$ points, then they are equal everywhere. Further, for even values of $\lambda = 2, \ldots, 2k-2$ we have $\lambda/2 - 1 \in \mathbb{N}_0$, $\lambda/2 - 1 \leq k-2$ and therefore due to properties of factorial degree

$$\hat{H}_k(\lambda) = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\lambda}{2}-1\right)^{[k-1]} = 0.$$

However, the polynomial $H_k(\lambda)$ has the same $k-1$ zeros, and therefore $\hat{H}_k(\lambda) = CH_k(\lambda)$. To find the constant C, let us put $\lambda = 0$ in this equality. Then we get

$$1 = h_k^0 = \hat{H}_k(0) = CH_k(0) = C\frac{(-1)^{k-1}}{(2k-2)!!}(-2)\cdots(-2k+2) = C$$

and that means $\hat{H}_k(\lambda) = H_k(\lambda)$. □

3.1.5 The value of polyharmonic function at the origin

Taking into account the research done, we can prove the following statement.

Theorem 3.2. *For any k-harmonic function $u \in C^{k-1}(\bar{S})$ in the unit ball S the following equality holds*

$$u(0) = \frac{1}{\omega_n} \int_{\partial S} \left(h_k^0 u + h_k^1 \frac{\partial u}{\partial \nu} + \cdots + h_k^{k-1} \frac{\partial^{k-1} u}{\partial \nu^{k-1}} \right) ds_x,$$

where the numbers h_k^s are taken from (3.1.16) *or* (3.1.18), *satisfy the recurrence equation* (3.1.20) *and are the expansion coefficients of the polynomial*

$$H_{k-1}(\lambda) = \frac{(-1)^{k-1}}{(2k-2)!!} (\lambda - 2) \cdots (\lambda - 2k + 2) \tag{3.1.25}$$

in factorial powers $\lambda^{[s]}$

$$H_{k-1}(\lambda) = h_k^{k-1} \lambda^{[k-1]} + h_k^{k-2} \lambda^{[k-2]} + \cdots + h_k^1 \lambda^{[1]} + h_k^0. \tag{3.1.26}$$

Proof. For any polyharmonic function $u \in C^{k-1}(\bar{S})$ in the unit ball S the numbers h_k^s from the equality (3.1.7) must satisfy the system of equations (3.1.10). By Lemma 3.3 the solution to this system is represented in the form (3.1.16), and from Lemma 3.4 it follows that they are also written in the form (3.1.18). Lemma 3.6 proves that h_k^s also satisfy the recurrence equation (3.1.20). Finally, by Lemma 3.7 the equality (3.1.26) holds. The theorem is proved. □

Remark 3.3. The equality (3.1.7) due to (3.1.26), using the operator Λ and the equality (3.1.3) can be rewritten in the form

$$u(0) = \frac{1}{\omega_n} \int_{\partial S} H_{k-1}(\Lambda) u(x) \, ds_x.$$

Example 3.2. For the 4-harmonic function $u \in C^3(\bar{S})$, in accordance with the 4th row of the triangle H from (3.1.24), the equality holds

$$u(0) = \frac{1}{\omega_n} \int_{\partial S} \left(u - \frac{11}{16} \frac{\partial u}{\partial \nu} + \frac{3}{16} \frac{\partial^2 u}{\partial \nu^2} - \frac{1}{48} \frac{\partial^3 u}{\partial \nu^3} \right) ds_x.$$

Consider the polynomial

$$H_{k-1}^{(m)}(\lambda) = \lambda(\lambda - 2) \cdots (\lambda - 2m + 2)(\lambda - 2m - 2) \cdots (\lambda - 2k + 2), \tag{3.1.27}$$

where the factor $\lambda - 2m$ is missing. Obviously, the equality $H_{k-1}(\lambda) = H_{k-1}^{(0)}(\lambda)/H_{k-1}^{(0)}(0)$ is true for it and $H_{k-1}^{(m)}(2m) \neq 0$.

Lemma 3.8. *Let the Almansi representation of the k-harmonic function $u \in C^{k-1}(\bar{S})$ in S have the form*

$$u(x) = u_0(x) + \cdots + |x|^{2k-2} u_{k-1}(x).$$

Then for $m \in \mathbb{N}_0$ and $m < k$ the following equality holds

$$u_m(0) = \frac{1}{\omega_n} \int_{\partial S} \frac{H_{k-1}^{(m)}(\Lambda)}{H_{k-1}^{(m)}(2m)} u(x)\, ds_x. \tag{3.1.28}$$

Proof. Let $\lambda \in \mathbb{R}$, $i \in \mathbb{N}_0$, $i < k$ and $v(x)$ be a function harmonic in S. It is easy to see that in S the equality is true

$$(\Lambda - \lambda)\Big(|x|^{2i} v(x)\Big) = |x|^{2i}\big((2i - \lambda)v(x) + \Lambda v(x)\big)$$

and therefore

$$H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i} v(x)\Big) = |x|^{2i}\Big(H_{k-1}^{(m)}(2i)v(x) + Q_{k-1}(\Lambda)v(x)\Big),$$

where $Q_{k-1}(\lambda)$ is some polynomial of $(k-1)$th degree, depending on $H_{k-1}^{(m)}$ and such that $Q_{k-1}(0) = 0$. The function $Q_{k-1}(\Lambda)v$ is also harmonic in S. Let S_r be a sphere of radius r with center at the origin. For any $r \in (0, 1)$ we have $Q_{k-1}(\Lambda)v \in C(\bar{S}_r)$. Then

$$\int_{\partial S_r} Q_{k-1}(\Lambda)v(x)\, ds_x = Q_{k-1}(0) \int_{\partial S_r} v(x)\, ds_x = 0.$$

Therefore, if $i \neq m$, then $H_{k-1}^{(m)}(2i) = 0$ and that means

$$\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i} v(x)\Big)\, ds_x$$
$$= H_{k-1}^{(m)}(2i) \int_{\partial S_r} v(x)\, ds_x + \int_{\partial S_r} Q_{k-1}(\Lambda)v(x)\, ds_x = 0.$$

If $i = m$, then similarly to the previous

$$\frac{1}{\omega_n^r} \int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2m} v(x)\Big)\, ds_x$$
$$= H_{k-1}^{(m)}(2m) \frac{1}{\omega_n^r} \int_{\partial S_r} v(x)\, ds_x = H_{k-1}^{(m)}(2m) v(0),$$

where ω_n^r is the area of ∂S_r. Therefore, for the function $u(x)$ the equality holds

$$\frac{1}{\omega_n^r}\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)u(x)\,ds_x = \sum_{i=0}^{k-1}\frac{1}{\omega_n^r}\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i}u_i(x)\Big)\,ds_x = H_{k-1}^{(m)}(2m)u_m(0). \quad (3.1.29)$$

Since $u \in C^{k-1}(\bar S)$, then dividing this equality by $H_{k-1}^{(m)}(2m) \neq 0$ and passing to the limit at $r \to 1-0$ we get (3.1.28). The lemma is proved. □

Theorem 3.3. *For every k-harmonic function $u \in C^k(\bar S)$ in the unit ball S the following equality is true*

$$\int_{\partial S} H_k^{(k)}(\Lambda)u(x)\,ds_x = 0,$$

where $H_k^{(k)}(\lambda) = \lambda(\lambda-2)\cdots(\lambda-2k+2)$.

Proof. It is easy to see that $\forall\, i < k$, $H_k^{(k)}(2i) = 0$. Therefore, due to the equality (3.1.29) from Lemma 3.8 for $r \in (0,1)$ we can write

$$\frac{1}{\omega_n^r}\int_{\partial S_r} H_k^{(k)}(\Lambda)u(x)\,ds_x = \sum_{i=0}^{k-1}\frac{1}{\omega_n^r}\int_{\partial S_r} H_k^{(k)}(\Lambda)\Big(|x|^{2i}u_i(x)\Big)\,ds_x = 0.$$

Passing to the limit at $r \to 1$ we obtain the asserting equality. The theorem is proved. □

A natural generalization of the well-known property $\int_{\partial S}\frac{\partial u}{\partial \nu}\,ds_x = 0$ of harmonic functions to polyharmonic functions is the following statement.

Corollary 3.1. If the numbers a_i are found from the equality

$$H_k^{(k)}(\lambda) = a_k\lambda^{[k]} + a_{k-1}\lambda^{[k-1]} + \cdots + a_1\lambda^{[1]},$$

then the function $u \in C^k(\bar S)$, k-harmonic in the unit ball S, satisfies the equality

$$\int_{\partial S}\left(a_1\frac{\partial u}{\partial \nu} + \cdots + a_k\frac{\partial^k u}{\partial \nu^k}\right)ds_x = 0.$$

To prove Corollary 3.1, it is enough to recall (3.1.3) and use the statement of Theorem 3.3. Theorem 3.2 can be generalized as follows.

Theorem 3.4. *For any k-harmonic function $u \in C^{k-1}(\bar{S})$ in the unit ball S the following equality holds*

$$\Delta^m u(0) = \frac{(2,2)_m (n,2)_m}{\omega_n} \int_{\partial S} \frac{H_{k-1}^{(m)}(\Lambda)}{H_{k-1}^{(m)}(2m)} u(x)\, ds_x, \tag{3.1.30}$$

where the polynomial $H_{k-1}^{(m)}(\lambda)$ is found from (3.1.27) *at $m = 0, \dots, k-1$.*

Proof. Let

$$u(x) = u_0(x) + \cdots + |x|^{2k-2} u_{k-1}(x) \tag{3.1.31}$$

be the Almansi representation of a k-harmonic function $u(x)$ in S and such that $u \in C^{k-1}(\bar{S})$, then for $m \in \mathbb{N}_0$ and $m < k$ the equality (3.1.28) holds. Moreover, if v is a harmonic function in S, then

$$\Delta\Big(|x|^{2m} v(x)\Big) = |x|^{2m-2} 2m(2m + n - 2 + 2\Lambda) v(x).$$

Therefore, for $i < m$ we have

$$\Delta^i\Big(|x|^{2m} v(x)\Big) = |x|^{2m-2i} \prod_{j=m-i+1}^{m} 2j(2j + n - 2 + 2\Lambda) v(x)$$

which means that $\Delta^i\Big(|x|^{2m} v(x)\Big)|_{x=0} = 0$. If $i = m$, then we have

$$\begin{aligned}\Delta^m\Big(|x|^{2m} v(x)\Big) &= \prod_{j=1}^{m} 2j(2j + n - 2 + 2\Lambda) v(x) \\ &= \prod_{j=1}^{m} 2j(2j + n - 2) v(x) + P_k(\Lambda) v(x) = 2m!!\, n \cdots (n + 2m - 2) v(x) \\ &\quad + P_k(\Lambda) v(x) = (2,2)_m (n,2)_m v(x) + P_k(\Lambda) v(x),\end{aligned} \tag{3.1.32}$$

where $P_k(\lambda)$ is some polynomial of kth degree such that $P_k(0) = 0$. Therefore we have

$$\Delta^m\Big(|x|^{2m} v(x)\Big)|_{x=0} = (2,2)_m (n,2)_m v(0).$$

From (3.1.31) it follows that for $i > m$

$$\Delta^i\Big(|x|^{2m} v(x)\Big) = (2,2)_m (n,2)_m \Delta^{i-m} v(x) + \Delta^{i-m} P_k(\Lambda) v(x) = 0.$$

Therefore, acting on the equality (3.1.31) with the operator Δ^m for $m \in \mathbb{N}_0$ and $m < k$, then setting $x = 0$ and using (3.1.28) we obtain

$$\Delta^m u(x)|_{x=0} = \sum_{i=0}^{k-1} \Delta^m \Big(|x|^{2i} u_i(x)\Big)|_{x=0} = (2,2)_m (n,2)_m u_m(0)$$
$$= \frac{1}{\omega_n} \frac{(2,2)_m (n,2)_m}{H_{k-1}^{(m)}(2m)} \int_{\partial S} H_{k-1}^{(m)}(\Lambda) u(x)\, ds_x.$$

The equality (3.1.30) is proved which means the theorem is true. □

Remark 3.4. It is easy to see that if the function $u \in C^k(\bar{S})$ is $(k+1)$-harmonic in the unit ball, then for the numbers a_i from Corollary 3.1, according to Theorem 3.4, the equality holds true

$$\Delta^k u(0) = \frac{(n,2)_k}{\omega_n} \int_{\partial S} \Big(a_1 \frac{\partial u}{\partial \nu} + \cdots + a_k \frac{\partial^k u}{\partial \nu^k} \Big)\, ds_x.$$

Remark 3.5. In Theorem 1.9 the values of derivatives of the form $G_{(\nu)}(D) u(x)|_{x=0}$ are calculated for the harmonic function $u(x)$, where $G_{(\nu)}(x)$ is a harmonic polynomial (1.3.6) of the complete system $\{G_{(\nu)}(x)\}$ on $L_2(\partial S)$.

Example 3.3. Let the function $u(x)$ be 3-harmonic in S and $u \in C^2(\bar{S})$. It is not hard to see that

$$H_2^{(2)}(\lambda) = \lambda(\lambda - 2) = -\lambda^{[1]} + \lambda^{[2]},$$

$H_2^{(2)}(4) = 8$, $(2,2)_2 = 8$, $(n,2)_2 = n(n+2)$ and that means the equality is true

$$\Delta^2 u(0) = \frac{n(n+2)}{\omega_n} \int_{\partial S} \Big(-\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2} \Big)\, ds_x.$$

If the function $u(x)$ is biharmonic in S, then by Corollary 3.1 we obtain

$$\int_{\partial S} \Big(-\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2} \Big)\, ds_x = 0 \Rightarrow \int_{\partial S} \frac{\partial u}{\partial \nu}\, ds_x = \int_{\partial S} \frac{\partial^2 u}{\partial \nu^2}\, ds_x.$$

From the last equality it immediately follows that if a function $u \in C^2(\bar{S})$ biharmonic in S satisfies the Neumann conditions on the boundary

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = f_1(x), \quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = f_2(x),$$

then the functions $f_1(x)$ and $f_2(x)$ must satisfy the equality

$$\int_{\partial S} f_1(x)\, ds_x = \int_{\partial S} f_2(x)\, ds_x. \tag{3.1.33}$$

3.2 Neumann boundary value problem for homogeneous equation

In this section, we study the conditions for the solvability of the Neumann problem for a homogeneous polyharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^k u(x) = 0,\ x \in S; \quad \frac{\partial^i u}{\partial \nu^i}\Big|_{\partial S} = f_i(s),\ s \in \partial S,\ i = \overline{1,k}, \tag{3.2.1}$$

where ν is the unit vector of the outer normal to ∂S.

In Section 1.1 a more general boundary value problem is considered, containing high-order polynomials of normal derivatives in boundary conditions. Theorem 1.2 on page 27 on the solvability of this boundary value problem and on the representation of the solution is proved. A special case of this theorem is the following statement.

Theorem 3.5. *A solution to the Neumann problem* (3.2.1) *for* $f_i \in C^{k-i+\varepsilon}(\partial S)$, $i = \overline{1,k}$ *exists if and only if for any vector* b_k *which is a solution to the equation* $N_k^T(0)b_k = 0$ *the condition is satisfied*

$$\int_{\partial S} (b_k^1 f_1 + \cdots + b_k^k f_k)\, ds = 0,$$

where

$$b_k = (b_k^1, \ldots, b_k^k), \quad N_k(\lambda) = \Big((\lambda + 2j - 2)^{[i]}\Big)_{i,j=\overline{1,k}} \cdot C,$$

$C = \Big((-1)^i \binom{j}{i}\Big)_{i,j=\overline{0,k-1}}$, *and* $\binom{j}{i} = 0$ *when* $j < i$.

The subject of this section is the study of the numbers b_k^s for $s = \overline{1,k}$ and $k \in \mathbb{N}$, which appeared in Theorem 3.5. It is clear that if we multiply the numbers $b_k^1, \ldots, b_k^k$ by the same number, then the conditions of the theorem is not change. We normalize these numbers so that $b_k^1 > 0$ and $|b_k^k| = 1$. Using Lemmas 3.9-3.15 it is possible to obtain a recurrence equation for the numbers b_k^s, and then solve the resulting recurrence equation. The necessary condition for the solvability of the Neumann problem is given explicitly in Theorem 3.6. Finally, in Theorem 3.7, based on matrices $\mathbb{H}_m$ of the form (2.4.16) associated with Theorem 3.4, necessary and sufficient conditions for the solvability of the Neumann problem are given and an explicit representation of its solution is found. Using Lemmas 3.9-3.15, the equivalence of the solvability conditions for the Neumann problem from Theorems 3.6 and 3.7 is established. Illustrative Example 3.4 is provided. The main results of Section 3.2 is published in papers [87, 88].

Let us return to the numbers b_k^s. For example, for $k = 1$ we have $b_1^1 = 1$ and for $k = 2$, according to the equality (3.1.33) from Example 3.3, we have $b_2^1 = -b_2^2 = 1$. Let us arrange the numbers b_k^s in the form of a number triangle similar to Pascal's triangle [35]

$$B = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & -1 & & & \\ & & b_3^1 & & b_3^2 & & b_3^3 & & \\ & b_4^1 & & b_4^2 & & b_4^3 & & b_4^4 & \\ b_5^1 & & b_5^2 & & b_5^3 & & b_5^4 & & b_5^5 \\ & & & & \ldots & & & & \end{array}. \tag{3.2.2}$$

At the moment it is not clear that the triangle B consists of integers, but this become clear later.

3.2.1 Integer triangle B

Let us find solutions to the homogeneous system $N_k^T(0)b_k = 0$ for $k > 1$. Let us write down the matrix $N_k^T(0)$. It looks like

$$N_k^T(0) = C^T \cdot \left((2i-2)^{[j]}\right)_{i,j=\overline{1,k}},$$

where the last matrix can be written in more detail in the form

$$\left((2i-2)^{[j]}\right)_{i,j=\overline{1,k}} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2^{[1]} & 2^{[2]} & \cdots & 2^{[k]} \\ \vdots & \vdots & \ddots & \vdots \\ (2k-2)^{[1]} & (2k-2)^{[2]} & \cdots & (2k-2)^{[k]} \end{pmatrix}.$$

Since the matrix C is non-singular, we arrive at a system of equations of the form

$$\begin{array}{c} 2^{[1]}b_k^1 + \cdots + 2^{[k-1]}b_k^{k-1} = -2^{[k]}b_k^k \\ \vdots \\ (2k-2)^{[1]}b_k^1 + \cdots + (2k-2)^{[k-1]}b_k^{k-1} = -(2k-2)^{[k]}b_k^k \end{array}. \tag{3.2.3}$$

We denote the determinant of the matrix on the left by Δ_k^0, and the determinant of the matrix obtained from this matrix by replacing the sth column at $1 \le s \le k-1$ with the column $\left(2^{[k]},\ldots,(2k-2)^{[k]}\right)^T$ we denote by Δ_k^s (see Lemma 3.1 on page 161), i.e.

$$\Delta_k^s = \\ = \begin{vmatrix} 2^{[1]} & \cdot & 2^{[s-1]} & 2^{[k]} & 2^{[s+1]} & \cdot & 2^{[k-1]} \\ 4^{[1]} & \cdot & 4^{[s-1]} & 4^{[k]} & 4^{[s+1]} & \cdot & 4^{[k-1]} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (2k-2)^{[1]} & \cdot & (2k-2)^{[s-1]} & (2k-2)^{[k]} & (2k-2)^{[s+1]} & \cdot & (2k-2)^{[k-1]} \end{vmatrix}.$$

It is easy to see that

$$\Delta_k^0 \\ = \begin{vmatrix} 2^{[1]} & \dots & 2^{[k-1]} \\ 4^{[1]} & \dots & 4^{[k-1]} \\ \vdots & \ddots & \vdots \\ (2k-2)^{[1]} & \cdots & (2k-2)^{[k-1]} \end{vmatrix} = 2\cdot 4\cdots(2k-2)\begin{vmatrix} 1 & \dots & 1^{[k-2]} \\ 1 & \dots & 3^{[k-2]} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (2k-3)^{[k-2]} \end{vmatrix} \\ = (2k-2)!!\begin{vmatrix} 1 & \dots & 1^{k-2} \\ 1 & \dots & 3^{k-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (2k-3)^{k-2} \end{vmatrix} = (2k-2)!!W[1,3,\ldots,(2k-3)] \\ = (2k-2)!!(2k-4)!!\cdots 2!!,$$

where $W[\lambda_1,\ldots,\lambda_{k-1}]$ denotes the Vandermonde determinant of order $k-1$. Therefore, the system (3.2.3) is uniquely solvable for any b_k^k. Let us choose $b_k^k = \pm 1$, but so that $b_k^1 > 0$.

Let us calculate the determinants Δ_k^s. We consider a more general case. Let $\lambda_1,\ldots,\lambda_{k-1}\in\mathbb{C}$ and then we denote

$$\Delta_k^s(\lambda) \equiv \Delta_k^s(\lambda_1,\ldots,\lambda_{k-1}) \\ = \begin{vmatrix} \lambda_1^{[1]} & \dots & \lambda_{k-1}^{[1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[s-1]} & \dots & \lambda_{k-1}^{[s-1]} \\ \lambda_1^{[k]} & \dots & \lambda_{k-1}^{[k]} \\ \lambda_1^{[s+1]} & \dots & \lambda_{k-1}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[k-1]} & \dots & \lambda_{k-1}^{[k-1]} \end{vmatrix} = (-1)^{k-s-1}\begin{vmatrix} \lambda_1^{[1]} & \dots & \lambda_{k-1}^{[1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[s-1]} & \dots & \lambda_{k-1}^{[s-1]} \\ \lambda_1^{[s+1]} & \dots & \lambda_{k-1}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[k-1]} & \dots & \lambda_{k-1}^{[k-1]} \\ \lambda_1^{[k]} & \dots & \lambda_{k-1}^{[k]} \end{vmatrix}. \quad (3.2.4)$$

and $\Delta_k^0(\lambda) = \det\left(\lambda_j^{[i]}\right)_{i,j=\overline{1,k-1}}$.

The following statement follows from Lemma 3.1.

Lemma 3.9. *For s integer and such that $1 \le s \le k$ the equality holds*

$$s^{[1]}\Delta_k^1(\lambda) + \cdots + s^{[s]}\Delta_k^s(\lambda) = -\Delta_k^0(\lambda) s \prod_{i=1}^{k-1}(s - \lambda_i). \tag{3.2.5}$$

Proof. Let us apply Lemma 3.1 to determinants of the form $\Delta_k^s(\lambda_1 - 1, \dots, \lambda_{k-1} - 1)$. We multiply the resulting equality of type (3.1.12) by the number $(s+1)\lambda_1 \dots \lambda_{k-1}$, where $0 \le s \le k-1$. Then we have

$$(s+1)^{[1]}\Delta_k^1(\lambda) + \cdots + (s+1)^{[s+1]}\Delta_k^s(\lambda) = -\Delta_k^0(\lambda)(s+1)\prod_{i=1}^{k-1}(s+1-\lambda_i).$$

If we now replace $s+1$ with s, then we get (3.2.5) for $1 \le s \le k$. The lemma is proved. □

Let us use Lemma 3.9 to find the numbers b_k^s. The following statement is similar to Lemma 3.2.

Lemma 3.10. *For any integers $k > 1$ and s such that $1 \le s \le k$ the equality holds*

$$s^{[1]}b_k^1 + \cdots + s^{[s]}b_k^s = (-1)^{k+1}2^k\left(\frac{s}{2}\right)^{[k]}. \tag{3.2.6}$$

Proof. By virtue of the notation made above, $\Delta_k^s(2, \dots, 2k-2) = \Delta_k^s$ and therefore, by Lemma 3.9 with $\lambda_i = 2i$ we have

$$s^{[1]}\Delta_k^1 + \cdots + s^{[s]}\Delta_k^s = -\Delta_k^0\prod_{i=0}^{k-1}(s-2i) = -2^k\left(\frac{s}{2}\right)^{[k]}\Delta_k^0. \tag{3.2.7}$$

Hence, for $s = 1$ and $k > 1$ we get

$$\Delta_k^1 = -\Delta_k^0 2^k\left(\frac{1}{2}\right)^{[k]} = (-1)^k\Delta_k^0(2k-3)!!.$$

Remembering the system (3.2.3) we find $b_k^s = -b_k^k\dfrac{\Delta_k^s}{\Delta_k^0}$. For $s = 1$ we have

$$b_k^1 = -b_k^k\frac{\Delta_k^1}{\Delta_k^0} = (-1)^{k+1}b_k^k(2k-3)!!$$

and that means $b_k^1 > 0$ for $b_k^k = (-1)^{k+1}$. Thus $b_k^s = (-1)^k \dfrac{\Delta_k^s}{\Delta_k^0}$. Therefore, dividing both sides of the equality (3.2.7) by $\Delta_k^0 \neq 0$ and multiplying by $(-1)^k$ we obtain (3.2.6). The lemma is proved. □

From the proof of Lemma 3.10 it follows that

$$b_k^1 = (2k-3)!! \text{ at } k > 1; \quad b_k^k = (-1)^{k+1} \text{ at } k \geq 1. \tag{3.2.8}$$

Thus, we have determined the left and right sides of the integer triangle (3.2.2), consisting of the numbers b_k^s.

3.2.2 Formulas to determine b_k^s

Using the idea of the proof of Lemma 3.3 we establish the following result.

Lemma 3.11. *For $k \in \mathbb{N}$ and an integer s such that $1 \leq s \leq k-1$ the numbers b_k^s are defined by the equality*

$$b_k^s = (-1)^{k+s} \frac{2^k}{s!} \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2}\right)^{[k]}. \tag{3.2.9}$$

Proof. Let us denote $\tilde{b}_k^s = s! b_k^s$ and assume that $\binom{a}{b} = 0$ for $b > a$. Then (3.2.6) can be rewritten as

$$\binom{s}{1}\tilde{b}_k^1 + \cdots + \binom{s}{s}\tilde{b}_k^s + \cdots + \binom{s}{k-1}\tilde{b}_k^{k-1} = (-1)^{k+1} 2^k \left(\frac{s}{2}\right)^{[k]}$$

and therefore the vector $\tilde{b}_k = (\tilde{b}_k^1, \ldots, \tilde{b}_k^{k-1})$ is a solution to the system

$$C\tilde{b}_k = (-1)^{k+1} 2^k \left(\left(\frac{j}{2}\right)^{[k]}\right)_{j=\overline{1,k-1}}$$

with matrix $C = \left(\binom{i}{j}\right)_{i,j=\overline{1,k-1}}$. It is easy to verify that the triangular matrix C is invertible and

$$C^{-1} = \left((-1)^{i+j}\binom{i}{j}\right)_{i,j=\overline{1,k-1}}.$$

Indeed, the common element of the product of these matrices for $i \geq j$ has the form

$$
\begin{aligned}
\left(C \cdot C^{-1}\right)_{i,j} &= \sum_{s=1}^{k-1} (-1)^{s+j} \binom{i}{s}\binom{s}{j} \\
&= (-1)^j \sum_{s=j}^{i} (-1)^s \binom{i}{s}\binom{s}{j} = (-1)^j \frac{i!}{j!} \sum_{k=0}^{i-j} \frac{(-1)^{k+j}}{(i-j-k)!k!} \\
&= \frac{i!}{(i-j)!j!} \sum_{k=0}^{i-j} (-1)^k \frac{(i-j)!}{(i-j-k)!k!} = \binom{i}{j}(1-1)^{i-j} = \delta_{i,j}.
\end{aligned}
$$

If $i < j$, then since $\binom{i}{s}\binom{s}{j} = 0$ we have $\left(C \cdot C^{-1}\right)_{i,j} = 0$. That is why

$$
\tilde{b}_k = (-1)^{k+1} 2^k \left(\sum_{j=1}^{k-1} (-1)^{i+j} \binom{i}{j} \binom{\frac{j}{2}}{}^{[k]} \right)_{i=\overline{1,k-1}}
$$

and therefore, taking into account that $\tilde{b}_k^s = s! b_k^s$ we get

$$
b_k^s = (-1)^{k+s} \frac{2^k}{s!} \sum_{j=1}^{k-1} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2}\right)^{[k]}.
$$

This immediately follows (3.2.9). The lemma is proved. □

Let us check the resulting formula (3.2.9) for $s = 1$. We have

$$
b_k^1 = (-1)^{k+1} 2^k \left(\frac{1}{2}\right)^{[k]} = (2k-3)!!,
$$

which coincides with what is previously found. Let us calculate the unknown numbers b_k^2. For $k > 1$ from (3.2.9) we find

$$
b_k^2 = (-1)^{k+2} 2^k \left(\frac{1}{2}\right)^{[k]} = -b_k^1 = -(2k-3)!!. \tag{3.2.10}
$$

So, we have found the value of the first side column parallel to the left side of the triangle (3.2.2). For $k = 2$ we have the known value $b_2^2 = -1$. Let us try to transform the formula (3.2.9) to a more compact form similar to Lemma 3.4.

Lemma 3.12. *For $k \in \mathbb{N}$ and an integer s such that $1 \le s \le k$ the equality holds*

$$
b_k^s = (-1)^{k+1} \frac{2^k}{s!} \left((\sqrt{t} - 1)^s \right)^{(k)} \Big|_{t=1}. \tag{3.2.11}
$$

Proof. It is easy to see that

$$\left(\frac{j}{2}\right)^{[k]}=\frac{j}{2}\left(\frac{j}{2}-1\right)\dots\left(\frac{j}{2}-k+1\right)=\left(t^{\frac{j}{2}}\right)^{(k)}\Big|_{t=1}.$$

Therefore, for integer $k>1$ and s such that $1\le s\le k-1$, in accordance with (3.2.9), the equalities hold

$$\begin{aligned}b_k^s=(-1)^{k+s}\frac{2^k}{s!}\sum_{j=1}^{s}(-1)^{j+1}\binom{s}{j}\left(\frac{j}{2}\right)^{[k]}&\\ =(-1)^{k+1}\frac{2^k}{s!}\sum_{j=1}^{s}(-1)^{s-j}\binom{s}{j}\left((\sqrt{t})^j\right)^{(k)}\Big|_{t=1}&\\ =(-1)^{k+1}\frac{2^k}{s!}\left(\sum_{j=1}^{s}(-1)^{s-j}\binom{s}{j}(\sqrt{t})^j\right)^{(k)}\Bigg|_{t=1}=(-1)^{k+1}\frac{2^k}{s!}\left((\sqrt{t}-1)^s\right)^{(k)}\Big|_{t=1}&,\end{aligned}$$

i.e. (3.2.11) is true for the specified k and s. The formula (3.2.11) is also valid for $s=k\ge 1$. Indeed, substituting $s=k$ into (3.2.11) we get

$$\begin{aligned}b_k^k=(-1)^{k+1}\frac{2^k}{k!}\left((\sqrt{t}-1)^k\right)^{(k)}\Big|_{t=1}&\\ =(-1)^{k+1}\frac{2^k}{k!}\left(\frac{k}{2\sqrt{t}}(\sqrt{t}-1)^{k-1}\right)^{(k-1)}\Bigg|_{t=1}&\\ =\frac{(-1)^{k+1}}{\sqrt{t}}\frac{2^{k-1}}{(k-1)!}\left((\sqrt{t}-1)^{k-1}\right)^{(k-1)}\Big|_{t=1}=-b_{k-1}^{k-1}=(-1)^{k-1}b_1^1&.\end{aligned}$$

If we substitute $s=k=1$ into (3.2.11), we get $b_1^1=1$ and therefore $b_k^k=(-1)^{k-1}=(-1)^{k+1}$. It was previously calculated that $b_k^k=(-1)^{k+1}$ and therefore the formula (3.2.11) is correct for $s=k$. The lemma is proved. □

3.2.3 Recurrence equation for b_k^s

For $k,s\in\mathbb{N}$ we introduce the notation

$$a_k^s=\left((\sqrt{t}-1)^s\right)^{(k)}\Big|_{t=1}. \tag{3.2.12}$$

Let us study the triangle A, similar to the triangle B (3.2.2)

$$A = \begin{array}{ccccccccc} & & & 1 & & 0 & & & \\ & & 0 & & a_1^1 & & 0 & & \\ & 0 & & a_2^1 & & a_2^1 & & 0 & \\ 0 & & a_3^1 & & a_3^2 & & a_3^3 & & 0 \\ & & & & \cdots & & & & \end{array},$$

but consisting of the numbers a_k^s for $k, s \in \mathbb{N}$ and $s \le k$. Here k is the row number, and s is the element number in the row. Let us supplement this triangle with values at $s = 0$ and $k = 0$. According to (3.2.12) we assume that $a_k^0 = a_0^s = 0$. In addition, it is easy to see that $a_1^1 = \frac{1}{2}$ and $a_0^0 = 1$.

Lemma 3.13. *For the numbers a_k^s the recurrent equality is true*

$$a_{k+1}^s = \left(\frac{s}{2} - k\right) a_k^s + \frac{s}{2} a_k^{s-1}, \tag{3.2.13}$$

where $a_k^s = 0$ for $s > k$ and $a_k^0 = 0$, $k \in \mathbb{N}$, but $a_0^0 = 1$.

Proof. It is easy to see that if the degree s is higher than the order of the derivative k, then from (3.2.12) it follows that $a_k^s = 0$. The values of a_1^1 and a_0^0 are calculated using the equality (3.2.12) before the lemma. Let us use the equality (3.1.19) from Lemma 3.5 with $f_i(t) = \sqrt{t} - 1$, $i = 1, \dots, s$. Since

$$(\sqrt{t} - 1)^{(m)} = \frac{1}{2}\left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - m + 1\right) t^{\frac{1}{2} - m} = \frac{(-1)^{m-1}}{2^m} (2m-3)!!\, t^{\frac{1}{2}-m},$$

then we can write

$$\begin{aligned} &\left((\sqrt{t} - 1)^s\right)^{(k)} \\ &\quad = \sum_{i_1+\cdots+i_s=k,\, i_j>0} \binom{k}{i_1 \dots i_s} (-1)^{k-s} \frac{(2i_1-3)!! \cdots (2i_s-3)!!}{2^k} t^{\frac{s}{2}-k} \\ &\qquad + s \sum_{i_1+\cdots+i_{s-1}=k,\, i_j>0} \binom{k}{i_1 \dots i_{s-1}} (-1)^{k-s+1} \frac{(2i_1-3)!! \cdots (2i_{s-1}-3)!!}{2^k} \\ &\qquad\quad \times t^{\frac{s-1}{2}-k} (\sqrt{t} - 1) + s(s-1) \sum_{i_1+\cdots+i_{s-2}=k,\, i_j>0} \binom{k}{i_1 \dots i_{s-2}} \\ &\quad \times (-1)^{k-s+2} \frac{(2i_1-3)!! \cdots (2i_{s-2}-3)!!}{2^k} t^{\frac{s-2}{2}-k} (\sqrt{t} - 1)^2 + \cdots = a_k^s t^{\frac{s}{2}-k} \end{aligned}$$

$$+ sa_k^{s-1} t^{\frac{s-1}{2}-k}(\sqrt{t}-1) + s(s-1)a_k^{s-2} t^{\frac{s-2}{2}-k}(\sqrt{t}-1)^2 + \cdots . \quad (3.2.14)$$

Therefore

$$\left((\sqrt{t}-1)^s\right)^{(k+1)} = a_{k+1}^s t^{\frac{s}{2}-k-1} + sa_{k+1}^{s-1} t^{\frac{s-1}{2}-k-1}(\sqrt{t}-1) + \cdots .$$

On the other hand, differentiating (3.2.14) we get

$$\left((\sqrt{t}-1)^s\right)^{(k+1)} = \left(\frac{s}{2}-k\right) a_k^s t^{\frac{s}{2}-k-1} + \frac{s}{2} a_k^{s-1} t^{\frac{s-2}{2}-k} + (\sqrt{t}-1)F(t).$$

Setting $t = 1$ in the previous equalities, we have

$$a_{k+1}^s = \left(\frac{s}{2}-k\right) a_k^s + \frac{s}{2} a_k^{s-1}.$$

Which is what needed to be proved. The lemma is proved. □

Remark 3.6. Note that the equation

$$a_k^{s+1} = \frac{1}{2} a_{k-1}^s + (s-2k+1)a_k^s,$$

similar to (3.2.13), is used in the construction of special polynomials (1.3.24).

The values of a_k^s using the equality (3.2.13) are calculated for several columns of the triangle A, parallel to the lateral sides, and based on the resulting equalities, an assumption is made about the values of a_k^s. Skipping these calculations, we prove the resulting formula.

Lemma 3.14. *Solution of the recurrent equation* (3.2.13) *under the boundary conditions* $a_k^0 = a_k^{k+1} = 0$*,* $k \in \mathbb{N}$ *and* $a_0^0 = 1$ *is unique and has the form*

$$a_k^s = (-1)^{k+s} \frac{(2k-s-1)!s}{2^k(2k-2s)!!}, \quad (3.2.15)$$

where $s, k \in \mathbb{N}$ *and* $s \le k$*.*

Proof. Let us prove the existence and uniqueness of a solution to the equation (3.2.13) under the specified boundary conditions.

1^0. Assuming $k = 0$ and $s = 1$ in the formula (3.2.13), due to the boundary conditions, we obtain $a_1^1 = \frac{1}{2} a_0^1 + \frac{1}{2} a_0^0 = \frac{1}{2}$. The equality (3.2.15) gives the same value.

2^0. Let us calculate the value of a_k^1 using the equality (3.2.13). Considering that $a_k^0 = 0$ we find

$$a_k^1 = \left(\frac{1}{2} - k + 1\right) a_{k-1}^1 + \frac{1}{2} a_{k-1}^0 = -\frac{2k-3}{2} a_{k-1}^1$$
$$= (-1)^{k-1} \frac{(2k-3)!!}{2^{k-1}} a_1^1 = (-1)^{k-1} \frac{(2k-3)!!}{2^k} = (-1)^{k+1} \frac{(2k-1)!!}{2^k(2k-1)}.$$

This equality is derived for $k > 1$, but is also valid for $k = 1$. Using the equality (3.2.15) we get the same value

$$a_k^1 = (-1)^{k+1} \frac{(2k-2)!}{2^k(2k-2)!!} = (-1)^{k+1} \frac{(2k-3)!!}{2^k}.$$

3^0. Let us calculate the value of a_k^k using the equality (3.2.13). Considering that $a_k^{k+1} = 0$ we find

$$a_k^k = \left(\frac{k}{2} - k + 1\right) a_{k-1}^k + \frac{k}{2} a_{k-1}^{k-1} = \frac{k}{2} a_{k-1}^{k-1} = \frac{k!}{2^{k-1}} a_1^1 = \frac{k!}{2^k}.$$

This equality is obtained for $k > 1$, but given that $a_1^1 = \frac{1}{2}$ it also is true for $k \geq 1$. Using the equality (3.2.15) we get the same value

$$a_k^k = \frac{(2k-k-1)!k}{2^k(2k-2k)!!} = \frac{k!}{2^k}.$$

So, the values of a_k^s on the sides of the triangle A are calculated and the equality (3.2.15) specifies the solution of the equation (3.2.13) on the rows of the triangle A. The structure of the equation (3.2.13) is such that, given the known values of a_k^s on the lateral sides of the triangle A, all values of a_k^s inside the triangle A are determined from the equation (3.2.13) definitely. Therefore, if the numbers a_k^s given by the equality (3.2.15) satisfy the equation (3.2.13) inside the triangle A, then the equality (3.2.15) specifies a solution to the equation (3.2.13). Let us prove it. Let a_k^s be found from (3.2.15). If substitute them into the right side of the equality (3.2.13), then we get

$$a_{k+1}^s = \left(\frac{s}{2} - k\right) a_k^s + \frac{s}{2} a_k^{s-1} = (-1)^{k+s} \left(\frac{s}{2} - k\right) \frac{(2k-s-1)!s}{2^k(2k-2s)!!}$$
$$+ (-1)^{k+s-1} \frac{s}{2} \frac{(2k-s)!(s-1)}{2^k(2k-2s+2)!!} = (-1)^{k+s-1} \frac{(2k-s)!s}{2^{k+1}(2k-2s+2)!!}$$

$$\times\left((2k-2s+2)+(s-1)\right)=(-1)^{k+s+1}\frac{(2k-s+1)!s}{2^{k+1}(2k-2s+2)!!},$$

that is a_{k+1}^s is also found from (3.2.15). The lemma is proved. □

3.2.4 Neumann problem for the homogeneous equation

Let us go back to the triangle B.

Lemma 3.15. *The numbers b_k^s, $s,k\in\mathbb{N}$ are defined by the equality*

$$b_k^s=(-1)^{s+1}\binom{2k-s-1}{s-1}\frac{(2k-2s+1)!!}{(2k-2s+1)} \tag{3.2.16}$$

and are the only solution to the recurrence equation

$$b_{k+1}^s=(2k-s)b_k^s-b_k^{s-1} \tag{3.2.17}$$

under boundary conditions $b_k^1=\dfrac{(2k-1)!!}{(2k-1)}$ and $b_k^k=(-1)^{k+1}$, $k\in\mathbb{N}$ in accordance with (3.2.8).

Proof. Using the equalities (3.2.11), (3.2.12) and (3.2.15) we write

$$b_k^s=(-1)^{k+1}\frac{2^k}{s!}a_k^s=(-1)^{s+1}\frac{(2k-s-1)!}{(2k-2s)!!(s-1)!}$$
$$=(-1)^{s+1}\binom{2k-s-1}{s-1}\frac{(2k-2s+1)!!}{(2k-2s+1)},$$

that is the equality (3.2.16) defines the triangle B. On the lateral sides of triangle B the equality (3.2.16) gives

$$b_k^1=\binom{2k-2}{0}\frac{(2k-1)!!}{(2k-1)}=(2k-3)!!,$$
$$b_k^k=(-1)^{k+1}\binom{2k-k-1}{k-1}\frac{(2k-2k+1)!!}{(2k-2k+1)}=(-1)^{k+1},$$

which coincides with the boundary conditions. If we substitute the value $a_k^s=(-1)^{k+1}\dfrac{s!}{2^k}b_k^s$ into the equation (3.2.13), then we have

$$(-1)^{k+2}\frac{s!}{2^{k+1}}b_{k+1}^s=\left(\frac{s}{2}-k\right)(-1)^{k+1}\frac{s!}{2^k}b_k^s+\frac{s}{2}(-1)^{k+1}\frac{(s-1)!}{2^k}b_k^{s-1}.$$

After reduction by $(-1)^{k+2}\dfrac{s!}{2^{k+1}}$ we obtain the equation (3.2.17)

$$b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1}.$$

The lemma is proved. □

An equation of the form (3.2.17) defines a number triangle similar to the Pascal, Euler or Stirling triangles (1.1.2). Such triangles are studied modulo prime numbers, for example in [63]. Let us calculate the value of b_k^s for the first slanted columns parallel to the lateral sides of the triangle B. We have

$$b_k^2 = -\binom{2k-3}{1}\frac{(2k-3)!!}{(2k-3)} = -(2k-3)!! = -b_k^1, \quad k \ge 2,$$

which corresponds to (3.2.10),

$$b_k^{k-1} = (-1)^k\binom{k}{k-2}\frac{3!!}{3} = (-1)^k\binom{k}{2}, \quad k \ge 2$$

and

$$b_k^{k-2} = (-1)^{k-1}\binom{k+1}{k-3}\frac{5!!}{5} = (-1)^{k-1}3\binom{k+1}{4}, \quad k \ge 3.$$

Calculating b_k^s using the equality (3.2.16) we expand the triangle B from (3.2.2) to six rows

$$B = \begin{array}{c} 1 \\ 1 \quad -1 \\ 3 \quad -3 \quad 1 \\ 15 \quad -15 \quad 6 \quad -1 \\ 105 \quad -105 \quad 45 \quad -10 \quad 1 \\ 945 \quad -945 \quad 420 \quad -105 \quad 15 \quad -1 \\ \cdots \quad b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1} \quad \cdots \end{array}. \tag{3.2.18}$$

Taking into account the research done, Theorem 3.5 can be rewritten in the form.

Theorem 3.6. *A solution to the Neumann problem* (3.2.1) *for* $f_i \in C^{k-i}(\partial S)$, $i = 1\ldots,k$ *exists if and only if the vector* $b_k = (b_k^1,\ldots,b_k^k)$ *equal to the kth row of the triangle B from* (3.2.18) *is orthogonal to the vector*

$$F_k = \left(\int_{\partial S} f_1(s)\,ds,\ldots,\int_{\partial S} f_k(s)\,ds\right),$$

that is

$$(b_k, F_k) = \int_{\partial S} \sum_{j=1}^{k} (-1)^{j+1} \binom{2k-j-1}{j-1} (2k-2j-1)!! f_j(s)\, ds = 0.$$

For example, for the 4-harmonic equation, in accordance with the 4th row of the triangle B, the necessary and sufficient conditions for the solvability of the Neumann problem in S have the form

$$\int_{\partial S} \big(15 f_1(s) - 15 f_2(s) + 6 f_3(s) - f_4(s)\big)\, ds = 0.$$

The solution to the Dirichlet problem for the 4-harmonic equation in a ball is given in Example 2.6 on page 147.

To construct a solution to the Neumann problem, a higher smoothness of the boundary functions is necessary since we are going to apply Theorem 2.13, in which the smoothness $u \in C^{m-1}(\bar{S})$ is required from the solution to the Dirichlet problem. Let us present a solution to the Neumann problem for the homogeneous polyharmonic equation (see also Section 3.3.3).

Theorem 3.7. *Let the harmonic functions $q_i(x)$ be solutions of the Dirichlet boundary value problems*

$$\Delta q_i(x) = 0,\ x \in S; \quad q_i|_{\partial S} = \psi_i(s),\ s \in \partial S, \tag{3.2.19}$$

where $\psi_k \in C^{2m-1-i+\varepsilon}(\partial S)$, $i = 0, \dots, m-1$, $\varepsilon > 0$. Then for the existence of a solution $u \in C^m(\bar{S})$ to the Neumann boundary value problem

$$\Delta^m u(x) = 0, \quad x \in S, \tag{3.2.20}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \psi_0(s), \quad \dots, \quad \frac{\partial^m u}{\partial \nu^m}\Big|_{\partial S} = \psi_{m-1}(s), \quad s \in \partial S, \tag{3.2.21}$$

it is necessary and sufficient to fulfill the condition

$$\sum_{j=0}^{m-1} h_m^{(j+1)} q_j(0) = 0, \tag{3.2.22}$$

where $h_m^{(j)}$ are coefficients from the representation $h_m(\lambda) \equiv \lambda(\lambda-2)\dots(\lambda-2m+2) = \sum_{j=0}^{m} h_m^{(j)} \lambda^{[j]}$. Solution of the Neumann problem can be written as

$$u(x) = \int_0^1 v(tx)\, \frac{dt}{t} + C, \tag{3.2.23}$$

where C is an arbitrary constant,

$$v(x) = \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(1-\Lambda; |x|^2-1) q_j(x), \tag{3.2.24}$$

and $K_{m-1}^{(j)}(\lambda; |x|^2-1)$ *is determined in* (2.4.30) *on page 142.*

Proof. Let us make in the Neumann boundary value problem (3.2.20)-(3.2.21) the change of variable $v = \Lambda u$. Then since $\Delta\Lambda u(x) = (\Lambda+2)\Delta u(x)$, whence follows $\Delta^m \Lambda u(x) = (\Lambda+2m)\Delta^m u(x) = 0$ we get the following boundary value problem for $v(x)$

$$\Delta^m v(x) = 0,\ x \in S;\ v|_{\partial S} = \psi_0(s); \ldots, (\Lambda-1)^{[m-1]} v\big|_{\partial S} = \psi_{m-1}(s),\ s \in \partial S.$$

By Theorem 2.13 on page 142, in which we formally replace the operator Λ by the operator $\Lambda - 1$, the solution to this problem can be written in the form

$$v(x) = \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(1-\Lambda; |x|^2-1) q_j(x),$$

where the harmonic functions $q_j(x)$ are solutions to the problems (3.2.19). Denote

$$v_1(x) = \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(1-\Lambda; |x|^2-1)(q_j(x) - q_j(0)),$$
$$v_0(x) = \sum_{j=0}^{m-1} \frac{q_j(0)}{j!} K_{m-1}^{(j)}(1; |x|^2-1).$$

It is easy to see that since $\Lambda(q_j(0)) = 0$ and $K_{m-1}^{(j)}(\lambda; |x|^2-1)$ is a polynomial in λ, then the following equalities hold true

$$v(x) = \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(1-\Lambda; |x|^2-1)(q_j(x) - q_j(0))$$
$$+ \sum_{j=0}^{m-1} \frac{1}{j!} K_{m-1}^{(j)}(1-\Lambda; |x|^2-1) q_j(0) = v_1(x) + v_0(x),$$

where $v_0(x)$ is the m-harmonic polynomial. Now it is necessary to solve the equation $\Lambda u = v$ in m-harmonic functions in S. The functions $v_i(x)$ are m-harmonic by construction. Note that for any m-harmonic in S function $p(x)$, the equality $(\Lambda p)(0) = 0$ is true and therefore $v_1(0) = 0$. Find the value $v_0(0)$. It is not hard to see that

$$v_0(0) = \sum_{j=0}^{m-1} \frac{q_j(0)}{j!} K_{m-1}^{(j)}(1;-1) = \sum_{j=0}^{m-1} \frac{q_j(0)}{j!} \Big(\sum_{k=0}^{m-1} (-1)^k H_k(\lambda) \Big)^{(j)}(1). \qquad (3.2.25)$$

Lemma 3.16. *Let in accordance with* (2.4.24)

$$H_k(\lambda) = \frac{1}{(2k)!!} \lambda(\lambda-2)\cdots(\lambda-2k+2),\ k \in \mathbb{N}; \quad H_0(\lambda) = 1.$$

Then, for $j \in \mathbb{N}_0$, the following equality holds true

$$\Big(\sum_{k=0}^{m-1} (-1)^k H_k(\lambda) \Big)^{(j)}(1) = (-1)^{m-1} \frac{2m}{j+1} H_m^{(j+1)}(0). \qquad (3.2.26)$$

Proof. Let us prove the equality (3.2.26) by the method of mathematical induction on m. For $m = 0$ we have zeros on the left-hand and on the right-hand sides of (3.2.26), and for $m = 1$ we also have the equality $H_0^{(j)}(1) = \frac{2}{j+1} H_1^{(j+1)}(1)$, where $j \in \mathbb{N}_0$. Under the assumption that the equality (3.2.26) is true for $m \geq 1$, we prove its validity for $m = m+1$. According to the definition of the derivative (2.4.10), we have $(f(\lambda)g(\lambda))^{(1)} = f^{(1)}(\lambda)g(\lambda+1) + f(\lambda)g^{(1)}(\lambda)$ and therefore taking into account that $H_{m+1}(\lambda) = H_m(\lambda)(\lambda-2m)/(2m+2)$ we write

$$H_{m+1}^{(1)}(\lambda) = \frac{1}{2m+2} \big(H_m^{(1)}(\lambda)(\lambda-2m+1) + H_m(\lambda) \big),$$

whence

$$H_{m+1}^{(2)}(\lambda) = \frac{1}{2m+2} \big(H_m^{(2)}(\lambda)(\lambda-2m+2) + 2H_m^{(1)}(\lambda) \big)$$

and therefore

$$H_{m+1}^{(j+1)}(\lambda) = \frac{1}{2m+2} \big(H_m^{(j+1)}(\lambda)(\lambda-2m+j+1) + (j+1)H_m^{(j)}(\lambda) \big).$$

Thus, the right-hand side of (3.2.26) for $m = m+1$ is transformed to the form

$$(-1)^m \frac{2m+2}{j+1} H_{m+1}^{(j+1)}(0) = \frac{(-1)^m}{j+1} \big(H_m^{(j+1)}(0)(-2m+j+1) + (j+1)H_m^{(j)}(0) \big)$$

$$= (-1)^{m-1}\frac{2m}{j+1}H_m^{(j+1)}(0) + (-1)^m\big(H_m^{(j+1)}(0) + H_m^{(j)}(0)\big)$$
$$= \sum_{k=0}^{m-1}(-1)^k H_k^{(j)}(1) + (-1)^m H_m^{(j)}(1) = \sum_{k=0}^{m}(-1)^k H_k^{(j)}(1),$$

which proves the induction step. Here, in the second line, the induction hypothesis is used. The lemma is proved. □

If in (3.2.25) we use the just-proved lemma, the following equality $H_m(\lambda) = h_m(\lambda)/(2m)!!$ and the condition (3.2.22), then we get

$$v_0(0) = (-1)^{m-1}2m\sum_{j=0}^{m-1} q_j(0)\frac{H_m^{(j+1)}(0)}{(j+1)!}$$
$$= \frac{(-1)^{m-1}}{(2m-2)!!}\sum_{j=0}^{m-1} q_j(0)\frac{h_m^{(j+1)}(0)}{(j+1)!} = \frac{(-1)^{m-1}}{(2m-2)!!}\sum_{j=0}^{m-1} q_j(0)h_m^{(j+1)} = 0.$$

The converse is also true, i.e. $v(0) = 0 \Rightarrow v_0(0) = 0 \Rightarrow$ (3.2.22). Lemma 3.23 proves that the equation $\Lambda u = v$ is solvable in m-harmonic functions if and only if $v(0) = 0$, but this condition is satisfied. The solution to the equation $\Lambda u = v$ can be written as (3.2.23) $u(x) = \int_0^1 v(tx)\,\frac{dt}{t} + C$. With smoothness imposed on the boundary functions $\psi_k \in C^{2m-1-k+\varepsilon}(\partial S)$, $k = 0, \dots, m-1$, similarly to Theorem 2.13, we have $q_k \in C^{2m-1-k}(\partial S)$ and hence, since in the equality (3.2.24) the differential operator applied to the function q_k has the order $m-1-k$, then $v = \Lambda u \in C^m(\bar S)$ and by (3.2.23) $u \in C^m(\bar S)$. Thus, the m-harmonic function (3.2.23) is a solution to the Neumann boundary value problem (3.2.20)-(3.2.21). If a solution to the Neumann boundary value problem $u(x)$ exists, then the function $v(x)$ from (3.2.24) must satisfy the equality $\Lambda u = v$, which is possible if and only if $v(0) = 0$, whence follows equality (3.2.22). The solution $u(x)$ is unique up to a constant. The theorem is proved. □

Remark 3.7. The solvability condition (3.2.22) for the Neumann boundary value problem obtained in Theorem 3.7 does not impose an explicit condition on the boundary functions ψ_k, but only connects the values of harmonic functions $q_k(0)$, $k = 0, \dots, m-1$. This condition does not formally coincide with the solvability condition (2.4.20) from Remark 2.11.

Lemma 3.17. *The solvability condition* (3.2.22) *and the solvability condition of Theorem 3.6 are equivalent.*

Proof. First, we rewrite the condition (2.4.20) for the m-harmonic equation and the Neumann boundary conditions taken in the form $\frac{\partial^j u}{\partial \nu^j}\big|_{\partial S} = \psi_j$, $j = 0, \ldots, m-1$ as

$$0 = \int_{\partial S} \sum_{j=1}^{m} h_{m+1,m}^{(j)} \psi_{j-1}\, ds_\xi = \int_{\partial S} \sum_{j=0}^{m-1} h_{m+1,m}^{(j+1)} \psi_j\, ds_\xi,$$

where $h_{m+1,m}(\lambda)$ are elements of the last row of the matrix $\mathbb{H}_{m+1}$. If we notice that $h_{m+1,m}(\lambda) = h_m(\lambda)/(2m)!!$ and take into account the mean value property for the harmonic functions $q_j(x)$, i.e. $\frac{1}{\omega_n}\int_{\partial S} \psi_j\, ds_\xi = q_j(0)$, then the condition (2.4.20) for the m-harmonic equation takes the form

$$0 = \frac{1}{(2m)!!}\int_{\partial S} \sum_{j=0}^{m-1} h_m^{(j+1)} \psi_j\, ds_\xi = \frac{\omega_n}{(2m)!!} \sum_{j=0}^{m-1} h_m^{(j+1)} q_j(0).$$

This condition is equivalent to the condition (3.2.22). Further, in accordance with Remarks 3.13 and 3.14 the condition (2.4.20) is equivalent to the solvability condition from Theorem 3.6. The lemma is proved. □

Example 3.4. Let us find a solution to the Neumann boundary value problem (3.2.20)-(3.2.21) for the 3-harmonic equation in the unit ball ($m = 3$). In accordance with the method of constructing a solution to the Neumann boundary value problem from Theorem 3.7, we use the solution to the Dirichlet boundary value problem from Example 2.5 and replace in this solution the operator Λ by the operator $\Lambda - 1$. After simple transformations, we get

$$\begin{aligned} v(x) = \sum_{j=0}^{2} \frac{1}{j!} K_\lambda^{(j)}(1-\Lambda; |x|^2 - 1) q_j(x) = \Big(1 + \frac{1}{2}(|x|^2 - 1) - \frac{1}{4!!}(|x|^2-1)^2 \\ - \frac{1}{2}(|x|^2-1)\Lambda + \frac{1}{4!!}(|x|^2-1)^2\Lambda^2\Big) q_0(x) + \Big(\frac{1}{2}(|x|^2-1) + \frac{1}{4!!}(|x|^2-1)^2 \\ - \frac{2}{4!!}(|x|^2-1)^2\Lambda\Big) q_1(x) + \frac{1}{4!!}(|x|^2-1)^2 q_2(x). \end{aligned}$$

Find $v(0)$. It is clear that $v(0) = \frac{1}{4!!}(3q_0(0) - 3q_1(0) + q_2(0))$. Since $h_3(\lambda) = \lambda(\lambda-2)(\lambda-4) = 3\lambda^{[1]} - 3\lambda^{[2]} + \lambda^{[3]}$, then the condition (3.2.22) has the form

$$\sum_{j=0}^{2} h_3^{(j+1)} q_j(0) = 3q_0(0) - 3q_1(0) + q_2(0) = 0,$$

which corresponds to the condition $v(0) = 0$. Under this condition, the solution (3.2.23) to the Neumann boundary value problem has the form

$$u(x) = \int_0^1 v(tx)\,\frac{dt}{t} = \int_0^1 \Big(\Big(1+\frac{1}{2}(|x|^2t^2-1) - \frac{1}{4!!}(|x|^2t^2-1)^2 \\ - \frac{1}{2}(|x|^2t^2-1)\Lambda + \frac{1}{4!!}(|x|^2t^2-1)^2\Lambda^2\Big)q_0(tx) + \Big(\frac{1}{2}(|x|^2t^2-1) + \frac{1}{4!!}(|x|^2t^2-1)^2 \\ - \frac{2}{4!!}(|x|^2t^2-1)^2\Lambda\Big)q_1(tx) + \frac{1}{4!!}(|x|^2t^2-1)^2 q_2(tx)\Big)\,\frac{dt}{t} + C.$$

In the particular case when the functions ψ_i, $i = 0,1,2$ are constant, we have $q_i(x) = \psi_i$ and therefore

$$u(x) = \frac{|x|^2}{8}\big(3\psi_0 + \psi_1 - \psi_2\big) + \frac{|x|^4}{32}\big(-\psi_0 + \psi_1 + \psi_2\big) + C,$$

provided that $3\psi_0 - 3\psi_1 + \psi_2 = 0$.

3.3 Neumann boundary value problem for inhomogeneous equation

Let us consider the Neumann boundary value problem for an inhomogeneous polyharmonic equations in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^k u(x) = f(x), \quad x \in S; \tag{3.3.1}$$

$$\frac{\partial^j u}{\partial \nu^j}\Big|_{\partial S} = \varphi_j(s), \quad s \in \partial S, \quad j = \overline{1,k}, \tag{3.3.2}$$

where ν is the unit vector of the outer normal to ∂S. This section is complementary to the results of the previous section on the solvability of the homogeneous problem (3.3.1)-(3.3.2), i.e. at $f = 0$. First, in Theorem 3.10, based on Theorem 3.8 and Lemmas 3.18-3.20, the case of polynomial right-hand side $f(x)$ is studied. After this, in Theorem 3.11, this result is extended to the case of $f \in C^1(\bar{S})$. Theorem 3.12 provides a method for finding a solution to the Neumann problem (3.3.1)-(3.3.2) for an inhomogeneous equation. In the case of polynomial data, a method is given for constructing a solution to the Neumann problem. The main results are illustrated by Examples 3.5-3.7. The main result of Section 3.3 is published in [84].

Let $f \in C^1(\bar{S})$ and $f(x) \neq 0$. Consider volume potential $V[f](x)$ with density $f(x)$

$$V[f](x) = -\frac{1}{\omega_n}\int_S E(x,\xi) f(\xi)\, d\xi,$$

where $E(x,\xi) = (n-2)^{-1}|\xi - x|^{2-n}$ $(n > 2)$ is an elementary solution to the Laplace equation, and ω_n is the area of the unit sphere in $\mathbb{R}^n$. Let us denote $V_k[f] = V[\ldots V[f]]$. It is known that $f \in C^1(\bar{S}) \Rightarrow V[f] \in C^1(\bar{S})$ and $\Delta V[f] = f$ in S [28], and therefore $V_k[f] \in C^1(\bar{S})$ and $\Delta^m V_{k-m}[f] = f$ for $m = 1, \ldots, k$. Let us represent the solution $u(x)$ of the Neumann problem (3.3.1)-(3.3.2) as the sum $u(x) = V_k[f] + w(x)$. Then

$$\Delta^k u(x) - f = \Delta^k V_k[f] + \Delta^k w(x) - f = \Delta^k w(x)$$

and therefore with respect to $w(x)$ we obtain the following problem

$$\Delta^k w(x) = 0, \quad x \in S; \tag{3.3.3}$$

$$\frac{\partial^j w}{\partial \nu^j}_{|\partial S} = \tilde{\varphi}_j(s),\ s \in \partial S, \quad j = \overline{1,k}, \tag{3.3.4}$$

where indicated

$$\tilde{\varphi}_j(s) = \varphi_j(s) - \frac{\partial^j V_k[f]}{\partial \nu^j}.$$

By virtue of Theorem 3.6 the solvability condition for the problem (3.3.3)-(3.3.4) has the form

$$\begin{aligned} 0 = \int_{\partial S} (b_k^1 \tilde{\varphi}_1 + \cdots + b_k^k \tilde{\varphi}_k)\, ds_x = \\ = \int_{\partial S} (b_k^1 \varphi_1 + \cdots + b_k^k \varphi_k)\, ds_x - \int_{\partial S}\left(b_k^1 \frac{\partial V_k[f]}{\partial \nu} + \cdots + b_k^k \frac{\partial^k V_k[f]}{\partial \nu^k}\right) ds_x, \end{aligned}$$

where b_k^s are taken from (3.2.16). Due to the validity of the equality $\frac{\partial^k w}{\partial \nu^k} = \Lambda^{[k]} w$ on ∂S (see Corollary 2.1) this condition can be rewritten as

$$\int_{\partial S} (b_k^1 \varphi_1 + \cdots + b_k^k \varphi_k)\, ds_x = \int_{\partial S} \left(b_k^1 \Lambda V_k[f] + \cdots + b_k^k \Lambda^{[k]} V_k[f]\right) ds_x. \tag{3.3.5}$$

Since a solution to the problem (3.3.1)-(3.3.2) exists if and only if there is a solution to the problem (3.3.3)-(3.3.4), then a necessary and sufficient condition for the solvability of the problem (3.3.1)-(3.3.2) looks like (3.3.5). Let us transform this condition. We denote

$$I(f)=\int_{\partial S}\left(b_k^1\Lambda V_k[f]+\cdots+b_k^k\Lambda^{[k]}V_k[f]\right)ds_x.$$

It is clear that the functional $I(f)$ is linear in f.

3.3.1 Polynomial right-hand side of the equation

Let $f(x)$ be a polynomial. According to the Almansi formula, the polynomial $f(x)$ can always be written in the form

$$f(x)=\sum_{m,s}|x|^{2m}u_s^{(m)}(x), \tag{3.3.6}$$

where $u_s^{(m)}(x)$ are homogeneous harmonic polynomials of degree s. It is clear that $u_0^{(m)}(x)=const=u_0^{(m)}$.

Theorem 3.8. *If the polynomial $f(x)$ is written in the form* (3.3.6)*, then*

$$I(f)=\int_{\partial S}\left(b_k^1\Lambda V_k[f]+\cdots+b_k^k\Lambda^{[k]}V_k[f]\right)ds_x$$
$$=(-1)^{k+1}\sum_m\frac{u_0^{(m)}}{(2m+n,2)_k}\,\omega_n. \tag{3.3.7}$$

Proof. Let us first assume that $f(x)$ is a polynomial of a special form $f(x)=f_{m,s}(x)=|x|^{2m}u_s(x)$, where $u_s(x)$ is a homogeneous harmonic polynomial of degree s. We use the following statement.

Theorem 3.9. *Let the polynomial $f(x)$ be written in the form*

$$f(x)=\sum_{m,s}|x|^{2m}u_s^{(m)}(x),$$

where $u_s^{(m)}(x)$ are homogeneous harmonic polynomials of degree s. Then for $x\in\bar S$ the following equality holds

$$V[f](x)$$
$$=\sum_{m,s}\frac{|x|^{2m+2}u_s^{(m)}(x)}{(2m+2)(2m+2s+n)}-\sum_{m,s}\frac{u_s^{(m)}(x)}{(2m+2)(2s+n-2)}. \tag{3.3.8}$$

The proof of Theorem 3.9 follows from the equality (3.7.12) on page 313. From this theorem it follows that for $f_{m,s}(x) = |x|^{2m}u_s(x)$ the equality is true

$$V_k[f_{m,s}] = V[\dots V[f_{m,s}]] = \frac{|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k} + P_{k-1}(|x|^2)u_s(x),$$

where $P_{k-1}(|x|^2)$ is some polynomial of degree $k-1$ in $|x|^2$, specific form which we are not interested in. Since for the operator Λ the following equalities are true

$$\begin{aligned}(\Lambda-\lambda)\big(P_{k-1}(|x|^2)u_s(x)\big) &= u_s(x)(\Lambda-\lambda)P_{k-1}(|x|^2) + P_{k-1}(|x|^2)\Lambda u_s(x)\\ &= \big((\Lambda-\lambda)P_{k-1}(|x|^2) + sP_{k-1}(|x|^2)\big)u_s(x) = u_s(x)(\Lambda+s-\lambda)P_{k-1}(|x|^2),\end{aligned}$$

where $\lambda \in \mathbb{R}$, and also

$$\Lambda\frac{|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k} = \frac{(2m+2k+s)|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k},$$

then we can write

$$\begin{aligned}&\big(b_k^1\Lambda+\cdots+b_k^k\Lambda^{[k]}\big)V_k[f_{m,s}]\\ &\qquad= \frac{|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k}\sum_{i=1}^k b_k^i(2m+2k+s)^{[i]}\\ &\qquad\qquad+\big(b_k^1\Lambda+\cdots+b_k^k\Lambda^{[k]}\big)P_{k-1}(|x|^2)u_s(x)\\ &\qquad\qquad\qquad\equiv \frac{B_{m,s}|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k} + Q_{k-1}(|x|^2)u_s(x),\end{aligned}$$

where we denoted

$$B_{m,s} = \sum_{i=1}^k b_k^i(2m+2k+s)^{[i]}, \tag{3.3.9}$$

$$Q_{k-1}(|x|^2) = \big(b_k^1(\Lambda+s)+\cdots+b_k^k(\Lambda+s)^{[k]}\big)P_{k-1}(|x|^2). \tag{3.3.10}$$

Therefore

$$\begin{aligned}I(f_{m,s}) &= \int_{\partial S}\frac{B_{m,s}|x|^{2m+2k}u_s(x)}{(2m+2,2)_k(2m+2s+n,2)_k}\,ds_x\\ &\qquad+\int_{\partial S}Q_{k-1}(|x|^2)u_s(x)\,ds_x \equiv I_1(f_{m,s})+I_2(f_{m,s}).\end{aligned} \tag{3.3.11}$$

For $s \in \mathbb{N}$ we have $u_s(0) = 0$ and by the mean value theorem for harmonic functions

$$I_1(f_{m,s}) = \frac{B_{m,s}}{(2m+2,2)_k(2m+2s+n,2)_k} \int_{\partial S} u_s(x)\, ds_x = 0,$$
$$I_2(f_{m,s}) = Q_{k-1}(1) \int_{\partial S} u_s(x)\, ds_x = 0.$$

Let $s = 0$. In this case, the polynomial $Q_{k-1}(|x|^2)$ from (3.3.10) has the form

$$Q_{k-1}(|x|^2) = \big(b_k^1 \Lambda + \cdots + b_k^k \Lambda^{[k]}\big) P_{k-1}(|x|^2).$$

If we write $P_{k-1}(|x|^2) = \sum_{i=0}^{k-1} p_i |x|^{2i}$, then we get

$$Q_{k-1}(|x|^2) = \sum_{i=1}^{k-1} p_i\big((2i)b_k^1 + \cdots + (2i)^{[k]} b_k^k\big)|x|^{2i} = 0$$

due to the fairness of the system (3.2.3). This means $I_2(f_{m,0}) = 0$ in this case as well.

Lemma 3.18. *The functional $I_1(f_{m,0})$ has the form*

$$I_1(f_{m,0}) = \frac{(-1)^{k+1}}{(2m+n,2)_k} u_0 \omega_n.$$

Proof. According to the definition of the functional $I_1(f_{m,s})$, using the equality (3.3.9) for $s = 0$ we can write

$$\begin{aligned} I_1(f_{m,0}) &= \frac{B_{m,0}}{(2m+2,2)_k(2m+n,2)_k} \int_{\partial S} u_0(x)\, ds_x \\ &= \frac{B_{m,0} u_0 \omega_n}{(2m+n,2)_k(2m+2,2)_k} = \frac{u_0 \omega_n}{(2m+n,2)_k(2m+2,2)_k} \\ &\times \sum_{i=1}^{k} b_k^i (2m+2k)^{[i]} = \frac{u_0\omega_n}{(2m+n,2)_k(2m+2,2)_k} R_k(m+k), \end{aligned} \tag{3.3.12}$$

where $R_k(x) = \sum_{i=1}^k b_k^i (2x)^{[i]}$ is denoted. By virtue of the system (3.2.3), the polynomial $R_k(x)$ has the following roots $x = 1, \ldots, x = k-1$. In addition, $x = 0$ is also its root. That is why

$$R_k(x) = 2^k b_k^k x(x-1)\cdots(x-k+1) = 2^k b_k^k x^{[k]}.$$

Substituting here the value $b_k^k = (-1)^{k+1}$ from the equality (3.2.9) we find

$$R_k(x) = (-1)^{k+1} 2^k x^{[k]}.$$

Using the obtained value $R_k(x)$ in the equality (3.3.12) we have

$$\begin{aligned} I_1(f_{m,0}) &= \frac{(-1)^{k+1}2^k(m+k)^{[k]}}{(2m+n,2)_k(2m+2,2)_k}u_0\omega_n \\ &= \frac{(-1)^{k+1}}{(2m+n,2)_k}\frac{2^k(m+k)(m+k-1)\cdots(m+1)}{(2m+2)\cdots(2m+2k)}u_0\omega_n = \frac{(-1)^{k+1}u_0\omega_n}{(2m+n,2)_k}. \end{aligned}$$

The lemma is proved. □

If we use the proved lemma and take into account that $I_2(f_{m,s}) = 0$ for $s \in \mathbb{N}_0$, then we obtain

$$I(f_{m,s}) = \begin{cases} \dfrac{(-1)^{k+1}u_0\omega_n}{(2m+n,2)_k}, & s = 0 \\ 0, & s \in \mathbb{N} \end{cases}.$$

Since the polynomial $f(x)$ is the sum of polynomials of the form $|x|^{2m}u_s^{(m)}(x)$ and the functional $I(f)$ is linear, then summing the functionals $I(f_{m,s})$ on m and s we get

$$I(f) = (-1)^{k+1}\sum_m \frac{u_0^{(m)}}{(2m+n,2)_k}\,\omega_n.$$

The proof Theorem 3.8 is complete. □

Let us also transform the resulting expression for the functional $I(f)$.

Lemma 3.19. *The equality takes place*

$$\frac{1}{(2m+n,2)_k} = \frac{1}{(2k-2)!!}\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{2m+n+2i}. \tag{3.3.13}$$

Proof. For a function $K(t)$ continuous for $t \geq 0$ the following equality holds

$$\int_0^t K(\tau)(t-\tau)^{k-1,!}\,d\tau = \int_0^t \cdots \int_0^{\tau_{k-1}} K(\tau_k)\,d\tau_k \dots d\tau_1,$$

where $t^{k,!} = t^k/k!$, since both functions (to the left and to the right of the equal sign) are solutions to the homogeneous Cauchy problem

$$y^{(k)} = K(t); \quad y(0) = 0,\ y^{(1)}(0) = 0, \dots, y^{(k-1)}(0) = 0,$$

and this is the only solution.

Let us transform the left side of (3.3.13). The following equalities are valid

$$\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{2m+n+2i}=\int_0^1\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i t^{2m+n+2i-1}\,dt$$
$$=\int_0^1 t^{2m+n-1}(1-t^2)^{k-1}\,dt \underset{t^2=\tau}{=} \frac{(k-1)!}{2}\int_0^1 \tau^{m+n/2-1}(1-\tau)^{k-1,!}\,d\tau$$
$$=\frac{(k-1)!}{2}\int_0^1\cdots\int_0^{\tau_{k-1}}\tau_k^{m+n/2-1}\,d\tau_k\ldots d\tau_1$$
$$=\frac{(k-1)!}{2(m+n/2)\cdots(m+n/2+k-1)}=\frac{2^{k-1}(k-1)!}{(2m+n,2)_k}=\frac{(2k-2)!!}{(2m+n,2)_k}.$$

This immediately implies the equality (3.3.13). The lemma is proved. □

Lemma 3.20. *For the polynomial $f_{m,0}(x)=|x|^{2m}u_0^{(m)}$ the equality holds*

$$\omega_n\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i u_0^{(m)}}{2m+n+2i}=\int_{|x|<1} f_{m,0}(x)\big(1-|x|^2\big)^{k-1}\,dx \tag{3.3.14}$$

and in addition for $s\in\mathbb{N}$

$$\int_{|x|<1} f_{m,s}(x)\big(1-|x|^2\big)^{k-1}\,dx=0. \tag{3.3.15}$$

Proof. The following equalities are valid, proving the equality (3.3.14)

$$\omega_n\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i u_0^{(m)}}{2m+n+2i}$$
$$=\int_0^1\rho^{2m}\int_{|x|=1}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\rho^{2i}u_0^{(m)}\,ds_x\rho^{n-1}\,d\rho$$
$$=\int_0^1\rho^{2m}\int_{|x|=\rho}(1-\rho^2)^{k-1}u_0^{(m)}\,ds_x\,d\rho=\int_{|x|<1}|x|^{2m}u_0^{(m)}\big(1-|x|^2\big)^{k-1}\,dx$$
$$=\int_{|x|<1} f_{m,0}(x)\big(1-|x|^2\big)^{k-1}\,dx.$$

To prove (3.3.15) we write the previous equalities in reverse order and use the mean value theorem

$$\int_{|x|<1} f_{m,s}(x)\big(1-|x|^2\big)^{k-1}\,dx = \int_0^1 \rho^{2m}(1-\rho^2)^{k-1}\int_{|x|=\rho} u_s^{(m)}(x)\,ds_x\,d\rho = 0,$$

since $u_s^{(m)}(0) = 0$ for $s \in \mathbb{N}$. The lemma is proved. □

Theorem 3.10. *If $f(x)$ is a polynomial, then the equality holds*

$$I(f) = \frac{1}{(2k-2)!!}\int_{|x|<1}\big(|x|^2-1\big)^{k-1} f(x)\,dx \tag{3.3.16}$$

and necessary and sufficient condition for the solvability of the Neumann problem (3.3.1)-(3.3.2) *for $\varphi_j \in C^{k-j+\varepsilon}(\partial S)$, $j = 1,\dots,k$ has the form*

$$\int_{\partial S}\sum_{j=1}^{k}(-1)^{j+1}\binom{2k-j-1}{j-1}(2k-2j-1)!!\varphi_j(s)\,ds_x = \int_S \frac{(|x|^2-1)^{k-1}}{(2k-2)!!} f(x)\,dx, \tag{3.3.17}$$

where we assume that $(-1)!! = 1$.

Proof. Let $f(x) = \sum_{m,s} f_{m,s}(x) \equiv |x|^{2m}u_s^{(m)}(x)$. By Theorem 3.8, Lemma 3.19 and the equality (3.3.14) from Lemma 3.20 we get

$$\begin{aligned} I(f) &= (-1)^{k+1}\sum_m \frac{u_0^{(m)}}{(2m+n,2)_k}\,\omega_n \\ &= \frac{(-1)^{k+1}}{(2k-2)!!}\sum_m \omega_n \sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i u_0^{(m)}}{2m+n+2i} \\ &= \frac{(-1)^{k+1}}{(2k-2)!!}\int_{|x|<1}\sum_m f_{m,0}(x)\big(1-|x|^2\big)^{k-1}\,dx. \end{aligned}$$

If we now use (3.3.15), adding zero terms to the formula obtained above, we obtain

$$I(f) = \frac{(-1)^{k+1}}{(2k-2)!!}\int_{|x|<1}\sum_{m,s} f_{m,s}(x)\big(1-|x|^2\big)^{k-1}\,dx,$$

which is equivalent to (3.3.16). Therefore, the condition(3.3.5) for the solvability of the Neumann problem (3.3.1)-(3.3.2) take the form (3.3.17). The theorem is proved. □

Example 3.5. Consider the following Neumann problem

$$\Delta^3 u(x) = 1,\ x \in S;$$
$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0, \quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = 0, \quad \frac{\partial^3 u}{\partial \nu^3}\Big|_{\partial S} = 0.$$

According to Theorem 3.10 this problem is not solvable, since the condition (3.3.17) is not satisfied

$$\frac{1}{8}\int_S (|x|^2 - 1)^2\, dx > 0.$$

Indeed, let this problem be solvable and let $u(x)$ be its solution. Let us apply the operator $\Lambda + 6$ to the equation. Then, in accordance with the equalities

$$(\Lambda + 6)\Delta^3 u = \Delta(\Lambda + 4)\Delta^2 u = \Delta^2(\Lambda + 2)\Delta u = \Delta^3 \Lambda u$$

the function $v = \Lambda u$ is a solution to the following Dirichlet problem

$$\Delta^3 v(x) = 6, \quad x \in S;$$
$$v|_{\partial S} = 0, \quad \Lambda v|_{\partial S} = 0, \quad \Lambda^2 v|_{\partial S} = 0.$$

The solution to this problem using the formula (2.1.16) from Example 2.1 on page 77 is a polynomial of the form

$$v(x) = 6\frac{(|x|^2 - 1)^3}{(2,2)_3(n,2)_3} = \frac{(|x|^2 - 1)^3}{8n(n+2)(n+4)}.$$

However, there is no 4-harmonic function $u(x)$ in S that satisfies the equation

$$\Lambda u = v = \frac{1}{8n(n+2)(n+4)}\big(|x|^6 - 3|x|^4 + 3|x|^2 - 1\big), \quad x \in S,$$

because the function Λu, analytic in S, in its expansion at zero, does not have a term of the form cx^0, but the right side of this equation has such a term $-1/(8n(n+2)(n+4))$. Therefore, the original Neumann problem is not solvable.

3.3.2 General case of the right side of the equation

The following statement holds.

Theorem 3.11. *If $f \in C^1(\bar S)$, then for $f(x)$ the equality* (3.3.16) *is valid and the necessary and sufficient condition for the solvability of the Neumann problem* (3.3.1)-(3.3.2) *for $\varphi_j \in C^{k-j+\varepsilon}(\partial S)$, $j=1,\dots,k$ has the form* (3.3.17)

$$\int_{\partial S}\sum_{j=1}^{k}(-1)^{j+1}\binom{2k-j-1}{j-1}(2k-2j-1)!!\varphi_j(s)\,ds_x \\ = \int_S \frac{(|x|^2-1)^{k-1}}{(2k-2)!!}f(x)\,dx,$$

where as above $(-1)!!=1$.

Proof. Let us look at the functional

$$I(f)=\int_{\partial S}\big(b_k^1\Lambda+\cdots+b_k^k\Lambda^{[k]}\big)V_k[f]\,ds_x. \tag{3.3.18}$$

We prove that $\Lambda^k V_k[f]\in C(\bar S)$. According to the divergence theorem we get

$$\int_S u\Lambda v\,d\xi=\int_S\sum_{i=1}^{n}(\xi_i uv)_{\xi_i}\,d\xi-n\int_S uv\,d\xi-\int_S v\Lambda u\,d\xi \\ =\int_{\partial S}\sum_{i=1}^{n}\xi_i^2 uv\,ds_\xi-\int_S v(\Lambda+n)u\,d\xi=\int_{\partial S}uv\,ds_\xi-\int_S v(\Lambda+n)u\,d\xi,$$

even if Λv has an integrable singularity in S. That is why

$$\Lambda_x\int_S E(x,\xi)f(\xi)\,d\xi=\int_S\sum_{i=1}^{n}(x_i-\xi_i)(E(x,\xi))_{x_i}f(\xi)\,d\xi \\ -\int_S f(\xi)\Lambda_\xi E(x,\xi)\,d\xi=(2-n)\int_S E(x,\xi)f(\xi)\,d\xi \\ -\int_{\partial S}f(\xi)E(x,\xi)\,ds_\xi+\int_S E(x,\xi)(\Lambda+n)f(\xi)\,d\xi \\ =\int_S E(x,\xi)(\Lambda+2)f(\xi)\,d\xi-\int_{\partial S}E(x,\xi)f(\xi)\,ds_\xi.$$

Hence, keeping in mind that

$$V[f](x)=-\frac{1}{\omega_n}\int_S E(x,\xi)f(\xi)\,d\xi,\quad W[f](x)=\frac{1}{\omega_n}\int_{\partial S}E(x,\xi)f(\xi)\,ds_\xi$$

we get

$$\Lambda V_k[f] = V[(\Lambda+2)V_{k-1}[f]] + W[V_{k-1}[f]]. \tag{3.3.19}$$

Let us find out the action of the operator Λ on the second term in the equality (3.3.19). From the formula for representing the solution to the Poisson equation it is easy to obtain

$$\int_{\partial S} f(\xi)\Lambda_\xi E(x,\xi)\,ds_\xi = \int_{\partial S} E(x,\xi)\Lambda_\xi f(\xi)\,ds_\xi - \omega_n f(x) - \int_S E(x,\xi)\Delta f(\xi)\,d\xi,$$

which means

$$\Lambda_x \int_{\partial S} E(x,\xi) f(\xi)\,ds_\xi = (2-n)\int_{\partial S} E(x,\xi)f(\xi)\,ds_\xi \\ - \int_{\partial S} E(x,\xi)\Lambda_\xi f(\xi)\,ds_\xi + \omega_n f(x) + \int_S E(x,\xi)\Delta f(\xi)\,d\xi.$$

From here, with $f = V_{k-1}[f]$ we find

$$\Lambda W[V_{k-1}[f]] = (2-n)W[V_{k-1}[f]] - W[\Lambda V_{k-1}[f]] \\ = -W[(\Lambda+n-2)V_{k-1}[f]]. \tag{3.3.20}$$

Consider the expression

$$\Lambda^i V_k[f] = \Lambda^i V[V[\underset{k-3}{\ldots} V[f]]] = \Lambda^i \int_{|\xi_1|<1}\int_{|\xi_2|<1}\cdots\int_{|\xi_k|<1} E(x,\xi_1)\times \\ \times E(\xi_1,\xi_2)\cdots E(\xi_{k-1},\xi_k) f(\xi_k)\,d\xi_k\,d\xi_{k-1}\ldots d\xi_1.$$

According to the equality (3.3.19) the operator Λ can be entered under the sign of the integral over $\xi_i \in S$ as $\Lambda_{\xi_{i-1}}$, but in this case surface integrals of the type $W[f]$ over ∂S arise. According to the equality (3.3.20) the operator Λ can be entered under the integral sign in this case as well. Let us distribute the operators of the product $\Lambda\cdots\Lambda = \Lambda^i$ among the functions $E(\xi_{j-1},\xi_j)$ so that there is no more than one Λ operator per function $E(\xi_{j-1},\xi_j)$ and if the integral is taken over $\xi_j \in \partial S$, then the operator $\Lambda_{\xi_{j-1}}$ does not apply to the corresponding function $E(\xi_{j-1},\xi_j)$. As a result of these actions, $\Lambda^i V_k[f]$ is equal to a linear combination of expressions of the form

$$\int_{|\xi_1|=1}\cdots\int_{|\xi_i|<1}\cdots\int_{|\xi_k|<1} E(x,\xi_1)\cdots\Lambda_{\xi_{i-1}}E(\xi_{i-1},\xi_i)\cdots E(\xi_{k-1},\xi_k)$$

$$\times f(\xi_k)\,d\xi_k\,d\xi_i\ldots ds_{\xi_1} = W[W[\ldots V[\Lambda V[\ldots \Lambda V[\ldots V[f]]]]]],$$

where the integral over the last variable ξ_k is always (for any $i = 1, \ldots, k$) taken over the ball S.

Let us associate this expression with the index $\alpha = (\alpha_1 \ldots, \alpha_k)$, where $\alpha_i = 0, 1, 2$ so that if the i-th integral is taken over $\xi_i \in \partial S$ and the kernel is equal to $E(\xi_{i-1}, \xi_i)$, then $\alpha_i = 0$ if the ith integral is taken over $\xi_i \in S$, but the kernel is equal to $\Lambda_{\xi_{i-1}} E(\xi_{i-1}, \xi_i)$, then $\alpha_i = 1$, otherwise $\alpha_i = 2$. The functions $\Lambda_{\xi_{j-1}} E(\xi_{j-1}, \xi_j)$ are integrable over $\xi_j \in S$, and the functions $E(\xi_{i-1}, \xi_i)$ are integrable over $\xi_i \in \partial S$. Therefore, according to Fubini's theorem, these integrals can be rearranged in any order. Let us rearrange the integrals in the equality (3.3.18) so that we obtain the expression

$$I(f) = \int_{|\xi_k|<1} \Big(\sum_\alpha C_\alpha \int_{|\xi_{k-1}|<1} \cdots \int_{|\xi_i|<1} \cdots \int_{|\xi_1|=1} \int_{|x|=1} E(x, \xi_1) \cdots$$
$$\cdots \Lambda_{\xi_{i-1}} E(\xi_{i-1}, \xi_i) \cdots E(\xi_{k-1}, \xi_k)\, ds_x\, ds_{\xi_1} \ldots d\xi_{k-1} \Big) f(\xi_k)\, d\xi_k,$$

where to the integral expression under the sum sign corresponds the index α. It is easy to see that the integral of the form $\int_{|\xi|=1} E(x, \xi) P_m(\xi)\, ds_\xi$ (see Lemma 3.35) is a polynomial of degree m, where $P_m(x)$ is a polynomial of degree m, and integrals of the form

$$\int_{|\xi|<1} E(x, \xi) P_m(\xi)\, d\xi, \quad \Lambda \int_{|\xi|<1} E(x, \xi) P_m(\xi)\, d\xi$$

are polynomials of degree $m + 2$ (see the equality (3.7.12)), which means the expression in parentheses is the polynomial $Q_{2k}(\xi)$. For example,

$$\Lambda_x \int_{|\xi|<1} E(x, \xi)\, d\xi = -\omega_n \frac{|x|^2}{n}.$$

Since in the equality (3.3.18) there is no zero degree Λ under the integral, then the possible highest degree $2k$ of this polynomial is given by terms of the form

$$b_k^1 \int_{|\xi_k|<1} \cdots \int_{|\xi_1|<1} \int_{|x|=1} \Lambda_x E(x, \xi_1) E(\xi_1, \xi_2) \cdots E(\xi_{k-1}, \xi_k)\, ds_x d\xi_1 \ldots d\xi_{k-1},$$

or

$$b_k^k \int_{|\xi_k|<1} \cdots \int_{|\xi_1|<1} \int_{|x|=1} \Lambda_x E(x, \xi_1) \Lambda_{\xi_1} E(\xi_1, \xi_2) \cdots$$

$$\cdots \Lambda_{\xi_{k-1}} E(\xi_{k-1}, \xi_k)\, ds_x d\xi_1 \ldots d\xi_{k-1}.$$

Therefore we have

$$I(f) = \int_{|\xi|<1} Q_{2k}(\xi) f(\xi)\, d\xi.$$

The degree of $Q_{2k}(x)$ is probably less than $2k$, but that does not matter. If one select

$$f(\xi) = f_0(\xi) \equiv Q_{2k}(\xi) - \frac{1}{(2k-2)!!}(|\xi|^2 - 1)^{k-1},$$

then by virtue of Theorem 3.10

$$0 = I(f_0) = \int_{|\xi|<1} Q_{2k}(\xi) f_0(\xi)\, d\xi - \\ - \int_{|\xi|<1} \frac{(|\xi|^2-1)^{k-1}}{(2k-2)!!} f_0(\xi)\, d\xi = \int_{|\xi|<1} f_0^2(\xi)\, d\xi$$

and that means $f_0(\xi) = 0$ or

$$Q_{2k}(\xi) = \frac{1}{(2k-2)!!}(|x|^2 - 1)^{k-1}.$$

Thus, the equality (3.3.16) is also valid for $f \in C^1(\bar{S})$ and the necessary and sufficient condition for the solvability of the Neumann problem (3.3.1)-(3.3.2) has the form (3.3.17). The theorem is proved. □

Remark 3.8. For $k = 1$, from Theorem 3.11 we obtain the well-known result of the solvability of the Neumann problem for the Poisson equation [28] and the solvability condition (3.3.17) has the form

$$\int_{\partial S} \varphi_1(s)\, ds_x = \int_S f(x)\, dx.$$

Example 3.6. Recalling the triangle B from (3.2.18) we write the solvability conditions for the Neumann problem for the 4-harmonic equation in a ball in the form

$$\int_{\partial S} \big(15\varphi_1(s) - 15\varphi_2(s) + 6\varphi_3(s) - \varphi_4(s)\big)\, ds_x = \frac{1}{48}\int_S \big(|x|^2 - 1\big)^3 f(x)\, dx.$$

3.3.3 Construction of a solution to the Neumann problem

In Section 3.2 the solution to the Neumann problem for the homogeneous polyharmonic equation is given. Let us present a solution to the Neumann problem with a nonzero right-hand side of the equation.

Theorem 3.12. *The solution $u(x)$ to the Neumann problem* (3.3.1)-(3.3.2) *can be represented as*

$$u(x) = \int_0^1 v(tx)\frac{dt}{t} + C, \tag{3.3.21}$$

where $v(x)$ is the solution to the following auxiliary Dirichlet problem

$$\Delta^k v(x) = (\Lambda + 2k)f(x), \quad x \in S; \tag{3.3.22}$$

$$v|_{\partial S} = \varphi_1(s),\ \frac{\partial^j v}{\partial \nu^j}\Big|_{\partial S} = j\varphi_j(s) + \varphi_{j+1}(s),\ s \in \partial S,\ j = \overline{1, k-1}. \tag{3.3.23}$$

The function $u(x)$ is defined if $v(0) = 0$.

Proof. Let $u(x)$ be a solution to the Neumann problem. It is easy to check that $\Delta\Lambda u = (\Lambda + 2)\Delta u$ and therefore

$$\Delta^k\Lambda u = \Delta^{k-1}\Lambda\Delta u + 2\Delta^k u = \Delta^{k-2}\Lambda\Delta^2 u + 4\Delta^k u$$
$$= \cdots = \Lambda\Delta^k u + 2k\Delta^k u = (\Lambda + 2k)\Delta^k u.$$

If we act with the operator $(\Lambda + 2k)$ on both sides of the equation (3.3.1), then we have

$$(\Lambda + 2k)\Delta^k u(x) = \Delta^k\Lambda u(x) = (\Lambda + 2k)f(x)$$

and therefore the function $v = \Lambda u$ satisfies the equation (3.3.22) $\Delta^k v(x) = (\Lambda + 2k)f(x)$. By virtue of the equality $\frac{\partial^k w}{\partial \nu^k} = \Lambda^{[k]}w$ on ∂S the Neumann conditions (3.3.2) take the form

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \Lambda u|_{\partial S} = v|_{\partial S} = \varphi_1(s),$$
$$\frac{\partial^j u}{\partial \nu^j}\Big|_{\partial S} = \Lambda^{[j]}u|_{\partial S} = (\Lambda - 1)\cdots(\Lambda - j + 1)v|_{\partial S} = \varphi_j(s), \quad j = 2, \ldots, k$$

and that means

$$v|_{\partial S} = \varphi_1(s), \quad (\Lambda - 1)\cdots(\Lambda - j + 1)v|_{\partial S} = \varphi_j(s), \quad j = 2, \ldots, k.$$

Since

$$(\Lambda - 1)\cdots(\Lambda - j + 1) = \Lambda(\Lambda - 1)\cdots(\Lambda - j + 2) - (j - 1)(\Lambda - 1)\cdots(\Lambda - j + 2),$$

then the boundary conditions for $j = 2, \ldots, k$ are transformed to the form

$$\frac{\partial^{j-1} v}{\partial \nu^{j-1}}\Big|_{\partial S} - (j-1)\frac{\partial^{j-1} u}{\partial \nu^{j-1}}\Big|_{\partial S} = \varphi_j(s)$$

or the function $v(x)$ satisfies the Dirichlet conditions (3.3.23)

$$\frac{\partial^{j-1} v}{\partial \nu^{j-1}}\Big|_{\partial S} = (j-1)\varphi_{j-1}(s) + \varphi_j(s), \quad j = 2, \ldots, k,$$

where $v|_{\partial S} = \varphi_1(s)$.

Suppose that the function $v(x)$ is found as a solution to the problem (3.3.22)-(3.3.23). Then to find $u(x)$ we need to resolve the equation $v = \Lambda u$ in $\bar{S}$. As is easy to see, this equation has a solution

$$u(x) = \int_0^1 v(tx)\frac{dt}{t} + C.$$

Due to the differentiability of the function $v(x)$ at zero $v(tx) - v(0) \approx t dv(0)$ and therefore the integral converges only in the case $v(0) = 0$. If $u_1(x)$ is another solution to the Neumann problem, then, due to the uniqueness of the solution to the Dirichlet problem $v(x)$ (Theorem 2.16), we have $v = \Lambda u = \Lambda u_1$ and therefore $\Lambda(u - u_1) = 0$. Since $u(x) - u_1(x)$ is a polyharmonic function in S, then, similarly to the proof of Theorem 2.16, we conclude that $u(x) - u_1(x) = 0$ in S. Therefore, the solution (3.3.21) is unique. The theorem is proved. □

Remark 3.9. The solvability condition for the Neumann problem (3.3.1)-(3.3.2) can be written in a more compact but less verifiable form $v(0) = 0$, where $v(x)$ is the solution to the problem (3.3.22)-(3.3.23) (see Theorem 3.19 on page 250).

Remark 3.10. In Section 2.1 on page 70 the construction of a solution to the homogeneous Dirichlet problem for polynomial data is considered, and in Section 2.4 on page 129 the solution to the Dirichlet problem for a homogeneous polyharmonic equation is given. Based on these results and using Theorem 3.12, the corresponding Neumann problems can be solved for polynomial data. It is only necessary to construct a polynomial solution to the auxiliary Dirichlet problem of type (3.3.22)-(3.3.23).

Example 3.7. Let us illustrate Remark 3.10 using the following Neumann problem

$$\Delta^2 u(x) = 1,\ x \in S; \tag{3.3.24}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \frac{1}{2n+4}, \quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = \frac{1}{2n}. \tag{3.3.25}$$

According to Theorem 3.10, using the triangle B from (3.2.18), we check the conditions for the solvability of the problem (3.3.24)-(3.3.25). They look like

$$\int_{\partial S} \big(\varphi_1(s) - \varphi_2(s)\big)\, ds_x = \frac{1}{2}\int_S \big(|x|^2 - 1\big) f(x)\, dx.$$

It is easy to calculate that

$$\int_{\partial S} \big(\varphi_1(s) - \varphi_2(s)\big)\, ds_x = -\frac{\omega_n}{n(n+2)} = \frac{1}{2}\int_S \big(|x|^2 - 1\big) f(x)\, dx,$$

which means the condition (3.3.17) is satisfied. The auxiliary Dirichlet problem (3.3.22)-(3.3.23) has the form

$$\Delta^2 v(x) = (\Lambda + 4) f(x),\ x \in S;$$
$$v\big|_{\partial S} = \varphi_1(s), \quad \frac{\partial v}{\partial \nu}\Big|_{\partial S} = \varphi_1(s) + \varphi_2(s),\ s \in \partial S.$$

Using Theorem 2.13 on page 142 and Theorem 2.2 on page 84 ($l = 2$) we can write the solution $v(x)$ of this problem in the form

$$\begin{aligned} v(x) = q_0(x) &+ \frac{|x|^2 - 1}{2}\big(q_1(x) - \Lambda q_0(x)\big) \\ &+ \frac{(|x|^2 - 1)^2}{4}\int_0^1 \sum_{s=0}^{\infty} \frac{(1 - \alpha|x|^2)^s (1-\alpha)^{s+1}}{(2s)!!(2s+4)!!} \\ &\times \Big(\Delta^s (\Lambda + 4) f\Big)(\alpha x)\, \alpha^{n/2-1}\, d\alpha. \end{aligned}$$

Substituting here the values of the functions $f = 1$, $q_0(x) = \frac{1}{2n+4}$ and $q_1(x) = \frac{1}{2n+4} + \frac{1}{2n}$ are the solutions of the corresponding Dirichlet problems for the Laplace equation (see Theorem 2.15 on page 149) we get

$$v(x) = \frac{1}{2n+4} + \frac{|x|^2 - 1}{2}\Big(\frac{1}{2n+4} + \frac{1}{2n}\Big) + \frac{(|x|^2 - 1)^2}{8}\int_0^1 (1-\alpha)\alpha^{n/2-1}\, d\alpha$$

$$= \frac{1}{2(n+2)} + (|x|^2 - 1)\frac{n+1}{2n(n+2)} + \frac{(|x|^2-1)^2}{2n(n+2)} = \frac{1}{2n(n+2)}\Big((n-1)|x|^2 + |x|^4\Big).$$

It can be seen that $v(0) = 0$. Using (3.3.21) we obtain the solution to the Neumann problem under consideration in the form

$$u(x) = \frac{1}{2n(n+2)}\left(\frac{n-1}{2}|x|^2 + \frac{1}{4}|x|^4\right) + C.$$

3.4 Neumann type boundary value problems

The class of boundary value problems considered in this chapter is a natural generalization of the classical formulation of the Neumann problem proposed for the polyharmonic equation by A.V. Bitsadze [30]. In Section 3.3, using the integer triangle B, the conditions for the solvability of the Neumann problem are found, and in [99] the necessary conditions for the solvability of the class of Neumann type problems are found. In this section we prove that the previously found set of necessary conditions for the solvability of Neumann-type problems is also a set of sufficient conditions for the solvability of these problems.

In the unit ball S, consider the following class of boundary value problems of Neumann type $\mathcal{N}_k$, depending on the parameter $k \in \mathbb{N}_0$ (we will assume the values of the parameters n and m to be fixed in further study) for the homogeneous polyharmonic equations

$$\Delta^m u = 0, \quad x \in S; \tag{3.4.1}$$

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial S} = \varphi_1(x), \ldots, \frac{\partial^{k+m-1} u}{\partial \nu^{k+m-1}}\Big|_{\partial S} = \varphi_m(x),\ x \in \partial S, \tag{3.4.2}$$

where $\dfrac{\partial}{\partial \nu}$ is the outer normal derivative to the unit sphere, $m \in \mathbb{N}$. The smoothness of the boundary functions $\varphi_i(s)$, $i = 1, \ldots, m$, defined on ∂S is indicated below. Problem $\mathcal{N}_0$ is a Dirichlet problem that is unconditionally solvable, and problem $\mathcal{N}_1$ coincides with the Neumann problem [30], [88]. In the work [29] A.V. Bitsadze write out necessary and sufficient conditions for the solvability of the problem $\mathcal{N}_1$ for $m = 1, 2$ and shows that it can be solved in quadratures.

Studies of the solvability of certain formulations of Neumann-type problems in the unit ball, in addition to the papers listed above, can also be found for the biharmonic equation (in particular, problems $\mathcal{N}_1$ and $\mathcal{N}_2$) in [92, 95, 96, 170], and for the polyharmonic equation in [111]. In [132], [164] for boundary value problems for the polyharmonic equation with normal derivatives in boundary conditions, a sufficient condition for the Fredholm property of these problems and a new form of the Fredholm property criterion, equivalent to the complementarity condition, are obtained.

The section is organized as follows. Lemma 3.21 from Section 3.4.1 investigates the smoothness of the harmonic components of the solution to the Dirichlet problem $\mathcal{N}_0$. In Lemmas 3.22 and 3.23 from Section 3.4.2 sufficient conditions for the solvability of the problem $\mathcal{N}_k$ are found in terms of the function $v(x)$ – a solution to the auxiliary Dirichlet problem and depending on the integer triangle B. Theorem 3.15 from Section 3.4.3 provides necessary conditions for the solvability of the $\mathcal{N}_k$ problem in terms of the boundary functions of the problem. In Lemma 3.29 and Theorem 3.17 from Section 3.4.4 the conditions from Theorem 3.15 in the case of even $k-l$ are expressed through the auxiliary function $v(x)$, in the case of odd $k-l$ this is done in Theorem 3.18. Finally, Theorem 3.19 from Section 3.4.5 proves that for $k \le m$ the necessary conditions of Theorem 3.15 are also sufficient conditions for the solvability of the problem $\mathcal{N}_k$. In Example 3.9 the dynamics of the emergence of solvability conditions in problems $\mathcal{N}_k$ depending on k for the 7-harmonic equation is presented. In Example 3.10 the problem $\mathcal{N}_3$ for the inhomogeneous 7-harmonic equation is considered. More detailed results of the author on Neumann-type problems can be found in [94, 98, 99, 113, 114, 118].

3.4.1 Dirichlet boundary value problem $\mathcal{N}_0$

Let us first consider the Dirichlet boundary value problem $\mathcal{N}_0$ (see Section 2.4) for the homogeneous polyharmonic equation. If we use the notations $\Lambda = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$, $t^{[m]} = t(t-1)\dots(t-m+1)$ with the convention $t^{[0]} = 1$ and the equality $\Lambda^{[i]}u = \frac{\partial^i u}{\partial \nu^i}$ on ∂S, then this problem can be rewritten as

$$\begin{gathered} \Delta^m u = 0, \quad x \in S; \\ u|_{\partial S} = \varphi_1(x), \dots, \Lambda^{[m-1]}u|_{\partial S} = \varphi_m(x), \quad x \in \partial S. \end{gathered} \tag{3.4.3}$$

Note that the above-mentioned properties of the operator $\Lambda = x \cdot \nabla$ are noticed earlier in the work of A.V. Bitsadze [30]. It is known [162] that if the function $u(x)$ is m-harmonic in the starlike domain $D \subset \mathbb{R}^n$, then it can be expanded using the Almansi formula

$$u(x) = u^{(0)}(x) + |x|^2 u^{(1)}(x) + \dots + |x|^{2m-2} u^{(m-1)}(x), \tag{3.4.4}$$

where $x \in D$ and $u^{(i)}(x)$ are the "harmonic components" of the m-harmonic function $u(x)$.

In the further presentation we need the smoothness up to the boundary ∂S of the harmonic components of the solution to the Dirichlet problem (3.4.3). Let $\varepsilon \in (0, 1)$.

Lemma 3.21. *Let $u(x)$ be the solution to the Dirichlet problem* (3.4.3) *and $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$. Then the harmonic components $u^{(i-1)}(x)$ of this solution from the equality* (3.4.4) *are such that $u^{(i-1)} \in C^{m-1+\varepsilon}(\bar{S})$, $i = 1, \dots, m$. The solution $u(x)$ also has smoothness $u \in C^{m-1+\varepsilon}(\bar{S})$.*

Proof. Let $u(x)$ be a solution to the Dirichlet problem (3.4.3) and it is expanded according to the equality (3.4.4). Let also $w_i(x)$ be solutions to Dirichlet problems for the Laplace equation in S with the boundary condition $w_i|_{\partial S} = \varphi_i(x)$, where $i = 1, \dots, m$. Then we have in S

$$\Lambda^{[i]} u(x) = \Lambda^{[i]} \sum_{j=1}^{m} |x|^{2j-2} u^{(j-1)}(x) = \sum_{j=1}^{m} |x|^{2j-2} (\Lambda + 2j - 2)^{[i]} u^{(j-1)}(x),$$

and therefore for the ith boundary condition of the Dirichlet problem (3.4.3) we write

$$\begin{aligned} w_i|_{\partial S} &= \varphi_i(x) = \Lambda^{[i-1]} u|_{\partial S} \\ &= \sum_{j=1}^{m} |x|^{2j-2} (\Lambda + 2j - 2)^{[i-1]} u^{(j-1)}|_{\partial S} = \sum_{j=1}^{m} (\Lambda + 2j - 2)^{[i-1]} u^{(j-1)}|_{\partial S}. \end{aligned}$$

Since the operator Λ preserves the harmonicity of the function, then, due to the uniqueness of the solution to the Dirichlet problem, we obtain the following equations in functions harmonic in S

$$\sum_{j=1}^{m} (\Lambda + 2j - 2)^{[i-1]} u^{(j-1)}(x) = w_i(x), \quad x \in S, \tag{3.4.5}$$

where $i = 1, \dots, m$. Let us denote the matrix of the system of differential equations (3.4.5) by

$$A(\Lambda) = ((\Lambda + 2j - 2)^{[i-1]})_{i,j=1,\dots,m} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \Lambda & \Lambda + 2 & \dots & \Lambda + 2m - 2 \\ \Lambda^{[2]} & (\Lambda + 2)^{[2]} & \dots & (\Lambda + 2m - 2)^{[2]} \\ \dots & & \dots & \\ \Lambda^{[m-1]} & (\Lambda + 2)^{[m-1]} & \dots & (\Lambda + 2m - 2)^{[m-1]} \end{pmatrix}.$$

Let $W(\lambda_1, \lambda_2, \dots, \lambda_m) = (\lambda_i^{j-1})_{i,j=1,\dots,m}$ be the Vandermonde matrix of size $m \times m$, then the matrix $W[\lambda_1, \lambda_2, \dots, \lambda_m] = (\lambda_i^{[j-1]})_{i,j=1,\dots,m}$ we call factorial Vandermonde matrix. It is not hard to see that

$$A(\Lambda) = W[\Lambda, \Lambda + 2, \dots, \Lambda + 2m - 2].$$

The first two rows of the determinants $W(\lambda_1, \dots, \lambda_m)$ and $W[\lambda_1, \dots, \lambda_m]$ are the same and if the second row of the determinant $W[\lambda_1, \dots, \lambda_m]$ add it to its third row, then we get the third row of the determinant $W(\lambda_1, \dots, \lambda_m)$. Continuing similar actions, we make sure that $\det W(\lambda_1, \dots, \lambda_m) = \det W[\lambda_1, \dots, \lambda_m]$. That is why

$$\det W[\lambda_1, \dots, \lambda_m] = \prod_{1 \le i < j \le m} (\lambda_j - \lambda_i),$$

which means

$$\det A(\Lambda) = \det W[\Lambda, \Lambda + 2, \dots, \Lambda + 2m - 2] = (2m-2)!! \cdot (2m-4)!! \dots 2 \equiv a_m.$$

Let $A^*(\Lambda) = (A_{j,i}(\Lambda))_{i,j=1,\dots,m}$ be the adjoint matrix of $A(\Lambda)$, where $A_{i,j}(\Lambda)$ are algebraic complements to the elements of the original matrix. Let us prove that $\deg A_{j,i}(\Lambda) = m - j$.

First we examine the operators $A_{m,s}(\Lambda)$. To do this, consider the algebraic complement to the elements of the mth row in the Vandermonde factorial matrix $W^*_{m,s}[\lambda_1, \dots, \lambda_m]$ for $s = 1, \dots, m$. It is not hard to see that

$$W^*_{m,s}[\lambda_1, \dots, \lambda_m] = (-1)^{m-s} \prod_{1 \le i < j \le m;\, i,j \ne s} (\lambda_j - \lambda_i),$$

whence we get

$$A_{m,s}(\Lambda) = \frac{(-1)^{m-s} a_m}{(2s-2)!!(2m-2s)!!}$$

and that means $\deg A_{m,s}(\Lambda) = 0$. Next, consider the operator $A_{k,s}(\Lambda)$ for $k = 1 \dots, m-1$, $s = 1, \dots, m$ and assume that $\deg A_{i,s}(\Lambda) = m - i$ for $i = k + 1 \dots, m$ and $s = 1, \dots, m$. Let us replace the operator Λ with a numeric parameter λ and, for notational simplicity, consider the last element in the kth row of the matrix $A^*(\lambda)$

$$A_{k,m}(\lambda)=\begin{vmatrix} 1 & 1 & \dots & 1\\ \lambda & \lambda+2 & \dots & \lambda+2m-4\\ \dots & & \dots & \\ \lambda^{[k-2]} & (\lambda+2)^{[k-2]} & \dots & (\lambda+2m-4)^{[k-2]}\\ \lambda^{[k]} & (\lambda+2)^{[k]} & \dots & (\lambda+2m-4)^{[k]}\\ \dots & & \dots & \\ \lambda^{[m-1]} & (\lambda+2)^{[m-1]} & \dots & (\lambda+2m-4)^{[m-1]} \end{vmatrix}.$$

To find the degree of the polynomial $A_{k,m}(\lambda)$, consider the discrete derivative of the polynomial $P(\lambda)$ in the form $P^{(1)}(\lambda)=P(\lambda+1)-P(\lambda)$ [15, p.220]. It is not hard to see that

$$\Big(\prod_{i=1}^m P_i(\lambda)\Big)^{(1)}=\sum_{i=1}^m\Big(\prod_{j=1}^{i-1}P_j(\lambda+1)\,P_i^{(1)}(\lambda)\prod_{j=i+1}^m P_j(\lambda)\Big)$$

and $\big(\lambda^{[k]}\big)^{(1)}=k\lambda^{[k-1]}$ for $k>0$, but $\big(\lambda^{[0]}\big)^{(1)}=0$. Therefore, the derivative $A_{k,m}^{(1)}(\lambda)$ is equal to the sum of $m-1$ determinants, in which the derivatives are taken for each row separately. When taking derivatives from the first $k-1$ rows, we obtain determinants in which the row by which the derivative is taken linearly depends on the previous rows, which means these determinants are equal to zero. When taking derivatives of the last $m-k-1$ rows, we obtain determinants that are linear combinations of determinants $A_{i,m}(\lambda)$ for $i>k+1$. The derivative of the kth row $A_{k,m}(\lambda)$ is equal to

$$\begin{vmatrix} 1 & 1 & \dots & 1\\ \lambda+1 & \lambda+3 & \dots & \lambda+2m-3\\ \dots & & \dots & \\ (\lambda+1)^{[k-2]} & (\lambda+3)^{[k-2]} & \dots & (\lambda+2m-3)^{[k-2]}\\ k\lambda^{[k-1]} & k(\lambda+2)^{[k-1]} & \dots & k(\lambda+2m-4)^{[k-1]}\\ \dots & & \dots & \\ \lambda^{[m-1]} & (\lambda+2)^{[m-1]} & \dots & (\lambda+2m-4)^{[m-1]} \end{vmatrix}=kA_{k+1,m}(\lambda),$$

whence by assumption $\deg A_{k,m}(\lambda)=\deg A_{k+1,m}(\lambda)+1=m-k-1+1=m-k$. Since the value of $W^*_{m,s}[\lambda_1,\dots,\lambda_m]$ depends on the difference $\lambda_j-\lambda_i$, then by analogy with what is done $\deg A_{k,s}(\lambda)=m-k$ for $s=1,\dots,m$. Thus it is proved that

$\deg A_{j,i}(\Lambda) = m - j$. For example, for $m = 3$ we have

$$A^*(\Lambda) = 2\begin{pmatrix} (\Lambda+2)(\Lambda+4) & -2\Lambda-5 & 1 \\ -2\Lambda(\Lambda+4) & 4\Lambda+6 & -2 \\ \Lambda(\Lambda+2) & -2\Lambda-1 & 1 \end{pmatrix},$$

which means the solution to the Dirichlet problem for $m = 3$ has the form

$$\begin{aligned} u(x) = \frac{2}{4!!2!!}\big[&((8+6\Lambda+\Lambda^2)w_1(x) - (5+2\Lambda)w_2(x) + w_3(x)) \\ &- |x|^2\big((8\Lambda+2\Lambda^2)w_1(x) - (6+4\Lambda)w_2(x) + 2w_3(x)\big) \\ &+ |x|^4\big((2\Lambda+\Lambda^2)w_1(x) - (1+2\Lambda)w_2(x) + w_3(x)\big)\big], \end{aligned} \tag{3.4.6}$$

which coincides with the solution obtained earlier in Example 2.5 on page 146, but written differently. If we denote the columns of the matrix $A^*(\Lambda)$ by $A_1^*(\Lambda), \dots, A_m^*(\Lambda)$, then we can write

$$\begin{pmatrix} u^{(0)}(x) \\ \dots \\ u^{(m-1)}(x) \end{pmatrix} = \frac{1}{a_m}\big(A_1^*(\Lambda)w_1(x) + \ldots + A_m^*(\Lambda)w_m(x)\big),$$

where $\deg(A_i^*)_j(\Lambda) = m - i$ for all $j = 1, \dots, m$. Therefore, in order for the inclusion to be true

$$u^{(i-1)}(x) = \frac{1}{a_m}\sum_{j=1}^{m}(A_j^*)_i(\Lambda)w_j(x) \in C^{\varepsilon}(\bar{S}), \quad i = 1, \dots, m$$

it is enough to require $w_j \in C^{m-j+\varepsilon}(\bar{S})$, which means that $u^{(i-1)} \in C^{m-1+\varepsilon}(\bar{S})$ is sufficient for $w_j \in C^{2m-j-1+\varepsilon}(\bar{S})$, and this holds, for example, if $\varphi_j \in C^{2m-j-1+\varepsilon}(\partial S)$ [4, Lemma 2.7]. The lemma is proved. □

Remark 3.11. In Corollary 2.2 from Section 2.4, the solvability of the problem $\mathcal{N}_0$ is established with significantly less smoothness of the boundary functions $\varphi_j \in C^{m-1-j+\varepsilon}(\partial S)$, but in this case from the solution $u(x)$ it is required only that $\Lambda^j u|_{\partial S} = \varphi_j$ for $j = 0, \dots, m-1$. This is not the case for harmonic components $u^{(i-1)}(x)$. For example, for $m = 3$ it is clear from (3.4.6) that to include $u^{(i-1)} \in C^{2+\varepsilon}(\bar{S})$ there must be $w_j \in C^{5-j+\varepsilon}(\bar{S})$.

3.4.2 Neumann type problem $\mathcal{N}_k$

Let us now consider the general problem $\mathcal{N}_k$ for the homogeneous polyharmonic equation (3.4.1)-(3.4.2). It can be rewritten in the form

$$\begin{gathered}\Delta^m u = 0,\ x \in S;\\ \Lambda^{[k]}u|_{\partial S} = \varphi_1(x), \dots, \Lambda^{[m+k-1]}u|_{\partial S} = \varphi_m(x),\ x \in \partial S.\end{gathered} \tag{3.4.7}$$

From the equality $(\Lambda + 2m)\Delta^m u = \Delta^m \Lambda u$ it follows that if a function u is m-harmonic in S, then the function Λu is also m-harmonic in S. Consider the m-harmonic function $v = \Lambda^{[k]}u$ in S. With respect to this function we obtain the following boundary value problem

$$\begin{gathered}\Delta^m v = 0, \quad x \in S,\\ v|_{\partial S} = \varphi_1(x), \dots, (\Lambda - k)^{[m-1]}v|_{\partial S} = \varphi_m(x),\ x \in \partial S.\end{gathered} \tag{3.4.8}$$

Lemma 3.22. *The boundary value problem* (3.4.8) *is equivalent to the following Dirichlet problem*

$$\begin{gathered}\Delta^m v = 0, \quad x \in S,\\ v|_{\partial S} = \psi_1(x), \Lambda^{[1]}v|_{\partial S} = \psi_2(x), \dots, \Lambda^{[m-1]}v|_{\partial S} = \psi_m(x),\ x \in \partial S,\end{gathered} \tag{3.4.9}$$

where the functions $\psi_i(x)$, $i = 1, \dots, m$ are found from recurrence equalities

$$\psi_i(x) = \varphi_i(x) - \sum_{j=1}^{i-1} \binom{i-1}{j-1} (-k)^{[i-j]} \psi_j(x). \tag{3.4.10}$$

If $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, then $\psi_i \in C^{2m-i-1+\varepsilon}(\partial S)$ for $i = 1, \dots, m$, which means $v^{(i)} \in C^{m-1+\varepsilon}(\bar{S})$.

Proof. Let $v(x)$ be the solution to problem (3.4.8). Let us consider the ith boundary condition of this problem

$$(\Lambda - k)^{[i-1]} v|_{\partial S} = \varphi_i(x).$$

Using the binomial theorem on factorial powers [15] we write

$$(\Lambda - k)^{[i-1]} = \sum_{j=0}^{i-1} \binom{i-1}{j} (-k)^{[i-1-j]} \Lambda^{[j]}.$$

If we denote $\Lambda^{[j]}v|_{\partial S} = \psi_{j+1}(x)$ and use the resulting equality in the ith boundary condition, shifting the summation index $j \to j-1$, then we find

$$\varphi_i(x) = (\Lambda - k)^{[i-1]}v|_{\partial S} = \sum_{j=0}^{i-1}\binom{i-1}{j}(-k)^{[i-1-j]}\psi_{j+1}(x) = \sum_{j=0}^{i-2}\binom{i-1}{j} \times (-k)^{[i-1-j]}\psi_{j+1}(x) + \psi_i(x) = \sum_{j=1}^{i-1}\binom{i-1}{j-1}(-k)^{[i-j]}\psi_j(x) + \psi_i(x),$$

where $i = 1, \dots, m$. It is easy to see that $\psi_i(x)$ is found from the equations (3.4.10) in a unique way and therefore $v(x)$ is a solution to problem (3.4.9). The converse is also true, since we have

$$\Lambda^{[i-1]} = \sum_{j=0}^{i-1}\binom{i-1}{j}k^{[i-1-j]}(\Lambda - k)^{[j]}.$$

To complete the proof, recall the smoothness of the harmonic components $v^{(i)}(x)$ of the function $v(x)$ from Lemma 3.21. The lemma is proved. □

The Dirichlet problem from Lemma 3.22 is unconditionally solvable and its solution can be found by Lemma 3.21.

Let us study the equation $v = \Lambda^{[k]}u$ with respect to the unknown function $u(x)$ in m-harmonic functions in S. Sufficient conditions for the solvability of this equation are sufficient conditions for the solvability of the problem $\mathcal{N}_k$.

Lemma 3.23. *Let $v(x)$ be a solution to the problem* (3.4.8). *The equation $v(x) = \Lambda^{[k]}u(x)$ is solvable in m-harmonic functions in S if and only if the m-harmonic in S function $v(x)$ has no terms up to the $(k-1)$th order of smallness in its expansion in the neighborhood of the origin. The solution $u(x)$ is unique up to m-harmonic polynomials of degree $k-1$. If $v \in C^{m-1+\varepsilon}(\bar{S})$ and $u(x)$ exists, then $\Lambda^i u \in C^{m-1+\varepsilon}(\bar{S})$ for any $i = 0, \dots, k$.*

Proof. Sufficiency. Let us expand the function $v(x)$ into a series in a neighborhood of the origin $v(x) = \sum_{i=0}^{\infty} v_i(x)$, where $v_i(x)$ are homogeneous m-harmonic polynomials of degree i and the series converges uniformly in some neighborhood of the origin $U \subset S$. If the conditions of the lemma are satisfied, we have $v(x) = \sum_{i=k}^{\infty} v_i(x)$.

Consider the function

$$u(x) = \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{1}{a_i t^{i+1}}\, dt,$$

where $a_i = (-1)^{k-i-1} i!(k-i-1)!$. It is easy to see that for $0 \le i \le k-1$

$$\begin{aligned}(\Lambda - i)\int_0^1 v(tx)\,\frac{dt}{t^{i+1}} &= \int_0^1 D_t v(tx)\,\frac{dt}{t^i} - i\int_0^1 v(tx)\,\frac{dt}{t^{i+1}} = \frac{v(tx)}{t^i}\Big|_0^1 \\ &+ i\int_0^1 v(tx)\,\frac{dt}{t^{i+1}} - i\int_0^1 v(tx)\,\frac{dt}{t^{i+1}} = v(x) - \lim_{t\to+0}\frac{v(tx)}{t^i} \\ &= v(x) - \lim_{t\to+0}\sum_{j=k}^{\infty} t^{j-i} v_j(x) = v(x). \quad (3.4.11)\end{aligned}$$

Let us choose $\mu > 0$ such that $\mu S \subset U$. Since for $0 \le i < k$ the following integral converges

$$\int_0^\mu v(tx)\,\frac{dt}{t^{i+1}} = \int_0^\mu \sum_{j=k}^{\infty} v_j(tx)\,\frac{dt}{t^{i+1}} = \sum_{j=k}^{\infty}\frac{\mu^{j-i}}{j-i} v_j(x),$$

and the integral defining $u(x)$ has no other singularities, then the function $u(x)$ is defined in S. Similar arguments show that the function $u(x)$ is also m-harmonic in S. Let us check whether the equation $\Lambda^{[k]}u = v$ is satisfied in S. We denote

$$\Lambda_i^{[k]} = \Lambda(\Lambda-1)\ldots(\Lambda-i+1)(\Lambda-i-1)\ldots(\Lambda-k+1).$$

By virtue of (3.4.11) we have

$$\begin{aligned}\Lambda^{[k]}u(x) = \Lambda^{[k]}\int_0^1 v(tx)\sum_{i=0}^{k-1}\frac{1}{a_i}\frac{dt}{t^{i+1}} &= \sum_{i=0}^{k-1}\frac{1}{a_i}\Lambda_i^{[k]}(\Lambda - i)\int_0^1 v(tx)\,\frac{dt}{t^{i+1}} \\ &= \sum_{i=0}^{k-1}\frac{1}{a_i}\Lambda_i^{[k]} v(x) \equiv Q_{k-1}(\Lambda)v(x).\end{aligned}$$

Obviously, $Q_{k-1}(\lambda) - 1$ is a real polynomial of degree $k-1$. It has k roots. Indeed, if $0 \le j \le k-1$, then it is easy to see that $\Lambda_i^{[k]}\big|_{\Lambda=j} = a_i\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol and therefore

$$Q_{k-1}(j) - 1 = \sum_{i=0}^{k-1}\frac{a_i\delta_{i,j}}{a_i} - 1 = 0.$$

This means that $Q_{k-1}(\Lambda) - 1 \equiv 0$ and therefore $\Lambda^{[k]}u(x) = v(x)$ in S.

Let us study the smoothness of the functions $\Lambda^i u$. It is easy to see that the equality $\sum_{i=0}^{k-1} i^s/a_i = 0$ is true, for $s = 0, \dots, k-2$ and $\sum_{i=0}^{k-1} i^{k-1}/a_i = 1$. Due to (3.4.11) we can write

$$\begin{aligned}
\Lambda u(x) &= \sum_{i=0}^{k-1} \frac{1}{a_i} \int_0^1 (\Lambda - i) v(tx) \frac{dt}{t^{i+1}} + \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i}{a_i} \frac{dt}{t^{i+1}} \\
&= v(x) \sum_{i=0}^{k-1} \frac{1}{a_i} + \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i}{a_i} \frac{dt}{t^{i+1}} = \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i}{a_i t^{i+1}} dt
\end{aligned}$$

and similarly for $s = 0, \dots, k-1$

$$\begin{aligned}
\Lambda^s u(x) &= \sum_{i=0}^{k-1} \frac{i^{s-1}}{a_i} \int_0^1 (\Lambda - i) v(tx) \frac{dt}{t^{i+1}} + \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i^s}{a_i} \frac{dt}{t^{i+1}} \\
&= v(x) \sum_{i=0}^{k-1} \frac{i^{s-1}}{a_i} + \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i^s}{a_i} \frac{dt}{t^{i+1}} = \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i^s}{a_i t^{i+1}} dt.
\end{aligned}$$

Finally,

$$\Lambda^k u(x) = v(x) + \int_0^1 v(tx) \sum_{i=0}^{k-1} \frac{i^k}{a_i} \frac{dt}{t^{i+1}}.$$

Therefore, if $v \in C^{m-1+\varepsilon}(\bar{S})$, then $\Lambda^i u \in C^{m-1+\varepsilon}(\bar{S})$ for any $i = 0, \dots, k$.

Necessity. Let $u(x)$ be a solution to the equation $\Lambda^{[k]}u = v$ in m-harmonic functions in S and $u(x) = \sum_{i=0}^{\infty} u_i(x)$ is the series expansion of the function $u(x)$ in a neighborhood of the origin $U \subset S$. Then, from the uniqueness of the series expansion of the function $u(x)$ and the property $\Lambda^{[k]}u_i(x) = i^{[k]}u_i(x) = 0$, where $i = 0, \dots, k-1$ we get

$$\Lambda^{[k]}u(x) = \sum_{i=k}^{\infty} \Lambda^{[k]}u_i(x) = \sum_{i=k}^{\infty} v_i(x) = v(x), \quad x \in U,$$

that is $v_i(x) = 0$ for $i = 0, \dots, k-1$. Moreover, the homogeneous polynomials $u_i(x)$, $i \geq k$ are found uniquely from the equality $\Lambda^{[k]}u_i(x) \equiv i^{[k]}u_i(x) = v_i(x)$. Finally, if $u(x)$ is a solution to the equation $v(x) = \Lambda^{[k]}u(x)$, and $R_{k-1}(x)$ is some polynomial of degree $k-1$, then $u(x) + R_{k-1}(x)$ is also a solution to this equation. The lemma is proved. $\square$

3.4.3 Identities for the normal derivatives on a sphere

The solvability conditions for the problem $\mathcal{N}_k$ obtained in Lemma 3.23 are not explicitly related to the boundary functions of the problem. Let us try to obtain necessary conditions for the solvability of the problem $\mathcal{N}_k$ for the homogeneous polyharmonic equation, similar to the necessary conditions for the solvability of the problem $\mathcal{N}_1$. Corollary 3.1 on page 173 proves that if the numbers a_i are found from the equality

$$\lambda(\lambda-2)\cdots(\lambda-2k+2) = a_k\lambda^{[k]} + a_{k-1}\lambda^{[k-1]} + \cdots + a_1\lambda^{[1]},$$

then the k-harmonic in the unit ball S function $u \in C^k(\bar{S})$, satisfies the equality

$$\int_{\partial S}\left(a_1\frac{\partial u}{\partial \nu} + \cdots + a_k\frac{\partial^k u}{\partial \nu^k}\right) ds_x = 0,$$

which means the normal derivatives $\frac{\partial^i u}{\partial \nu^i}$ for $i = 1, \ldots, k$ of any k-harmonic function $u \in C^k(\bar{S})$ cannot take arbitrary values on the sphere ∂S. In addition, since the function $\Lambda u(x)$ for harmonic $u(x)$ is also harmonic, then for $u \in C^k(\bar{S})$ the equality is true

$$\int_{\partial S}\frac{\partial^k u}{\partial \nu^k}\, ds_x = 0, \quad k \in \mathbb{N}.$$

Let us explore what other restrictions similar to those written above exist for polyharmonic functions.

Let $P_m(t) = \sum_{i=0}^m p_i t^i$ be some polynomial of degree m with real coefficients $p_i \in \mathbb{R}$, $i = 0, \ldots, m$. We denote the linear space of such polynomials by $\mathcal{P}$. Consider the factorial power of the variable t of order i in the form $t^{[i]} = t(t-1)\ldots(t-i+1)$, and assume that $t^{[0]} \equiv 1$. We denote the factorial polynomial corresponding to the polynomial $P_m(t)$ as

$$P_{[m]}(t) = \sum_{i=0}^m p_i t^{[i]}. \tag{3.4.12}$$

Consider the linear mapping $\Phi : \mathcal{P} \to \mathcal{P}$, defined as $\Phi[P_m(t)] = P_{[m]}(t)$.

Lemma 3.24. *The mapping $\Phi : \mathcal{P} \to \mathcal{P}$ is an isomorphism.*

Proof. Obviously, Φ is a linear mapping: $\Phi[\lambda t^m + \mu t^k] = \lambda t^{[m]} + \mu t^{[k]} = \lambda\Phi[t^m] + \mu\Phi[t^k]$. Let us prove that Φ is bijective.

Surjectivity. Let the mapping Φ be non-surjective, and let $P_m(t) \in \mathcal{P}$ be a polynomial of the smallest degree without an inverse image. Consider a polynomial of the form $Q_{m-1}(t) = P_m(t) - p_m t^{[m]}$. Obviously, it has degree $m-1$, and therefore, by assumption, has an inverse image $R_{m-1}(t) \in \Phi^{-1}[Q_{m-1}(t)]$, i.e. $\Phi[R_{m-1}(t)] = R_{[m-1]}(t) = Q_{m-1}(t)$. Then

$$P_m(t) = p_m t^{[m]} + Q_{m-1}(t) = p_m t^{[m]} + R_{[m-1]}(t)$$
$$= p_m t^{[m]} + \sum_{k=0}^{m-1} r_k t^{[k]} = \Phi\Big[p_m t^m + \sum_{k=0}^{m-1} r_k t^k\Big],$$

which means the polynomial $P_m(t)$ has an inverse image $p_m t^m + \sum_{k=0}^{m-1} r_k t^k$. Contradiction.

Injectivity. Let $\Phi[P_m(t)] = \Phi[Q_n(t)]$, i.e. $P_{[m]}(t) = Q_{[n]}(t)$ and l is the largest index at which $p_l \neq q_l$, where p_k and q_k are coefficients of polynomials $P_m(t)$ and $Q_n(t)$, respectively. Then the equality is true

$$(p_l - q_l)t^{[l]} = -\sum_{k=0}^{l-1}(p_k - q_k)t^{[k]},$$

but this is impossible, since on the left there is a polynomial of the lth degree with the leading coefficient $p_l - q_l \neq 0$, and on the right there is a polynomial of the $(l-1)$th degree. The lemma is proved. □

Remark 3.12. An explicit form of the mapping Φ is given in (3.4.30).

Theorem 3.13. *In order for any k-harmonic function $u \in C^m(\bar{S})$ to satisfy the equality*

$$\int_{\partial S} P_m\Big(\frac{\partial}{\partial \nu}\Big) u \, ds_x = 0 \tag{3.4.13}$$

it is necessary and sufficient that the polynomial $\Phi[P_m](\lambda) = P_{[m]}(\lambda)$ has roots of the form $\lambda = 2i$ for $i = 0, 1, \ldots, k-1$.

Proof. Necessity. Let

$$\int_{\partial S} P_m\Big(\frac{\partial}{\partial \nu}\Big) u \, ds_x = 0$$

for any k-harmonic function $u \in C^m(\bar{S})$ in S. Due to the property of the operator Λ, we have $\int_{\partial S} P_{[m]}(\Lambda) u \, ds_x = 0$. Let us choose as $u(x)$ the following k-harmonic function $u(x) = |x|^{2i} v(x)$, where $v(x)$ is a function harmonic in S, $v \in C^m(\bar{S})$ and $i = 0, 1, \ldots, k-1$. Then

$$\int_{\partial S} P_{[m]}(\Lambda)\Big(|x|^{2i} v(x)\Big) \, ds_x = 0. \tag{3.4.14}$$

It is easy to see that in S the equality is true

$$(\Lambda-\lambda_1)\Big(|x|^{2i}v(x)\Big)=|x|^{2i}\big((2i-\lambda_1)v(x)+\Lambda v(x)\big), \tag{3.4.15}$$

where λ_1 is some complex number. Then, since the function Λv is harmonic in S, then similarly to (3.4.15) we have

$$\begin{aligned}(\Lambda-\lambda_2)(\Lambda-\lambda_1)&\Big(|x|^{2i}v(x)\Big)\\ &=|x|^{2i}\big((2i-\lambda_2)(2i-\lambda_1)v(x)+\big(2i-\lambda_2+2i-\lambda_1\big)\Lambda v(x)+\Lambda^2 v(x)\big),\end{aligned}$$

where $\lambda_2\in\mathbb{C}$. From here, acting in a similar way, it is easy to obtain

$$\begin{aligned}(\Lambda-\lambda_1)\dots(\Lambda-\lambda_m)&\Big(|x|^{2i}v(x)\Big)\\ &=|x|^{2i}\Big((\lambda-\lambda_1)\dots(\lambda-\lambda_m)v(x)+Q_m^{(i)}(\Lambda)v(x)\Big),\end{aligned}$$

where $Q_m^{(i)}(\lambda)$ is some mth degree polynomial in λ with coefficients depending on $\lambda_1,\dots,\lambda_m$ and i, and such that $Q_m^{(i)}(0)$=0. Let now $\lambda_1,\dots,\lambda_m$ be the roots of the polynomial $P_{[m]}(\lambda)$. Then we have

$$P_{[m]}(\Lambda)\Big(|x|^{2i}v(x)\Big)=|x|^{2i}\Big(P_{[m]}(2i)v(x)+Q_m^{(i)}(\Lambda)v(x)\Big),\quad x\in S, \tag{3.4.16}$$

where $Q_m^{(i)}(\Lambda)v(x)$ is a harmonic function in S. Let S_r be a sphere of radius r with center at the origin. For any $r\in(0,1)$ we have $Q_m^{(i)}(\Lambda)v\in C(\bar{S}_r)$. Then from (3.4.16), after integrating over ∂S_r and applying the mean value theorem, we obtain

$$\begin{aligned}\int_{\partial S_r}&P_{[m]}(\Lambda)\Big(|x|^{2i}v(x)\Big)\,ds_x\\ &=P_{[m]}(2i)r^{2i}\int_{\partial S_r}v(x)\,ds_x+r^{2i}\int_{\partial S_r}Q_m^{(i)}(\Lambda)v(x)\,ds_x\\ &=\omega_n P_{[m]}(2i)v(0)r^{2i+n-1}+r^{2i}\int_{\partial S_r}Q_m^{(i)}(\Lambda)v(x)\,ds_x,\end{aligned} \tag{3.4.17}$$

where ω_n is the area of the unit sphere ∂S. Let us calculate the integral in the resulting equality on the right. By Lemma 3.24 the polynomial $Q_m^{(i)}(\lambda)$ can be written in the form

$$Q_m^{(i)}(\lambda)=\sum_{k=0}^{m}q_k\lambda^{[k]},$$

where $q_0 = Q_m^{(i)}(0) = 0$. The function $w(x) = (\Lambda-1)\ldots(\Lambda-k+1)v(x)$ is harmonic in S, and therefore for $k \geq 1$ we have

$$\int_{\partial S_r} \Lambda^{[k]} v(x)\, ds_x = \int_{\partial S_r} \Lambda w(x)\, ds_x = r \int_{\partial S_r} \frac{\partial w}{\partial \nu}(x)\, ds_x = 0$$

and that is why

$$\int_{\partial S_r} Q_m^{(i)}(\Lambda) v(x)\, ds_x = Q_m^{(i)}(0) \int_{\partial S_r} v(x)\, ds_x = 0. \tag{3.4.18}$$

Therefore, from (3.4.17) we find

$$\int_{\partial S_r} P_{[m]}(\Lambda)\Big(|x|^{2i} v(x)\Big)\, ds_x = \omega_n P_{[m]}(2i) v(0) r^{2i+n-1}.$$

Passing to the limit at $r \to 1-0$ in the resulting equality, taking into account that $v \in C^m(\bar{S})$ and (3.4.14) we obtain

$$0 = \int_{\partial S} P_{[m]}(\Lambda)\Big(|x|^{2i} v(x)\Big)\, ds_x = \omega_n P_{[m]}(2i) v(0).$$

Passing to the limit at $r \to 1-0$ in the resulting equality, taking into account that $v \in C^m(\bar{S})$ and (3.4.14) we obtain

Sufficiency. Let $P_{[m]}(2i) = 0$ for $i = 0, 1, \ldots, k-1$. Then from (3.4.16) it follows that

$$P_{[m]}(\Lambda)\Big(|x|^{2i} v(x)\Big) = |x|^{2i} Q_m^{(i)}(\Lambda) v(x), \quad x \in S$$

for $i = 0, 1, \ldots, k-1$. This means that after integration over ∂S_r, we get

$$\int_{\partial S_r} P_{[m]}(\Lambda)\Big(|x|^{2i} v(x)\Big)\, ds_x = r^{2i} \int_{\partial S_r} Q_m^{(i)}(\Lambda) v(x)\, ds_x. \tag{3.4.19}$$

Consider a k-harmonic function in S of the form

$$u(x) = \sum_{i=0}^{k-1} |x|^{2i} v_i(x) \in C^m(\bar{S}).$$

For it, due to (3.4.19) and (3.4.18), the equalities are true

$$\int_{\partial S_r} P_{[m]}(\Lambda) u(x)\, ds_x = \sum_{i=0}^{k-1} r^{2i} \int_{\partial S_r} Q_m^{(i)}(\Lambda) v_i(x)\, ds_x$$

$$= \sum_{i=0}^{k-1} r^{2i} Q_m^{(i)}(0) \int_{\partial S_r} v_i(x)\, ds_x = 0.$$

Passing to the limit at $r \to 1-0$ and using the condition $u \in C^m(\bar{S})$, we obtain (3.4.13). The theorem is proved. □

Corollary 3.2. For any k-harmonic in S function $u \in C^m(\bar{S})$ the equality holds for $m \geq 2k-1$

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = 0.$$

Proof. Since the polynomial $\lambda^{[m]} = \lambda(\lambda-1)\ldots(\lambda-2k+2)\ldots(\lambda-m+1)$ has at least k roots $0, 2, \ldots, 2k-2$, then the corollary follows from Theorem 3.13. □

The following result generalizes Theorem 3.13.

Theorem 3.14. *Let $P_m(\lambda)$ be some polynomial of degree m and $H_l(x)$ some homogeneous harmonic polynomial of degree l. In order for any k-harmonic function $u \in C^m(\bar{S})$ to satisfy the equality*

$$\int_{\partial S} H_l(x) P_m\Big(\frac{\partial}{\partial \nu}\Big) u\, ds_x = 0 \tag{3.4.20}$$

it is necessary and sufficient that the polynomial $P_{[m]}(\lambda)$ has roots $\lambda = l + 2i$, $i = 0, 1, \ldots, k-1$.

Proof. Necessity. Let $\int_{\partial S} H_l(x) P_m\Big(\frac{\partial}{\partial \nu}\Big) u\, ds_x = 0$ for any k-harmonic in S function $u \in C^m(\bar{S})$ and some homogeneous harmonic polynomial $H_l(x)$ of degree l. Then, due to the property of the operator Λ, we have $\int_{\partial S} H_l(x) P_{[m]}(\Lambda) u\, ds_x = 0$. Let us choose as $u(x)$ a k-harmonic function of the form $u(x) = |x|^{2i} v(x)$, where $i = 0, 1, \ldots, k-1$, and $v(x)$ is a function $v \in C^m(\bar{S})$ harmonic in S. Then, according to Green's formula

$$\int_{\partial S} H_l(x) \Lambda v(x)\, ds_x = l \int_{\partial S} H_l(x) v(x)\, ds_x,$$

or

$$\int_{\partial S} H_l(x)(\Lambda - l) v(x)\, ds_x = 0 \tag{3.4.21}$$

and therefore

$$\int_{\partial S} H_l(x)\Lambda\Big(|x|^{2i}v(x)\Big)\,ds_x = 2i\int_{\partial S} H_l(x)v(x)\,ds_x \\ + \int_{\partial S} H_l(x)\Lambda v(x)\,ds_x = (2i+l)\int_{\partial S} H_l(x)v(x)\,ds_x.$$

Moreover

$$\int_{\partial S} H_l(x)\Lambda^k v(x)\,ds_x = l\int_{\partial S} H_l(x)(\Lambda^{k-1}v(x))\,ds_x = l^k\int_{\partial S} H_l(x)v(x)\,ds_x.$$

Similarly, using Green's formula, we write

$$\int_{\partial S} H_l(x)\Lambda^k\Big(|x|^{2i}v(x)\Big)\,ds_x = \int_{\partial S} H_l(x)\sum_{j=0}^{k}\binom{k}{j}(2i)^j\Lambda^{k-j}v(x)\,ds_x \\ = \sum_{j=0}^{k}\binom{k}{j}(2i)^j l^{k-j}\int_{\partial S} H_l(x)v(x)\,ds_x = \int_{\partial S}(2i+l)^k H_l(x)v(x)\,ds_x$$

and that means

$$0 = \int_{\partial S} H_l(x)P_{[m]}(\Lambda)\Big(|x|^{2i}v(x)\Big)\,ds_x \\ = P_{[m]}(2i+l)\int_{\partial S} H_l(x)v(x)\,ds_x. \quad (3.4.22)$$

If we now choose $v(x) = H_l(x)$, then since $\int_{\partial S} H_l^2(x)\,ds_x \neq 0$, we have $P_{[m]}(2i + l) = 0$ for any $i = 0, 1, \dots, k-1$.

Sufficiency. Let $P_{[m]}(l+2i) = 0$ for $i = 0, 1, \dots, k-1$. Then the equality $P_{[m]}(\lambda) = (\lambda - l - 2i)R^{(i)}(\lambda)$ is true, where $R^{(i)}(\lambda)$ is some polynomial. Let us use the equality (3.4.16) with $P_{[m]}(\lambda) = R^{(i)}(\lambda)$

$$R^{(i)}(\Lambda)\Big(|x|^{2i}v(x)\Big) = |x|^{2i}\Big(R^{(i)}(2i) + Q_m^{(i)}(\Lambda)\Big)v(x) \equiv |x|^{2i}v_i(x),$$

where the function $v_i(x)$ is harmonic in S. Therefore we have

$$P_{[m]}(\Lambda)\Big(|x|^{2i}v(x)\Big) = (\Lambda - l - 2i)R^{(i)}(\Lambda)\Big(|x|^{2i}v(x)\Big) = (\Lambda - l - 2i)\Big(|x|^{2i}v_i(x)\Big).$$

From here, using (3.4.21) we write

$$\int_{\partial S} H_l(x)P_{[m]}(\Lambda)\Big(|x|^{2i}v(x)\Big)\,ds_x = \int_{\partial S} H_l(x)(\Lambda - l - 2i)\Big(|x|^{2i}v_i(x)\Big)\,ds_x$$
$$= \int_{\partial S} H_l(x)\big((2i-l-2i)v_i(x) + \Lambda v_i(x)\big)\,ds_x = 0$$

and therefore, taking into account the property of the operator Λ and the Almansi representation of a polyharmonic function in S, we conclude that the equality (3.4.20) is true. The theorem is proved. □

In connection with the proven theorem, it becomes necessary to consider the following polynomials. Let $l \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$. Let us consider polynomials

$$H_i^{(l)}(t) = (t-l)(t-2-l)\dots(t-2i+2-l)$$

for $i \in \mathbb{N}$. Note that the polynomials $H_s(t) = \frac{1}{2s!!}H_s^{(0)}(t)$ (2.4.24) are used when solving the Dirichlet problem in Section 2.4.

Definition 3.1. We denote the inverse images of $H_i^{(l)}(t)$ under the isomorphism Φ as

$$P_i^{(l)}(t) = \Phi^{-1}[H_i^{(l)}(t)], \quad i \in \mathbb{N},$$

and that means $H_i^{(l)}(t) = P_{[i]}^{(l)}(t)$. We assume that $P_i^{(0)}(t) \equiv P_i(t)$. Then $H_i(t) = P_{[i]}(t)$.

Lemma 3.25. *Let $P_i(t) = \sum_{j=1}^{i} p_j^{(i)} t^j$, then the following recurrence equality is true for the coefficients $p_j^{(i)}$*

$$p_j^{(i)} = p_{j-1}^{(i-1)} + (j-2i+2)p_j^{(i-1)}, \quad 1 \le j \le i, \quad i \ge 2, \tag{3.4.23}$$

where $p_1^{(1)} = 1$ is the initial condition and $p_j^{(i)} = 0$ for $j > i$ and $0 \ge j$ are the boundary conditions for the equation (3.4.23).

Proof. Since $H_i(t) = (t-2i+2)H_{i-1}(t)$, we can write $P_{[i]}(t) = (t-2i+2)P_{[i-1]}(t)$. So

$$\sum_{j=1}^{i} p_j^{(i)} t^{[j]} = (t-2i+2)\sum_{j=1}^{i-1} p_j^{(i-1)} t^{[j]}$$
$$= \sum_{j=1}^{i-1} p_j^{(i-1)} t^{[j+1]} + \sum_{j=1}^{i-1}(j-2i+2)p_j^{(i-1)} t^{[j]} = \sum_{j=2}^{i} p_{j-1}^{(i-1)} t^{[j]}$$

$$+\sum_{j=1}^{i-1}(j-2i+2)p_j^{(i-1)}t^{[j]} = \sum_{j=1}^{i}\Big(p_{j-1}^{(i-1)} + (j-2i+2)p_j^{(i-1)}\Big)t^{[j]}$$

and therefore, by Lemma 3.24, the relation (3.4.23) is true. Here, the condition $p_{j-1}^{(i-1)} = 0$ for $j \le 1$ is first used, and then $p_j^{(i-1)} = 0$ for $j > i-1$. The lemma is proved. □

For polynomials $P_i^{(l)}(t)$ the equality [99, Lemma 7] is true

$$P_i^{(l)}(t) + 2iP_{i-1}^{(l)}(t) = P_i^{(l-2)}(t), \quad i \ge 2, \tag{3.4.24}$$

where we should assume that $P_0^{(l)}(t) = 1$, $P_1^{(l)}(t) = t-l$. In addition, the polynomials $P_i(t)$ and $P_i^{(1)}(t)$ satisfy the recurrence equalities [99, Lemma 4 and (36)]

$$\begin{gathered} P_i(t) + (2i-3)P_{i-1}(t) = t^2P_{i-2}(t), \\ P_i^{(1)}(t) + (2i-1)P_{i-1}^{(1)}(t) = t^2P_{i-2}^{(1)}(t),\ i \ge 2. \end{gathered} \tag{3.4.25}$$

and $P_{i+1}(t) = tP_i^{(1)}(t)$, $i \in \mathbb{N}$ [99, equality (34)].

Definition 3.2. Linear span of an infinite system of polynomials

$$\{P_m^{(l)}(t), P_{m+1}^{(l)}(t), P_{m+2}^{(l)}(t), \dots\},$$

starting with the polynomial $P_m^{(l)}(t)$ for $m \in \mathbb{N}$ we denote it by $\mathcal{L}_m^{(l)}$, i.e.

$$\mathcal{L}_m^{(l)} = \operatorname{lin}\{P_m^{(l)}(t), P_{m+1}^{(l)}(t), P_{m+2}^{(l)}(t), \dots\}. \tag{3.4.26}$$

We place the coefficients of the polynomials $P_i(t)$, $l \in \mathbb{N}$ (they have no free terms) from left to right from 1 to i in increasing index order in the ith row of the next integer triangle, which we call the Neumann triangle, since it plays an important role in the study of problems $\mathcal{N}_k$

$$\mathbb{P} = \begin{array}{c} 1 \\ -1 \quad 1 \\ 3 \quad -3 \quad 1 \\ -15 \quad 15 \quad -6 \quad 1 \\ 105 \quad -105 \quad 45 \quad -10 \quad 1 \\ \dots \\ \dots \quad p_j^{(i)} = p_{j-1}^{(i-1)} + (j-2i+2)p_j^{(i-1)} \quad \dots \end{array}, \tag{3.4.27}$$

where $p_j^{(i)}$ for $1 \le j \le i$, $i \in \mathbb{N}$ is the element of the ith row of the Neumann triangle $\mathbb{P}$ standing on jth place on the left and $p_1^{(1)} = 1$ is the initial condition, and $p_j^{(i)} = 0$ for $j > i$ and $0 \ge j$ are the boundary conditions for the given recurrence equation.

Remark 3.13. Rows of the triangle $\mathbb{P}$, which arose from the identities on the sphere for normal derivatives of polyharmonic functions (Theorem 3.14) and the integer triangle B from (3.2.18), which arose in the study of the Neumann problem using Theorems 3.5 and 1.2, coincide up to a common factor $(-1)^{i+1}$, where i is the number of row.

Indeed, as is easy to notice, the elements of triangle B and triangle $\mathbb{P}$ are related by the equality $p_j^{(i)} = (-1)^{i+1} b_i^j$. This means that $\mathbb{P}$ is obtained from B by multiplying even rows by -1, i.e. triangle $\mathbb{P}$ is a differently normalized triangle B. Therefore, from Lemma 3.15 it follows that

$$p_j^{(i)} = (-1)^{i+j} \binom{2i-j-1}{j-1} \frac{(2i-2j+1)!!}{2i-2j+1}, \tag{3.4.28}$$

and also that the degree of the polynomial $P_i(t)$ is equal to i, and the lowest degree of the monomials included in $P_i(t)$ is equal to 1 (there are no free terms).

In the following presentation, it is more convenient to work with the triangle $\mathbb{P}$ than with the triangle B. By virtue of the definition of the polynomials $P_i(t)$ and the factorial polynomial, the following equality is true

$$P_{[i]}(t) \equiv \sum_{j=1}^{i} p_j^{(i)} t^{[j]} = (t, -2)_i \equiv t(t-2)\dots(t-2i+2), \tag{3.4.29}$$

where $t^{[i]} = t(t-1)\dots(t-i+1)$ is the factorial degree of t, and $(a,b)_i = a(a+b)\cdots\big(a+(i-1)b\big)$ is the generalized Pochhammer symbol with the agreement $(a,b)_0 = 1$. Obviously, $t^{[i]} = (t,-1)_i$.

Remark 3.14. If we recall the polynomials $h_{m,i}(\lambda)$ from (2.4.14) and represent them in the form

$$h_{m,i}(\lambda) = \sum_{j=0}^{m-1} h_{m,i}^{(j)} \lambda^{[j]}, \quad i = 0, \dots, m-1,$$

then the coefficients $h_{m,i}^{(j)}$ form the matrix $\mathbb{H}_m$ from Lemma 2.19 of the form (2.4.22). Theorem 3.4 on page 174 on the value of $\Delta^k u(0)$ for an m-harmonic in S function $u(x)$ involves the coefficients $h_{m,k}^{(j)}$ of polynomials $H_{m-1}^{(k)}(\lambda) \equiv h_{m,k}(\lambda)$. Since $p_j^{(m-1)} = h_{m,m-1}^{(j)}$, then by Remark 3.13 the last row of the matrix $\mathbb{H}_m$ contains, up to a constant multiplier coefficients from the solvability conditions of the Neumann problem for the homogeneous $(m-1)$-harmonic equation.

In accordance with the Neumann triangle $\mathbb{P}$ from (3.4.27) several polynomials $P_i(t)$ can be written explicitly

$$P_1(t) = t, \quad P_2(t) = -t + t^2, \quad P_3(t) = 3t - 3t^2 + t^3,$$
$$P_4(t) = -15t + 15t^2 - 6t^3 + t^4,\ P_5(t) = 105t - 105t^2 + 45t^3 - 10t^4 + t^5.$$

Note that polynomials $P_m(t)$ can be obtained using Stirling numbers of the 2nd kind $S(m,k)$ from the triangle (1.1.3). For example,

$$P_m(t) = \sum_{k=0}^{m} p_k t^k \Rightarrow P_{[m]}(t) = \sum_{k=0}^{m}\Big(\sum_{i=k}^{m} S(i,k)p_i\Big)t^{[k]}. \tag{3.4.30}$$

Let $P_i^{(l)}(t) = \sum_{j=0}^{i} a_j^{(i,l)} t^j$, then for the coefficients $a_j^{(i,l)}$ the following recurrent equality [99, Lemma 5] is true

$$a_j^{(i,l)} = a_{j-1}^{(i-1,l)} + (j - 2i + 2 - l)a_j^{(i-1,l)}, \quad 0 \le j \le i, \quad i \ge 2,$$

where $a_0^{(1,l)} = -l$, $a_1^{(1,l)} = 1$ are initial conditions and $a_j^{(i,l)} = 0$ for $j > i$ and $0 > j$ are boundary conditions. Based on this, it is not difficult to write the first four polynomials $P_i^{(l)}(t)$, $l \in \mathbb{N}_0$. They look like

$$\begin{gathered} P_0^{(l)}(t) = l,\ P_1^{(l)}(t) = -l + t,\ P_2^{(l)}(t) = l(l+2) - (2l+1)t + t^2, \\ P_3^{(l)}(t) = -l(l+2)(l+4) + 3(l^2 + 3l + 1)t - 3(l+1)t^2 + t^3. \end{gathered} \tag{3.4.31}$$

The polynomials $P_i^{(l)}(t)$ play an important role in the results obtained below. In [99, Corollary 3], based on the results from [85], the following statement is established

Lemma 3.26. *Let m-harmonic in S function $u \in C^p(\bar{S})$ satisfy on ∂S the equalities*

$$Q_i\Big(\frac{\partial}{\partial \nu}\Big)u|_{\partial S} = \varphi_i(s), \quad i = 1, \dots, q, \tag{3.4.32}$$

where $Q_i(t)$ are some polynomials over $\mathbb{R}$ and $p = \max_i \deg Q_i(t)$. Then if

$$\mathcal{L}_Q^{(l)} \equiv \mathcal{L}_m^{(l)} \cap \operatorname{lin}\{Q_1(t), \dots, Q_q(t)\} \neq \{0\},$$

then the functions $\varphi_1(s), \dots, \varphi_q(s)$ must satisfy the equalities

$$\int_{\partial S} H_l(x) \sum_{i=1}^{q} \alpha_i^{(j)} \varphi_i(x)\, ds_x = 0, \quad j = 1, \dots, r,$$

where $r = \dim \mathcal{L}_Q^{(l)}$, the numbers $\alpha_i^{(j)}$ are such that the polynomials $S_j(t) = \sum_{i=1}^{q} \alpha_i^{(j)} Q_i(t)$ for $j = 1, \dots, r$ form a basis in $\mathcal{L}_Q^{(l)}$, and $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l.

Corollary 3.3. If under the conditions of Lemma 3.26

$$R(t) = \sum_{i=1}^{q} \alpha_i Q_i(t) \in \mathcal{L}_Q^{(l)},$$

then the functions $\varphi_i(s)$ must satisfy the equality

$$\int_{\partial S} H_l(x) \sum_{i=1}^{q} \alpha_i \varphi_i(s)\, ds_x = 0. \tag{3.4.33}$$

Proof. Indeed, if the polynomials $R_j(t) = \sum_{i=1}^{q} \alpha_i^{(j)} Q_i(t)$ for $j = 1, \dots, r$ form a basis in $\mathcal{L}_Q^{(l)}$ and $R(t) = \sum_{j=1}^{r} \beta_j R_j(t) \in \mathcal{L}_Q^{(l)}$, then by the property of the basis

$$\sum_{j=1}^{r} \beta_j R_j(t) = \sum_{j=1}^{r} \beta_j \sum_{i=1}^{q} \alpha_i^{(j)} Q_i(t) = \sum_{i=1}^{q} Q_i(t) \sum_{j=1}^{r} \alpha_i^{(j)} \beta_j \equiv \sum_{i=1}^{q} \alpha_i Q_i(t)$$

and therefore

$$\begin{aligned} \int_{\partial S} H_l(x) \sum_{i=1}^{q} \alpha_i \varphi_i(s)\, ds_x &= \int_{\partial S} H_l(x) \sum_{i=1}^{q} \varphi_i(s) \sum_{j=1}^{r} \alpha_i^{(j)} \beta_j\, ds_x \\ &= \int_{\partial S} H_l(x) \sum_{j=1}^{r} \beta_j \sum_{i=1}^{q} \alpha_i^{(j)} \varphi_i(s)\, ds_x = 0, \end{aligned}$$

which proves (3.4.33). □

In addition, we use one more statement from [99, Theorem 5], which is based on the recurrent equalities (3.4.24) and (3.4.25).

Lemma 3.27. *Let $l \in \mathbb{N}$ and $P_i(t) \equiv P_i^{(0)}(t)$, then the equality holds*

$$\mathcal{L}_m^{(2l)} = \operatorname{lin}\{P_m^{(2l)}(t), P_{m+1}^{(2l-2)}(t), \dots, P_{m+l}(t), t^2 P_{m+l-1}(t), \\ t^4 P_{m+l-2}(t), \dots, t^{2l-2} P_{m+1}(t), t^{2l} P_m(t), t^{2l+2} P_{m-1}(t), \\ \dots, t^{2m+2l-2} P_1(t), t^{2m+2l}, t^{2m+2l+1}, \dots\}, \quad (3.4.34)$$

and taking into account (3.4.25) *we have*

$$\mathcal{L}_m^{(2l-1)} = \operatorname{lin}\{P_m^{(2l-1)}(t), P_{m+1}^{(2l-3)}(t), \dots, P_{m+l-1}^{(1)}(t), t^2 P_{m+l-2}^{(1)}(t), \\ t^4 P_{m+l-3}^{(1)}(t), \dots, t^{2l-2} P_m^{(1)}(t), t^{2l} P_{m-1}^{(1)}(t), t^{2l+2} P_{m-2}^{(1)}(t), \\ \dots, t^{2m+2l-2} P_0^{(1)}(t), t^{2m+2l-1}, t^{2m+2l}, \dots\}. \quad (3.4.35)$$

Example 3.8. Let the biharmonic in S function $u \in C^3(\bar{S})$ satisfy the equalities on the unit sphere ∂S

$$\frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = \varphi_1(s), \quad \frac{\partial^3 u}{\partial \nu^3}\Big|_{\partial S} = \varphi_2(s).$$

Following the notations of Lemma 3.26, $m = 2$, $q = 2$, $Q_1(t) = t^2$ and $Q_2(t) = t^3$. According to Lemma 3.27 it is easy to see that

$$\mathcal{L}_Q^{(l)} = \{\alpha_2 P_2^{(l)}(t) + \alpha_3 P_3^{(l)}(t) : (\alpha_2, \alpha_3) \in I_Q \subset \mathbb{R}^2\},$$

where the set I_Q is such that the polynomial $\alpha_2 P_2^{(l)}(t) + \alpha_3 P_3^{(l)}(t)$ does not contain monomials of zero and first degrees since they do not exist in $\operatorname{lin}\{Q_1(t), Q_2(t)\}$. Using polynomials (3.4.31) we find

$$\alpha_2 P_2^{(l)}(t) + \alpha_3 P_3^{(l)}(t) = l(l+2)(\alpha_2 - (l+4)\alpha_3) \\ + (-(2l+1)\alpha_2 + 3\alpha_3(l^2 + 3l + 1))t + (\alpha_2 - 3(l+1)\alpha_3)t^2 + \alpha_3 t^3.$$

Therefore, the vector $(\alpha_2, \alpha_3) \in I_Q$ must be such that

$$l(l+2)(\alpha_2 - (l+4)\alpha_3) = 0, \; -(2l+1)\alpha_2 + 3\alpha_3(l^2 + 3l + 1) = 0. \quad (3.4.36)$$

1. Let $l = 0$, then from the second equation we find $\alpha_2 = 3\alpha_3$ and therefore

$$\mathcal{L}_Q^{(0)} = \{\alpha_2 P_2^{(0)}(t) + \alpha_3 P_3^{(0)}(t) : \alpha_2 = 3\alpha_3\} = \{\alpha_3 t^3 : \alpha_3 \in \mathbb{R}\} = \mathcal{L}_Q.$$

2. Let $l \in \mathbb{N}$, then since the determinant of the homogeneous system (3.4.36) in this case is reduced to the form

$$\Delta = \begin{vmatrix} 1 & -l-4 \\ -2l-1 & 3l^2 + 9l + 3 \end{vmatrix} = 3l^2 + 9l + 3 - (2l+1)(l+4) = l^2 - 1,$$

then for $l \in \mathbb{N}$ and $l \neq 1$ we have $\mathcal{L}_Q^{(l)} = \{0\}$. If $l = 1$, then from the first equation (3.4.36) we find $\alpha_2 = 5\alpha_3$ and therefore

$$\mathcal{L}_Q^{(1)} = \{\alpha_2 P_2^{(1)}(t) + \alpha_3 P_3^{(1)}(t) : \alpha_2 = 5\alpha_3\} = \{\alpha_3(-t^2 + t^3) : \alpha_3 \in \mathbb{R}\},$$

since according to (3.4.31)

$$5\alpha_3 P_2^{(1)}(t) + \alpha_3 P_3^{(1)}(t) \\ = \alpha_3(15 - 15t + 5t^2 - 15 + 15t - 6t^2 + t^3) = \alpha_3(-t^2 + t^3).$$

Hence $r = 1$. The basis in $\mathcal{L}_Q^{(1)}$ can be taken in the form $\{t^3 - t^2\}$, and therefore, since $S_1(t) = t^3 - t^2 = Q_2(t) - Q_1(t)$, then $\alpha_1^{(1)} = -1$, $\alpha_2^{(1)} = 1$ and by virtue of Lemma 3.26 we obtain one condition ($r = 1$) of the form

$$\int_{\partial S} H_1(x)(\varphi_2(x) - \varphi_1(x))\, ds_x = 0,$$

where $H_1(x) = \sum_{i=1}^{n} \lambda_i x_i$ is an arbitrary homogeneous harmonic polynomial of the 1st degree. This necessary condition for the existence of a solution to the Neumann problem is also obtained in [170].

There are no other identities obtained from Lemma 3.26 for biharmonic functions satisfying the specified conditions on the boundary.

Let us consider a more complex problem.

Theorem 3.15. *Let the functions $\varphi_i(s)$ for $i = 1, 2, \ldots, m$ be such that a solution $u(x)$ to the problem $\mathcal{N}_k$ for the homogeneous polyharmonic equation* (3.4.1) *exists and it is such that $u \in C^{k+m-1}(\bar{S})$. Then, for any $l \in \mathbb{N}_0$ such that $l < k$ the following orthogonality conditions must be satisfied*

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{\delta_\lambda+1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)}\varphi_{m-\sigma_\lambda}(s)\Big)\, ds_x = 0, \\ \lambda = \lambda_0, \ldots, \min\{k-l, m\} - 1, \tag{3.4.37}$$
$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = 0, \quad i = \max\{2m + l - k, 0\} + 1, \ldots, m,$$

where $\lambda_0 = [(k-l)/2]$, $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l, $p_j^{(i)}$ are the coefficients of the polynomial $P_i(t)$ from (3.4.27)*, having the form* (3.4.28)*,*

$$\delta_\lambda = 2\lambda - k + l + 1, \quad \sigma_\lambda = k - l - \lambda - 1. \tag{3.4.38}$$

The number of conditions (3.4.37) *for* $l < k$ *is equal to*

$$N_{m,k,l} = \min\{k - l - \lambda_0, m\} = \min\{[(k - l + 1)/2], m\}.$$

In the formulation of Theorem 3.15 one should keep in mind the following generally accepted convention for formulas with indices: if in a formula the lower limit of the index change area becomes greater than the upper limit, then this formula should not be taken into account. For example, the first formula in (3.4.37) is valid when $k > l$ and $[(k - l)/2] + 1 \le m$, and the second for $k > m + l$.

Proof. Let us fix a number $k \in \mathbb{N}$, which we call the index of problem $\mathcal{N}_k$, and let a solution to this problem exist. Let us apply Lemmas 3.26 and 3.27 to the problem $\mathcal{N}_k$. Let us represent the conditions (3.4.2) in the form (3.4.32). To do this, we set $Q_i(t) = t^{k+i-1}$ for $i = 1, \dots, m$ and $q = m$. In addition, we denote

$$\mathcal{D}_k = \mathrm{lin}\{t^k, t^{k+1}, \dots, t^{m+k-1}\}, \quad \mathcal{L}^{(l)}_{\mathcal{N}_k} \equiv \mathcal{L}^{(l)}_Q = \mathcal{L}^{(l)}_m \cap \mathcal{D}_k.$$

Since polynomials from the system $\mathcal{D}_k$ have maximum degree $m + k - 1$, then

$$\mathcal{L}^{(l)}_{\mathcal{N}_k} = \mathcal{D}_k \cap \mathrm{lin}\{P^{(l)}_m(t), P^{(l)}_{m+1}(t), P^{(l)}_{m+2}(t), \dots, P^{(l)}_{m+k-1}(t)\}.$$

Let us study the set $\mathcal{L}^{(l)}_{\mathcal{N}_k}$ for various relations between k and l.

1^0. Let $k \le l$. We us prove an auxiliary statement.

Lemma 3.28. *For* $k \le l$ *the equality* $\mathcal{L}^{(l)}_{\mathcal{N}_k} = \{0\}$ *holds true.*

Proof. Let for some $\alpha_m, \dots, \alpha_{k+m-1} \in \mathbb{R}$

$$\hat{P}_{k+m-1}(t) \equiv \alpha_m P^{(l)}_m(t) + \ldots + \alpha_{k+m-1} P^{(l)}_{k+m-1}(t) \in \mathcal{D}_k.$$

Let us apply the transformation Φ from (3.4.12) to the polynomial $\hat{P}_{k+m-1}(t)$. Then we get

$$\hat{P}_{[k+m-1]}(t) \in \mathrm{lin}\{t^{[k]}, t^{[k+1]}, \dots, t^{[m+k-1]}\}$$

and therefore the polynomial $\hat{P}_{[k+m-1]}(t)$ must have roots $0, 1, \dots, k - 1$. Due to the equality $P^{(l)}_{[i]}(t) = H^{(l)}_i(t) = (t - l, -2)_i$ we have

$$\hat{P}_{[k+m-1]}(t) = \alpha_m (t - l, -2)_m + \ldots + \alpha_{k+m-1}(t - l, -2)_{k+m-1}.$$

It follows that the coefficients $\alpha_m, \dots, \alpha_{k+m-1}$ must satisfy the following system of homogeneous equations

$$\begin{pmatrix} (0-l,-2)_m & (0-l,-2)_{m+1} & \dots & (0-l,-2)_{k+m-1} \\ (1-l,-2)_m & (1-l,-2)_{m+1} & \dots & (1-l,-2)_{k+m-1} \\ \dots & & \dots & \\ (k-1-l,-2)_m & (k-1-l,-2)_{m+1} & \dots & (k-1-l,-2)_{k+m-1} \end{pmatrix} \times \begin{pmatrix} \alpha_m \\ \alpha_{m+1} \\ \dots \\ \alpha_{k+m-1} \end{pmatrix} = 0. \quad (3.4.39)$$

Let us denote the determinant of this system as Δ_k and calculate it. Then since $(a,b)_{i+j} = (a,b)_i(a+ib)_j$ we have

$$\Delta_k = (-l,-2)_m(1-l,-2)_m \dots (k-1-l,-2)_m \times \begin{vmatrix} 1 & (-l-2k,-2)_1 & \dots & (-l-2k,-2)_{k-1} \\ 1 & (1-l-2k,-2)_1 & \dots & (1-l-2k,-2)_{k-1} \\ \dots & & \dots & \\ 1 & (k-1-l-2k,-2)_1 & \dots & (k-1-l-2k,-2)_{k-1} \end{vmatrix}.$$

If we denote $\lambda = -l-2k$ and

$$C_k = (-l,-2)_m(1-l,-2)_m \dots (k-1-l,-2)_m \neq 0,$$

then we get

$$\Delta_k = C_k \begin{vmatrix} 1 & (\lambda,-2)_1 & \dots & (\lambda,-2)_{k-1} \\ 1 & (1+\lambda,-2)_1 & \dots & (1+\lambda,-2)_{k-1} \\ \dots & & \dots & \\ 1 & (k-1+\lambda,-2)_1 & \dots & (k-1+\lambda,-2)_{k-1} \end{vmatrix}.$$

Multiply the $(k-1)$th column of the determinant by $\lambda-2(k-2)$ and subtract it from the kth column, then multiply the $(k-2)$th column of the determinant by $\lambda-2(k-3)$ and subtract it from the $(k-1)$th column, etc. and finally multiply the first column by λ and subtract it from the 2th column, then expand the resulting determinant on the first row and take out the factor i from the ith row of the resulting determinant

$$\Delta_k = C_k \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & (1+\lambda,-2)_{k-2} \\ \dots & & \dots & \\ 1 & k-1 & \dots & (k-1)(k-1+\lambda,-2)_{k-2} \end{vmatrix}$$

$$= (k-1)!C_k \begin{vmatrix} 1 & (1+\lambda,-2)_1 & \dots & (1+\lambda,-2)_{k-2} \\ 1 & (2+\lambda,-2)_1 & \dots & (2+\lambda,-2)_{k-2} \\ \dots & & \dots & \\ 1 & (k-1+\lambda,-2)_1 & \dots & (k-1+\lambda,-2)_{k-2} \end{vmatrix}.$$

The resulting determinant differs from the previous one by replacing $\lambda+1$ with λ and reducing the dimension by 1, and therefore by induction

$$\Delta_k = (k-1)!(k-2)!\dots 2!C_k \begin{vmatrix} 1 & (k-2+\lambda,-2)_1 \\ 1 & (k-1+\lambda,-2)_1 \end{vmatrix} \neq 0.$$

Therefore, for $k \leq l$ the system (3.4.39) has only a trivial solution $\alpha_m = \dots = \alpha_{k+m-1} = 0$, and therefore $\hat{P}_{k+m-1}(t) = 0$. The lemma is proved. □

From Lemma 3.28 it follows that the problem $\mathcal{N}_k$ for $k \leq l$ does not have necessary solvability conditions of the form (3.4.33).

2^0. Let $l+1 \leq k \leq m+l$ or $1 \leq k-l \leq m$. Let us first consider the case $k = l+1$. Note that the set $\mathcal{L}_m^{(l)}$ from (3.4.26), by Lemma 3.27, can also be found using the equalities (3.4.34) and (3.4.35), which, as is easy to see, can be uniformly written in the form

$$\begin{aligned} \mathcal{L}_m^{(l)} = \operatorname{lin}\{&P_m^{(l)}(t), P_{m+1}^{(l-2)}(t), \dots, P_{m+[l/2]-1}^{(2)}(t), P_{m+[l/2]}^{(\theta_l)}(t), t^2 P_{m+[l/2]-1}^{(\theta_l)}(t), \\ &\dots, t^{2[l/2]-2}P_{m+1}^{(\theta_l)}(t), t^{2[l/2]}P_m^{(\theta_l)}(t), t^{2[l/2]+2}P_{m-1}^{(\theta_l)}(t), \\ &\dots, t^{2m+2[l/2]-2}P_1^{(\theta_l)}(t), t^{2m+2[l/2]}, \dots\}. \end{aligned} \quad (3.4.40)$$

where $\theta_l = 0$ if l is even and $\theta_l = 1$ if l is odd.

Corollary 3.4. Nontrivial linear combination of l polynomials of the form

$$P_m^{(l)}(t), P_{m+1}^{(l-2)}(t), \dots, P_{m+[l/2]}^{(0)}(t), t^2 P_{m+[l/2]-1}^{(0)}(t), \dots, t^{2[l/2]-2}P_{m+1}^{(0)}(t)$$

for even l and polynomials of the form

$$P_m^{(l)}(t), P_{m+1}^{(l-2)}(t), \dots, P_{m+[l/2]}^{(1)}(t), t^2 P_{m+[l/2]-1}^{(1)}(t), \dots, t^{2[l/2]}P_m^{(1)}(t)$$

for odd l cannot have $t = 0$ as root of order l.

Indeed, from the recurrent equalities (3.4.24), (3.4.25) and Lemma 3.28 for $k = 2[l/2] = l$, when l is even, it follows that if

$$\alpha_0 P_m^{(l)}(t) + \alpha_1 P_{m+1}^{(l-2)}(t) \ldots + \alpha_{2[l/2]-1} t^{2[l/2]-2} P_{m+l-1}^{(0)}(t)$$
$$= \hat{\alpha}_0 P_m^{(l)}(t) + \hat{\alpha}_1 P_{m+1}^{(l)}(t) + \ldots + \hat{\alpha}_{2[l/2]-1} P_{m+2[l/2]-1}^{(l)}(t) \in \mathcal{D}_l,$$

then $\hat{\alpha}_0 = \ldots = \hat{\alpha}_{2[l/2]-1} = 0 \Rightarrow \alpha_0 = \ldots = \alpha_{2[l/2]-1} = 0$. A similar statement is true when l is odd. In this case $k = 2[l/2] + 1 = l$. The corollary is proved.

Let l be even. Since polynomials from $\mathcal{D}_k$ have maximum degree $m + k - 1$, then from (3.4.40) we find

$$\mathcal{L}_{\mathcal{N}_{l+1}}^{(l)} = \mathcal{D}_{l+1} \cap \operatorname{lin}\{P_m^{(l)}(t), P_{m+1}^{(l-2)}(t),$$
$$\ldots, t^{2[l/2]-2} P_{m+1}^{(0)}(t), t^{2[l/2]} P_m^{(0)}(t)\}. \quad (3.4.41)$$

Further, it is easy to notice that

$$t^{2[l/2]} P_m^{(0)}(t) \in \mathcal{D}_{l+1} = \operatorname{lin}\{t^{l+1}, t^{l+2}, \ldots, t^{m+l}\},$$

since the degree of the polynomial $P_m(t)$ is equal to m, and the lowest degree of the monomials included in $P_m(t)$ is equal to 1. Now let us use Corollary 3.4. Since linear combinations of the first l polynomials from (3.4.41) cannot be included in $\mathcal{D}_{l+1}$, then we have $\mathcal{L}_{\mathcal{N}_{l+1}}^{(l)} = \operatorname{lin}\{t^{2[l/2]} P_m^{(0)}(t)\} = \operatorname{lin}\{t^l P_m(t)\}$.

If l is odd, then similar to the case of even l

$$\mathcal{L}_{\mathcal{N}_{l+1}}^{(l)} = \mathcal{D}_{l+1} \cap \operatorname{lin}\{P_m^{(l)}(t), P_{m+1}^{(l-2)}(t), \ldots, t^{2[l/2]} P_m^{(1)}(t), t^{2[l/2]+2} P_{m-1}^{(1)}(t)\}.$$

Due to the fact that by Corollary 3.4 linear combinations of the first l polynomials above cannot be included in $\mathcal{D}_{l+1}$ and since

$$t^{2[l/2]+2} P_{m-1}^{(1)}(t) \in \mathcal{D}_{l+1} = \operatorname{lin}\{t^{l+1}, t^{l+2}, \ldots, t^{m+l}\},$$

then we have an equality similar to the case of even l

$$\mathcal{L}_{\mathcal{N}_{l+1}}^{(l)} = \operatorname{lin}\{t^{2[l/2]+2} P_{m-1}^{(1)}(t)\} = \operatorname{lin}\{t^{l+1} P_{m-1}^{(1)}(t)\} = \operatorname{lin}\{t^l P_m(t)\}.$$

Due to the obtained equality, by Corollary 3.3 the condition must be satisfied

$$\int_{\partial S} H_l(x) \sum_{i=1}^{m} p_i^{(m)} \varphi_i(s)\, ds_x = 0,$$

where $p_i^{(m)}$ are the coefficients of the polynomial $P_m(t)$, having the form (3.4.28).

Let $1 \le k - l \le m$. As shown above, in the chain of polynomials from (3.4.40) for any l there is a polynomial of the form $t^l P_m(t)$. Therefore, taking into account the structure of the polynomials from (3.4.40), by virtue of Corollary 3.4, and taking into account that $k + m - 1 \le 2m + l - 1$, we can write

$$\mathcal{L}_{\mathcal{N}_k}^{(l)} = \mathcal{D}_k \cap \mathrm{lin}\{t^l P_m(t), t^{l+2} P_{m-1}(t), \dots, t^{2m+l-2} P_1(t)\}. \qquad (3.4.42)$$

For example, the polynomial from (3.4.40) next in the chain after $t^l P_m(t)$, in the case of even l, is $t^{l+2} P_{m-1}(t)$, and, in the case of odd l, it has the same form $t^{2[l/2]+4} P_{m-2}^{(\theta_l)}(t) = t^{l+3} P_{m-2}^{(1)}(t) = t^{l+2} P_{m-1}(t)$. Therefore, let us find all polynomials of the form $t^{l+2\lambda} P_{m-\lambda}(t)$, where $0 \le \lambda \le m-1$ from the linear span above, which also appear in $\mathcal{D}_k$. The lowest degree of monomials included in $t^{l+2\lambda} P_{m-\lambda}(t)$ is equal to $l + 2\lambda + 1$, and the highest degree of these monomials is equal to $l + 2\lambda + m - \lambda = l + \lambda + m$. Therefore, if the inequalities $l + 2\lambda + 1 < k$ or $k + m - 1 < l + \lambda + m$ are satisfied, which is equivalent to $k - 1 < l + \lambda$, then the polynomial $t^{l+2\lambda} P_{m-\lambda}(t)$ cannot be a linear combination of monomials from $\mathcal{D}_k$, and therefore cannot be included in $\mathcal{D}_k$. Therefore, the condition for the polynomial $t^{l+2\lambda} P_{m-\lambda}(t)$ to belong to the set $\mathcal{D}_k$ has the form $l+2\lambda+1 \ge k$ and $k+m-1 \ge l+\lambda+m$ or $l+\lambda \le k-1 \le l+2\lambda$, where $0 \le \lambda \le m-1$, i.e.

$$t^{l+2\lambda} P_{m-\lambda}(t) \in \mathcal{D}_k \Leftrightarrow \lambda \le k - l - 1 \le 2\lambda, \quad 0 \le \lambda \le m - 1. \qquad (3.4.43)$$

2.1^0. Suppose that the index k is additionally such that $[(k-l)/2] \le m-1$, where $[a]$ is the integer part of a number a. Then we denote $\lambda_0 = [(k-l)/2]$. With such $\lambda = \lambda_0$ the inequality $k-l-1 \le 2\lambda_0 = 2[(k-l)/2]$ is holds (for odd $k-l$ we have equality, and for even $k - l$ we have inequality), and another inequality $\lambda_0 = [(k - l)/2] \le k - l - 1$ is also holds. Therefore, the conditions from (3.4.43) are met. Therefore, for $\lambda_0 = [(k - l)/2]$ the polynomial $t^{l+2\lambda_0} P_{m-\lambda_0}(t) = t^{l+2[(k-l)/2]} P_{m-[(k-l)/2]}(t)$ is included in $\mathcal{D}_k$. If we consider the value $\lambda = \lambda_0 - 1 = [(k - l)/2] - 1$, then the inequality $k - l - 1 > 2\lambda = 2[(k - l)/2] - 2$ (or equivalent to it inequality $k-l+1 > 2[(k-l)/2]$) holds, i.e. conditions (3.4.43) are not satisfied, which means the polynomial $t^{l+2\lambda_0-2} P_{m-\lambda_0+1}(t)$ is not included in $\mathcal{D}_k$. Thus, moving along the system of polynomials in (3.4.42) from left to right, we see that the polynomial

$$t^{l+2\lambda_0} P_{m-\lambda_0}(t) = t^{2[(k-l)/2]} P_{m-[(k-l)/2]}(t)$$

is the first polynomial from this system belonging to $\mathcal{D}_k$. We take the polynomial $t^{l+2\lambda_0} P_{m-\lambda_0}(t)$ to be the first polynomial for the basis in $\mathcal{L}_{\mathcal{N}_k}^{(l)}$. For $k = l$ we get

$\lambda_0 = 0$. Let us find the coefficients $\alpha_i^{(l)}$ in (3.4.33) for the polynomial $t^{l+2\lambda_0} P_{m-\lambda_0}(t)$. Since $Q_i(t) = t^{k+i-1}$ for $i = 1, \dots, m$, then consider the equality

$$t^{l+2\lambda_0} P_{m-\lambda_0}(t) = \sum_{i=1}^{m-\lambda_0} p_i^{(m-\lambda_0)} t^{i+l+2\lambda_0}, \tag{3.4.44}$$

where $p_i^{(m-\lambda_0)}$ are the coefficients of the polynomial $P_{m-\lambda_0}(t)$. By construction, $t^{i+l+2\lambda_0} \in \mathcal{D}_k$ for $i = 1, \dots, m-\lambda_0$. Therefore (3.4.44) is the representation of the first basis polynomial from $\mathcal{L}_{\mathcal{N}_k}^{(l)}$. If we denote $\delta = 2\lambda_0 - k + l + 1$ and $\sigma = k - l - \lambda_0 - 1$, then putting $j = i + 2\lambda_0 - k + l + 1$ in (3.4.44) we obtain

$$t^{l+2\lambda_0} P_{m-\lambda_0}(t) = \sum_{j=\delta+1}^{m-\sigma} p_{j-2\lambda_0+k-l-1}^{(m-\lambda_0)} t^{k+j-1}.$$

Therefore, according to Corollary 3.3, the first condition of the form (3.4.33) can be written as

$$\int_{\partial S} H_l(x) \sum_{j=\delta+1}^{m-\sigma} p_{j-2\lambda_0+k-l-1}^{(m-\lambda_0)} \varphi_j(s)\, ds_x = 0.$$

Let us construct other linearly independent polynomials included in the set $\mathcal{L}_{\mathcal{N}_k}^{(l)}$. Consider polynomials of the form $t^{l+2\lambda} P_{m-\lambda}(t)$ for $\lambda_0 \le \lambda \le m-1$. According to (3.4.43) such polynomials belong to the set $\mathcal{D}_k$ if $\lambda \le k - l - 1 \le 2\lambda$. Since the right inequality holds for $\lambda_0 \le \lambda$, the condition $\lambda \le k - l - 1$ remains. Obviously, an equality similar to (3.4.44) holds

$$t^{l+2\lambda} P_{m-\lambda}(t) = \sum_{i=1}^{m-\lambda} p_i^{(m-\lambda)} t^{i+l+2\lambda},$$

where $p_i^{(m-\lambda)}$ are the coefficients of the polynomial $P_{m-\lambda}(t)$. Similar to the previous case, if we use the notations (3.4.38) $\delta_\lambda = 2\lambda - k + l + 1$ and $\sigma_\lambda = k - l - \lambda - 1$, we obtain

$$t^{l+2\lambda} P_{m-\lambda}(t) = \sum_{j=\delta_\lambda+1}^{m-\sigma_\lambda} p_{j-2\lambda+k-l-1}^{(m-\lambda)} t^{k+j-1}.$$

Therefore, in accordance with Remark 3.3, a condition of the form (3.4.33) with number λ, under the assumption that $\lambda \le k - l - 1$ and $\lambda_0 \le \lambda \le m - 1$ or equivalently $\lambda_0 \le \lambda \le \min\{k-l, m\} - 1$ can be written as

$$\int_{\partial S} H_l(x) \Big(p_1^{(m-\lambda)} \varphi_{\delta_\lambda+1}(s) + \dots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(s) \Big)\, ds_x = 0. \tag{3.4.45}$$

Finally, since in the case under consideration $\min\{k-l, m\} = k-l$, then for $\lambda \leq k-l-1$ the inequality $\lambda \leq m-1$ is also satisfied and we have $k-l-\lambda_0$th conditions of the form (3.4.45) with $\lambda = \lambda_0, \dots, k-l-1$.

3^0. Let $m < k-l$ and, hence, $\min\{k-l, m\} = m$. Then we have $m-\lambda_0$ conditions of the form (3.4.45) for $\lambda = \lambda_0, \dots, m-1$, but there are also other conditions of the same type. Returning to the condition (3.4.43) and consider the case when $\lambda \leq k-l-1$ (this is a mandatory condition for including the polynomial $t^{l+2\lambda}P_{m-\lambda}(t)$ in $\mathcal{L}^{(l)}_{\mathcal{N}_k}$), but when the inequality $0 \leq \lambda \leq m-1$ does not hold, i.e. $m \leq \lambda \leq k-l-1$. In this case, the polynomial $P_{m-\lambda}(t)$ is not defined, but as is not difficult to see from (3.4.40) it follows

$$\mathcal{L}^{(l)}_{\mathcal{N}_k} = \mathcal{D}_k \cap \operatorname{lin}\{t^l P_m(t), t^{l+2}P_{m-1}(t), \\ \dots, t^{2m+l-2}P_1(t), t^{2m+l}, \dots, t^{k+m-1}\} \quad (3.4.46)$$

and therefore, for $m \leq \lambda$, instead of the polynomial $t^{l+2\lambda}P_{m-\lambda}(t)$, $\mathcal{L}^{(l)}_m$ contains a monomial of the form $t^{l+m+\lambda}$, which also belongs to $\mathcal{D}_k$ for $\lambda \leq k-l-1$. Therefore this monomial belongs to $\mathcal{L}^{(l)}_{\mathcal{N}_k}$ when $m \leq \lambda \leq k-l-1$. Therefore, the orthogonality condition of the form (3.4.33) for λ satisfying the condition $m \leq \lambda \leq k-l-1$ can be written in the form

$$\int_{\partial S} H_l(x)\varphi_{m+\lambda-k+l+1}(s)\, ds_x = 0,$$

or for $i = m+\lambda-k+l+1$ in the form

$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = 0, \quad i = 2m-k+l+1, \dots, m. \quad (3.4.47)$$

There are $k-l-m$ conditions of this type. As a result, if $m < k-l$, then we also have $(m-\lambda_0)+(k-l-m) = k-l-\lambda_0$ orthogonality conditions of the form (3.4.37).

4^0. Finally, consider the case when $\lambda_0 = [(k-l)/2] \geq m$. In this case, $k-l \geq 2[(k-l)/2] \geq 2m$ and therefore any polynomial of the form $t^{l+2\lambda}P_{m-\lambda}(t)$ for $0 \leq \lambda \leq m-1$ has no monomials included in $\mathcal{D}_k = \operatorname{lin}\{t^k, t^{k+1}, \dots, t^{m+k-1}\}$, and which means by virtue of (3.4.40) that $\mathcal{D}_k \subset \mathcal{L}^{(l)}_m$. Therefore, we only have orthogonality conditions of the form (3.4.47)

$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = 0, \quad i = 1, \dots, m.$$

Both obtained equalities (3.4.45) and (3.4.47) can be combined in the form (3.4.37) and the total number of conditions found for $l < k$ is equal to $N_{m,k,l} = \min\{k - l - \lambda_0, m\}$.

Now note that for $k - l < m$, according to (3.4.46), polynomials of the form

$$\left\{t^{l+2\lambda} P_{m-\lambda}(t) : \lambda_0 \le \lambda \le m - 1\right\} \cup \left\{t^{l+m+\lambda} : m \le \lambda \le k - l - 1\right\} \tag{3.4.48}$$

are generating polynomials in $\mathcal{L}_m^{(l)}$ that also belong to $\mathcal{D}_k$. They are linearly independent polynomials because they have increasing degrees in λ. Moreover, no other generating polynomial from $\mathcal{L}_m^{(l)}$ can appear in $\mathcal{D}_k$: either its degree is higher than $m + k - 1$, or the least degree of its monomials is less than k. Therefore, the polynomials from (3.4.48) are a basis in $\mathcal{L}_{\mathcal{N}_k}^{(l)}$, which means the equality (3.4.37) describes all the integral conditions for the boundary data of the problem $\mathcal{N}_k$, following from Lemma 3.26. The theorem is proved. □

Remark 3.15. In boundary value problems for the polyharmonic equation, a restriction is usually imposed on the order of derivatives in the boundary conditions (the order of derivatives on the boundary is less than the order of the equation) [47], which for the $\mathcal{N}_k$ problem can be written as $k + m - 1 \le 2m - 1$ or $k \le m$. There are no such restrictions in Theorem 3.15.

Remark 3.16. If from the elements of the kth row of the Neumann triangle $\mathbb{P}$ we construct the following vector $\mathbb{P}_k = (p_1^{(k)}, \ldots, p_k^{(k)})$, in which the free term equal to zero and denote

$$\mathbb{F}(s) = (\varphi_1(s), \ldots, \varphi_m(s)), \quad \mathbb{F}_{(l,\sigma_\lambda)}^{(r,\delta_\lambda)}(s) = (\varphi_{\delta_\lambda+1}(s), \ldots, \varphi_{m-\sigma_\lambda}(s)),$$

where the upper index of the last vector indicates a shift of the initial index of the vector $\mathbb{F}(s)$ by δ_λ positions to the right, and the lower index of the last vector indicates a shift of the final index of the vector $\mathbb{F}(s)$ by σ_λ positions to the left, then the equality (3.4.45) can be rewritten as

$$\int_{\partial S} H_l(x)\, \mathbb{P}_{m-\lambda} \cdot \mathbb{F}_{(l,\sigma_\lambda)}^{(r,\delta_\lambda)}(s)\, ds_x = 0.$$

Remark 3.17. It is easy to see that for $f = 0$ the orthogonality conditions (3.4.37) for the problem $\mathcal{N}_k$ coincide with the orthogonality conditions (3.4.37) for the problem $\mathcal{N}_{k-l}$, but for $l = 0$ and the equality $N_{m,k,l} = N_{m,k-l,0}$ holds.

Remark 3.18. The smoothness of the solution in Theorem 3.15 can be weakened, i.e. it is sufficient to require $\Lambda^i u \in C(\bar{S})$ for $i = 0, \ldots, m + k - 1$. Therefore, if

$\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, then from Lemmas 3.22 and 3.23 it follows that $\Lambda^i u \in C^{m-1}(\bar{S})$ for $i = 0, \dots, k$, which means $\Lambda^i u \in C(\bar{S})$ for $i = 0, \dots, m+k-1$. Consequently, by Theorem 3.15, for such $\varphi_i(x)$ and $k \le m$ the existence of $u(x)$ implies the conditions (3.4.37), i.e. conditions (3.4.37) are necessary conditions for the solvability of the problem $\mathcal{N}_k$.

3.4.4 Integral conditions for the solvability of the problem $\mathcal{N}_k$

Let us study the necessary conditions for the problem $\mathcal{N}_k$ from Theorem 3.15 when $k \le m$. In this case, $2m + l - k + 1 \ge m + l + 1 > m$ and therefore there will be no second group of conditions of the form (3.4.37).

I. Consider the case when the difference $(k - l)$ is an even number, where $l \in \mathbb{N}_0$ and $l < k$. Let us find out what the conditions (3.4.37) mean for $k \le m$ in terms of the function $v(x) = \Lambda^{[k]} u(x)$.

Lemma 3.29. *Let $v(x)$ be a solution to problem* (3.4.8) *and the boundary data of problem $\mathcal{N}_k$ satisfy the conditions* (3.4.37) *for even $k - l$ and $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1, \dots, m$. Then $(k - l)/2$ of the following equalities must be satisfied*

$$
\begin{gathered}
\int_{\partial S} H_l(x)(\Lambda - k, -2)_{m-(k-l)/2} v \, ds_x = 0, \\
\int_{\partial S} H_l(x)(\Lambda - k + 2, -2)_{m-(k-l)/2+1} v \, ds_x = 0, \\
\dots \\
\int_{\partial S} H_l(x)(\Lambda - l - 2, -2)_{m-1} v \, ds_x = 0,
\end{gathered}
\tag{3.4.49}
$$

for an arbitrary homogeneous harmonic polynomial $H_l(x)$.

Proof. Suppose the conditions (3.4.37) are satisfied for even $k - l$. Then $\lambda_0 = (k - l)/2$, $\lambda = (k - l)/2, (k - l)/2 + 1, \dots, k - l - 1$, $\delta_\lambda = 1, 3, \dots, k - l - 1$, $\sigma_\lambda = (k-l)/2-1, (k-l)/2-2, \dots, 0$, the number of conditions (3.4.37) is $N_{m,k,l} = (k-l)/2$ and they look like

$$
\begin{gathered}
\int_{\partial S} H_l(x)\Big(p_1^{(m-(k-l)/2)}\varphi_2(x) + \dots + p_{m-(k-l)/2}^{(m-(k-l)/2)}\varphi_{m-(k-l)/2+1}(x)\Big)\, ds_x = 0, \\
\int_{\partial S} H_l(x)\Big(p_1^{(m-(k-l)/2-1)}\varphi_4(x) + \dots + p_{m-(k-l)/2-1}^{(m-(k-l)/2-1)}\varphi_{m-(k-l)/2+2}(x)\Big)\, ds_x = 0,
\end{gathered}
$$

$$\dots$$

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-k+l+1)}\varphi_{k-l}(x)+\ldots+p_{m-k+l+1}^{(m-k+l+1)}\varphi_m(x)\Big)\,ds_x=0.$$

By Lemmas 3.22 and 3.21 for $\varphi_i\in C^{2m-i-1+\varepsilon}(\partial S)$, where $i=1,\ldots,m$ the function $v(x)$ has the smoothness $v\in C^{m-1}(\bar S)$. Taking into account the equalities $\varphi_i=(\Lambda-k)^{[i-1]}v$ on ∂S, leaving the first equality unchanged, taking out the common factor $\Lambda^{[2]}$ in the second equality, etc. and $\Lambda^{[k-l-2]}$ in the last equality we obtain

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-(k-l)/2)}(\Lambda-k)+\ldots+p_{m-(k-l)/2}^{(m-(k-l)/2)}(\Lambda-k)\ldots$$
$$\ldots(\Lambda-m-(k-l)/2)\Big)v\,ds_x=0,$$
$$\int_{\partial S} H_l(x)\Big(p_1^{(m-(k-l)/2-1)}(\Lambda-k-2)+\ldots+p_{m-(k-l)/2-1}^{(m-(k-l)/2-1)}(\Lambda-k-2)\ldots$$
$$\ldots(\Lambda-m-(k-l)/2-1)\Big)(\Lambda-k)^{[2]}v\,ds_x=0,$$
$$\dots$$
$$\int_{\partial S} H_l(x)\Big(p_1^{(m-k+l+1)}(\Lambda-2k+l+2)+\ldots+p_{m-k+l+1}^{(m-k+l+1)}(\Lambda-2k+l+2)\ldots$$
$$\ldots(\Lambda-m-k+2)\Big)(\Lambda-k)^{[k-l-2]}u\,ds_x=0.$$

Let us denote $w=(\Lambda-k)(\Lambda-k-2)\ldots(\Lambda-2m-l+2)v=(\Lambda-k,-2)_{m-(k-l)/2}v$. Then, according to the equality $P_{[k]}(\lambda)=(\lambda,-2)_k$, the expression under the integral, without the factor $H_l(x)$, in the first equality is transformed to the form

$$\Big(p_1^{(m-(k-l)/2)}(\Lambda-k)^{[1]}+\ldots+p_{m-(k-l)/2}^{(m-(k-l)/2)}(\Lambda-k)^{[m-(k-l)/2]}\Big)v$$
$$=(\Lambda-k,-2)_{m-(k-l)/2}v=w,$$

in the second equality it takes the form

$$\Big(p_1^{(m-(k-l)/2-1)}(\Lambda-k-2)^{[1]}+\ldots+p_{m-(k-l)/2-1}^{(m-(k-l)/2-1)}(\Lambda-k-2)^{[m-(k-l)/2-1]}\Big)$$
$$\times(\Lambda-k)^{[2]}v=(\Lambda-k-2,-2)_{m-(k-l)/2-1}(\Lambda-k)(\Lambda-k-1)v$$
$$=(\Lambda-k,-2)_{m-(k-l)/2}(\Lambda-k-1)v=(\Lambda-k-1)w,$$

and finally in the last $(k-l)/2$-th equality it is transformed to the form

$$\Big(p_1^{(m-k+l+1)}(\Lambda-2k+l+2)^{[1]}+\ldots+p_{m-k+l+1}^{(m-k+l+1)}(\Lambda-2k+l+2)^{[m-k+l+1]}\Big)$$
$$\times(\Lambda-k)^{[k-l-2]}v=(\Lambda-2k+l+2,-2)_{m-k+l+1}(\Lambda-k,-1)_{k-l-2}v$$
$$=\big((\Lambda-k)(\Lambda-k-1)\ldots(\Lambda-2k+l+3)\big)\big((\Lambda-2k+l+2)(\Lambda-2k+l)\ldots$$
$$\ldots(\Lambda-2m-l+2)\big)v=\big((\Lambda-k)(\Lambda-k-2)\ldots$$
$$\ldots(\Lambda-2m-l+2)\big)\big((\Lambda-k-1)(\Lambda-k-3)\ldots(\Lambda-2k+l+3)\big)v$$
$$=(\Lambda-k,-2)_{m-(k-l)/2}(\Lambda-k-1,-2)_{(k-l)/2-1}v=(\Lambda-k-1,-2)_{(k-l)/2-1}w.$$

Then the $(k-l)/2$ equalities obtained above can be written in the form

$$\int_{\partial S}H_l(x)w\,ds_x=0,\quad \int_{\partial S}H_l(x)(\Lambda-k-1,-2)_1w\,ds_x=0,\ldots$$
$$\ldots,\int_{\partial S}H_l(x)(\Lambda-k-1,-2)_{(k-l)/2-1}w\,ds_x=0.$$

From the first and second equalities it follows

$$\int_{\partial S}H_l(x)w\,ds_x=\int_{\partial S}H_l(x)\Lambda w\,ds_x=0.$$

Then from the third equality $\int_{\partial S}H_l(x)(\Lambda-k-1,-2)_2w\,ds_x=0$, taking into account those already obtained, it follows $\int_{\partial S}H_l(x)\Lambda^2w\,ds_x=0$. Continuing this process we obtain equalities of the form $\int_{\partial S}H_l(x)\Lambda^iw\,ds_x=0$ for $i=0,\ldots,(k-l)/2-2$. Taking them into account, from $\int_{\partial S}H_l(x)(\Lambda-k-1,-2)_{(k-l)/2-1}w\,ds_x=0$ we obtain $\int_{partialS}H_l(x)\Lambda^iw\,ds_x=0$ for $i=0,\ldots,(k-l)/2-1$. From the found equalities, in turn, it follows in a similar way that

$$\int_{\partial S}H_l(x)w\,ds_x=0,\quad \int_{\partial S}H_l(x)(\Lambda-k+2,2)_1w\,ds_x=0,\ldots$$
$$\ldots,\int_{\partial S}H_l(x)(\Lambda-k+2,2)_{(k-l)/2-1}w\,ds_x=0.\quad(3.4.50)$$

Recalling the notation $w=(\Lambda-k,-2)_{m-(k-l)/2}v$ and taking into account that

$$(\Lambda-k+2,2)_i(\Lambda-k,-2)_{m-(k-l)/2}=(\Lambda-k+2i)\ldots(\Lambda-k+2)$$
$$\times(\Lambda-k)\ldots(\Lambda-2m-l+2)=(\Lambda-k+2i,-2)_{m-(k-l)/2+i},$$

where $i=0,\ldots,(k-l)/2-1$ from (3.4.50) we obtain (3.4.49). Note that the polynomials $H_l(x)$ can be selected from the complete system of homogeneous harmonic polynomials of degree l from Definition 1.5 on page 50. The lemma is proved. □

In further research Theorem 1.9 on page 51 is needed.

Theorem 3.16. *Let $w(x)$ be a function harmonic in S and continuous in $\bar{S}$, then the equality holds*

$$\int_{|\xi|=1} G_{(\nu)}(\xi)w(\xi)\,ds_\xi = g_\nu G_{(\nu)}(D)w(x)_{|x=0}, \tag{3.4.51}$$

where $\{G_{(\nu)}(x) : \nu \in \mathbb{N}_0^n, \nu_1 \geq \ldots \geq \nu_n, \nu_n = 0, 1\}$ is a complete system of harmonic polynomials, and $g_\nu > 0$ is some constant.

Let us denote by $v_l(x)$ the small lth order of smallness in the expansion of the m-harmonic in S function $v(x)$ in a neighborhood of the origin.

Theorem 3.17. *Let the given problems $\mathcal{N}_k$ satisfy the conditions from* (3.4.37) *for even $k - l$, have the smoothness $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1, \ldots, m$, and the m-harmonic function $v(x)$, which is a solution to the problem* (3.4.8)*, has the following Almansi expansion*

$$v(x) = v^{(0)}(x) + |x|^2 v^{(1)}(x) + \ldots + |x|^{2m-2} v^{(m-1)}(x), \quad x \in S.$$

Then for even $k - l$ the equalities hold true

$$v_l^{(0)}(x) = v_l^{(1)}(x) = \ldots = v_l^{((k-l)/2-1)}(x) = 0. \tag{3.4.52}$$

Proof. From Lemmas 3.22 and 3.21 it follows that for $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, where $i = 1, \ldots, m$ harmonic components of the function $v(x)$ have smoothness $v^{(i)} \in C^{m-1}(\bar{S})$, $i = 0, \ldots, m-1$, which means $v \in C^{m-1}(\bar{S})$. In addition, by Lemma 3.29, the equalities (3.4.49) hold true. Let us prove that in this case the equalities (3.4.49) imply the conditions (3.4.52).

Let us consider the last equality from (3.4.49). We apply the operator $(\Lambda - l - 2, -2)_{m-1}$ to the Almansi expansion of the function $v(x)$

$$\begin{aligned}(\Lambda - l - 2, -2)_{m-1} v(x) = (\Lambda - l - 2, -2)_{m-1} v^{(0)}(x) + |x|^2 (\Lambda - l, -2)_{m-1} v^{(1)}(x) \\ + \ldots + |x|^{2m-2} (\Lambda + 2m - l - 4, -2)_{m-1} v^{(m-1)}(x)\end{aligned}$$

and consider the harmonic in S function

$$\begin{aligned}h^{(0)}(x) = (\Lambda - l - 2, -2)_{m-1} v^{(0)}(x) + (\Lambda - l, -2)_{m-1} v^{(1)}(x) + \ldots \\ + (\Lambda + 2m - l - 4, -2)_{m-1} v^{(m-1)}(x),\end{aligned}$$

which, due to the smoothness of $v^{(i)} \in C^{m-1}(\bar{S})$ and the last equality from (3.4.49), satisfies the condition

$$0 = \int_{\partial S} H_l(x)(\Lambda - l - 2, 2)_{m-1} v \, ds_x = \int_{\partial S} H_l(x) h^{(0)}(x) \, ds_x,$$

where $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l. Then from Theorem 3.16 it follows that $h_l^{(0)}(x) = 0$. Due to the homogeneity of the operator Λ and the definition of the generalized Pochhammer symbol, we write

$$\begin{aligned} 0 = h_l^{(0)}(x) &= (\Lambda - l - 2, -2)_{m-1} v_l^{(0)} + (\Lambda - l, -2)_{m-1} v_l^{(1)} \\ &\quad \ldots + (\Lambda + 2m - l - 4, -2)_{m-1} v_l^{(m-1)} = (-2, -2)_{m-1} v_l^{(0)} \\ &\quad + (0, -2)_{m-1} v_l^{(1)} + \ldots + (2m - 4, -2)_{m-1} v_l^{(m-1)} = (-2, -2)_{m-1} v_l^{(0)}(x). \end{aligned}$$

Therefore $v_l^{(0)}(x) = 0$. Next we apply the operator $(\Lambda - l - 4, -2)_{m-2}$ to the Almansi expansion of the function $v(x)$

$$\begin{aligned} &(\Lambda - l - 4, -2)_{m-2} v(x) = (\Lambda - l - 4, -2)_{m-2} v^{(0)}(x) \\ &\quad + |x|^2 (\Lambda - l - 2, -2)_{m-2} v^{(1)}(x) \ldots + |x|^{2m-2} (\Lambda + 2m - l - 6, -2)_{m-2} v^{(m-2)}(x) \end{aligned}$$

and consider the harmonic in S function

$$\begin{aligned} h^{(1)}(x) &= (\Lambda - l - 4, -2)_{m-2} v^{(0)}(x) + (\Lambda - l - 2, -2)_{m-2} v^{(1)}(x) \\ &\quad \ldots + (\Lambda + 2m - l - 6, -2)_{m-2} v^{(m-1)}(x), \end{aligned}$$

which, due to the smoothness of $v^{(i)} \in C^{m-1}(\bar{S})$ and the penultimate equality from (3.4.49), satisfies the condition

$$0 = \int_{\partial S} H_l(x)(\Lambda - l - 4, 2)_{m-2} v \, ds_x = \int_{\partial S} H_l(x) h^{(1)}(x) \, ds_x$$

for an arbitrary homogeneous harmonic polynomial $H_l(x)$. Then from Theorem 3.16 it follows that, taking into account the equality $v_l^{(0)}(x) = 0$, we get

$$\begin{aligned} 0 = h_l^{(1)}(x) &= (\Lambda - l - 4, -2)_{m-2} v_l^{(0)} + (\Lambda - l - 2, -2)_{m-2} v_l^{(1)} \\ &\quad + \ldots + (\Lambda + 2m - l - 6, -2)_{m-2} v_l^{(m-1)} = (-2, -2)_{m-2} v_l^{(1)} \\ &\quad + \ldots + (2m - 6, -2)_{m-2} v_l^{(m-1)} = (-2, -2)_{m-2} v_l^{(1)}(x). \end{aligned}$$

Therefore $v_l^{(1)}(x) = 0$. Continuing this process we will have the equalities $v_l^{(i)}(x) = 0$ for $i = 0, \dots, (k-l)/2 - 2$. Finally, we apply the operator $(\Lambda - k, -2)_{m-(k-l)/2}$ to the Almansi expansion of the function $v(x)$

$$(\Lambda - k, -2)_{m-(k-l)/2} v(x) = \sum_{i=0}^{m-1} |x|^{2i} (\Lambda - k + 2i, -2)_{m-(k-l)/2} v^{(i)}(x)$$

and consider the harmonic in S function

$$h^{((k-l)/2-1)}(x) = \sum_{i=0}^{m-1} (\Lambda - k + 2i, -2)_{m-(k-l)/2} v^{(i)}(x),$$

which, due to the smoothness of $v^{(i)} \in C^{m-1}(\bar{S})$ and the first equality from (3.4.49), satisfies the equality

$$0 = \int_{\partial S} H_l(x)(\Lambda - k, 2)_{m-(k-l)/2} v \, ds_x = \int_{\partial S} H_l(x) h^{((k-l)/2-1)}(x) \, ds_x.$$

Then, according to Theorem 3.16, similar to the previous cases, taking into account the equalities $v_l^{(i)}(x) = 0$, where $i = 0, \dots, (k-l)/2 - 2$, have

$$\begin{aligned} 0 = h_l^{((k-l)/2-1)} &= \sum_{i=0}^{m-1} (l - k + 2i, -2)_{m-(k-l)/2} v_l^{(i)} \\ &= \sum_{i=(k-l)/2-1}^{m-1} (2i + l - k, -2)_{m-(k-l)/2} v_l^{(i)} = (-2, -2)_{m-(k-l)/2} v_l^{((k-l)/2-1)} \\ &+ (0, -2)_{m-(k-l)/2} v_l^{((k-l)/2)} + \dots + (2m - 2 + l - k, -2)_{m-(k-l)/2} v_l^{(m-1)} \\ &\qquad = (-2, -2)_{m-(k-l)/2} v_l^{((k-l)/2-1)}(x). \end{aligned}$$

Therefore $v_l^{((k-l)/2-1)}(x) = 0$. Thus, for even $k - l$, from the equalities (3.4.49) it follows that $v_l^{(0)}(x) = v_l^{(1)}(x) = \dots = v_l^{((k-l)/2-1)}(x) = 0$. The theorem is proved. $\square$

II. Let us now study the necessary conditions for the problem $\mathcal{N}_k$ obtained in Theorem 3.15 ($k \le m$) in the case of an odd difference $k - l$, $l \in \mathbb{N}_0$ and $l < k$.

Lemma 3.30. *Let $v(x)$ be a solution to problem* (3.4.8) *and the boundary data of problem $\mathcal{N}_k$ satisfy the conditions* (3.4.37) *for odd $k - l$ and $\varphi_i \in C^{2m-i-1}(\partial S)$ for*

$i = 1,\dots,m$. Then $[(k-l)/2]+1$ equalities of the form must be satisfied

$$\begin{gathered}
\int_{\partial S} H_l(x)(\Lambda-k-1,-2)_{m-[(k-l)/2]-1} v\, ds_x = 0,\\
\int_{\partial S} H_l(x)(\Lambda-k+1,-2)_{m-[(k-l)/2]} v\, ds_x = 0,\\
\dots\\
\int_{\partial S} H_l(x)(\Lambda-l-2,-2)_{m-1} v\, ds_x = 0,
\end{gathered} \tag{3.4.53}$$

for an arbitrary homogeneous harmonic polynomial $H_l(x)$.

Proof. Let the conditions (3.4.37) be satisfied for odd $k-l$. Then $\lambda_0 = [(k-l)/2]$, $\lambda = [(k-l)/2], [(k-l)/2]+1,\dots,k-l-1$, $\delta_\lambda = 0,2,\dots,k-1$, $\sigma_\lambda = [(k-l)/2], [(k-l)/2]-1,\dots,0$, the number of conditions (3.4.37) is $N_{m,k,l} = [(k-l)/2]+1$ and they look like

$$\begin{gathered}
\int_{\partial S} H_l(x)\Big(p_1^{(m-[(k-l)/2])}\varphi_1(x)+\dots+p_{m-[(k-l)/2]}^{(m-[(k-l)/2])}\varphi_{m-[(k-l)/2]}(x)\Big)\, ds_x = 0,\\
\int_{\partial S} H_l(x)\Big(p_1^{(m-[(k-l)/2]-1)}\varphi_3(x)+\dots+p_{m-[(k-l)/2]-1}^{(m-[(k-l)/2]-1)}\varphi_{m-[(k-l)/2]+1}(x)\Big)\, ds_x = 0,\\
\dots\\
\int_{\partial S} H_l(x)\Big(p_1^{(m-k+l+1)}\varphi_{k-l}(x)+\dots+p_{m-k+l+1}^{(m-k+l+1)}\varphi_m(x)\Big)\, ds_x = 0.
\end{gathered}$$

By Lemmas 3.22 and 3.21 for $\varphi_i \in C^{2m-i-1}(\partial S)$, where $i = 1,\dots,m$ the function $v(x)$ has the smoothness $v \in C^{m-1}(\bar S)$. Taking into account the equality $\varphi_i = (\Lambda-k)^{[i-1]}v = (\Lambda-k,-1)_{i-1}v$ on ∂S, taking out the common factor $(\Lambda-k+1)^{-1}$ in the first equality (this is not an inverse operator, we are simply symbolically simplifying the operator acting on v), $(\Lambda-k)^{[1]}$ in the second equality, etc., $(\Lambda-k)^{[k-l-2]}$ in the last equality, we get

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-[(k-l)/2])}(\Lambda-k+1)+\dots$$
$$+p_{m-[(k-l)/2]}^{(m-[(k-l)/2])}(\Lambda-k+1,-1)_{m-[(k-l)/2]}\Big)(\Lambda-k+1)^{-1}v\,ds_x=0,$$
$$\int_{\partial S} H_l(x)\Big(p_1^{(m-[(k-l)/2]-1)}(\Lambda-k-1)+\dots$$
$$+p_{m-[(k-l)/2]-1}^{(m-[(k-l)/2]-1)}(\Lambda-k-1,-1)_{m-[(k-l)/2]-1}\Big)(\Lambda-k)^{[1]}v\,ds_x=0,$$
$$\dots$$
$$\int_{\partial S} H_l(x)\Big(p_1^{(m-k+l+1)}(\Lambda-2k+l+2)+\dots$$
$$+p_{m-k+l+1}^{(m-k+l+1)}(\Lambda-2k+l+2,-1)_{m-k+l+1}\Big)(\Lambda-k)^{[k-l-2]}v\,ds_x=0.$$

Let us denote $w=(\Lambda-k-1)(\Lambda-k-3)\dots(\Lambda-2m-l+2)v=(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}v$. Then the expression under the integral in the first equality is transformed to the form

$$\Big(p_1^{(m-[(k-l)/2])}(\Lambda-k+1)^{[1]}+\dots+p_{m-[(k-l)/2]}^{(m-[(k-l)/2])}(\Lambda-k+1)^{[m-[(k-l)/2]]}\Big)(\Lambda-k+1)^{-1}v$$
$$=(\Lambda-k+1,-2)_{m-[(k-l)/2]}(\Lambda-k+1)^{-1}v=(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}v=w$$

(here in the expression $[m-[(k-l)/2]]$ the outer brackets denote the factorial degree, and the inner ones the integer part of the number), in the second equality it takes the form

$$\Big(p_1^{(m-[(k-l)/2]-1)}(\Lambda-k-1)^{[1]}+\dots+p_{m-[(k-l)/2]-1}^{(m-[(k-l)/2]-1)}(\Lambda-k-1)^{[m-[(k-l)/2]-1]}\Big)$$
$$\times(\Lambda-k)^{[1]}v=(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}(\Lambda-k)v$$
$$=(\Lambda-k)(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}v=(\Lambda-k)w,$$

and finally in the last $([(k-l)/2]+1)$th equality it is transformed to the form

$$\Big(p_1^{(m-k+l+1)}(\Lambda-2k+l+2)^{[1]}+\dots+p_{m-k+l+1}^{(m-k+l+1)}(\Lambda-2k+l+2)^{[m-k+l+1]}\Big)$$
$$\times(\Lambda-k)^{[k-l-2]}v=(\Lambda-2k+l+2,-2)_{m-k+l+1}(\Lambda-k,-1)_{k-l-2}v$$
$$=\big((\Lambda-k)(\Lambda-k-1)\dots(\Lambda-2k+l+3)\big)\big((\Lambda-2k+l+2)(\Lambda-2k+l)$$
$$\dots(\Lambda-2m-l+2)\big)v=\big((\Lambda-k)(\Lambda-k-2)\dots(\Lambda-2k+l+3)\big)\big((\Lambda-k-1)(\Lambda-k-3)$$

$$\ldots(\Lambda-2m-l+2)\big)v=(\Lambda-k,-2)_{[(k-l)/2]}(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}v$$
$$=(\Lambda-k,-2)_{[(k-l)/2]}w.$$

Then the $[(k-l)/2]+1$ equalities obtained above can be written in the form

$$\int_{\partial S}H_l(x)w\,ds_x=0,\ \int_{\partial S}H_l(x)(\Lambda-k,-2)_1w\,ds_x=0,$$
$$\ldots,\int_{\partial S}H_l(x)(\Lambda-k,-2)_{[(k-l)/2]}w\,ds_x=0.$$

The first and second equalities imply the equalities

$$\int_{\partial S}w\,ds_x=\int_{\partial S}\Lambda w\,ds_x=0.$$

Then from the third equality $\int_{\partial S}(\Lambda-k,-2)_2w\,ds_x=0$, taking into account the equalities already obtained, it follows $\int_{\partial S}\Lambda^2w\,ds_x=0$. Continuing this process we have the equalities $\int_{\partial S}\Lambda^iw\,ds_x=0$ for $i=0,\ldots,[(k-l)/2]-1$. Taking them into account, from $\int_{\partial S}(\Lambda-k,-2)_{[(k-l)/2]}w\,ds_x=0$ we find $\int_{\partial S}\Lambda^iw\,ds_x=0$, where $i=0,\ldots,[(k-l)/2]$. From the derived equalities, in turn, it is easy to obtain

$$\int_{\partial S}H_l(x)w\,ds_x=0,\quad \int_{\partial S}H_l(x)(\Lambda-k+1,2)_1w\,ds_x=0,\ldots$$
$$\ldots,\int_{\partial S}H_l(x)(\Lambda-k+1,2)_{[(k-l)/2]}w\,ds_x=0.$$

Recalling the notation $w=(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}v$ and taking into account that

$$(\Lambda-k+1,2)_i(\Lambda-k-1,-2)_{m-[(k-l)/2]-1}=(\Lambda-k-1+2i)\ldots(\Lambda-k+1)$$
$$\times(\Lambda-k-1)\ldots(\Lambda-2m+2)=(\Lambda-k-1+2i,-2)_{m+i-[(k-l)/2]-1}$$

for $i=0,\ldots,[(k-l)/2]$ we obtain the conditions (3.4.53). □

Theorem 3.18. *Let the given problems $\mathcal{N}_k$ satisfy the conditions from* (3.4.37) *for odd $k-l$, have smoothness $\varphi_i\in C^{2m-i-1+\varepsilon}(\partial S)$, $i=1,\ldots,m$, and the m-harmonic function $v(x)$, which is a solution to the problem* (3.4.8)*, has the following Almansi expansion in S*

$$v(x)=v^{(0)}(x)+|x|^2v^{(1)}(x)+\ldots+|x|^{2m-2}v^{(m-1)}(x),\quad x\in S.$$

Then for odd $k - l$ the following equalities hold

$$v_l^{(0)} = v_l^{(1)} = \ldots = v_l^{[(k-l)/2]} = 0. \tag{3.4.54}$$

Proof. From Lemmas 3.22 and 3.21 it follows that for $\varphi_i \in C^{2m-i-1}(\partial S)$, where $i = 1, \ldots, m$ the harmonic components of the function $v(x)$ have smoothness $v^{(i)} \in C^{m-1}(\bar{S})$, $i = 0, \ldots, m-1$, and therefore $v \in C^{m-1}(\bar{S})$. In addition, by Lemma 3.30, the equalities (3.4.53) hold true. Let us prove that in this case the equalities (3.4.53) imply the equalities (3.4.54).

Since the last equality from (3.4.53) coincides with the last equality from (3.4.49), then, similarly to Theorem 3.17, we have $v_l^{(i)}(x) = 0$, where $i = 0, \ldots, [(k-l)/2]-1$. Let us apply the operator $(\Lambda - k - 1, -2)_{m-[(k-l)/2]-1}$ to the Almansi expansion of the function $v(x)$

$$(\Lambda - k - 1, -2)_{m-[(k-l)/2]-1} v(x) = \sum_{i=0}^{m-1} |x|^{2i} (\Lambda - k - 1 + 2i, -2)_{m-[(k-l)/2]-1} v^{(i)}(x)$$

and consider the harmonic in S function

$$h^{([(k-l)/2])}(x) = \sum_{i=0}^{m-1} (\Lambda - k - 1 + 2i, -2)_{m-[(k-l)/2]-1} v^{(i)}(x),$$

which, due to the smoothness of $v^{(i)} \in C^{m-1}(\bar{S})$ and the first equality from (3.4.53), satisfies the condition

$$0 = \int_{\partial S} H_l(x)(\Lambda - k - 1, -2)_{m-[(k-l)/2]-1} v \, ds_x = \int_{\partial S} H_l(x) h^{([(k-l)/2])}(x) \, ds_x.$$

Then, by Theorem 3.16, similarly to Theorem 3.17, taking into account the equalities $v_l^{(i)}(x) = 0$, where $i = 0, \ldots, [(k-l)/2] - 1$, we have

$$\begin{aligned}
0 = h_l^{([(k-l)/2])}(x) &= \sum_{i=0}^{m-1} (\Lambda - k - 1 + 2i, -2)_{m-[(k-l)/2]-1} v_l^{(i)}(x) \\
&= \sum_{i=[(k-l)/2]}^{m-1} (l - k - 1 + 2i, -2)_{m-[(k-l)/2]-1} v_l^{(i)}(x) \\
&= (-2, -2)_{m-[(k-l)/2]-1} v_l^{([(k-l)/2])}(x) + (0, -2)_{m-[(k-l)/2]-1} v_l^{([(k-l)/2]+1)}(x) \\
&\quad + \ldots + (2m + l - k - 3, -2)_{m-[(k-l)/2]-1} v_l^{(m-1)}(x)
\end{aligned}$$

$$= (-2,-2)_{m-[(k-l)/2]-1} v_l^{([(k-l)/2])}(x).$$

Therefore $v_l^{([(k-l)/2])}(x) = 0$. Thus, for odd $k-l$, it follows from the equalities (3.4.53) that $v_l^{(0)}(x) = v_l^{(1)}(x) = \ldots = v_l^{([(k-l)/2])}(x) = 0$. The theorem is proved. □

3.4.5 Necessary and sufficient conditions for the solvability of the problem $\mathcal{N}_k$

Let us prove that the necessary conditions for the solvability of the problem $\mathcal{N}_k$, obtained in Theorem 3.15 for $k \le m$, are also sufficient conditions if these problems have the necessary smoothness.

Theorem 3.19. *Let $k \le m$ and $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1,\ldots,m$ then be necessary and sufficient conditions for the solvability of the problem $\mathcal{N}_k$ are $N_k = [(k+1)/2][(k+2)/2]$ conditions*

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{\delta_\lambda+1}(x) + \ldots + p_{m-\lambda}^{(m-\lambda)}\varphi_{m-\sigma_\lambda}(x)\Big)\, ds_x = 0 \qquad (3.4.55)$$

for $l = 0,1,\ldots,k-1$, $\lambda = [(k-l)/2],\ldots,k-l-1$, $\delta_\lambda = 2\lambda-k+l+1$, $\sigma_\lambda = k-l-\lambda-1$, where $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l. The solution to the problem $\mathcal{N}_k$ exists and is unique up to m-harmonic polynomials of degree $k-1$.

Proof. Sufficiency. Let $v(x)$ be the solution to problem (3.4.8)

$$\Delta^m v = 0,\ x \in S,$$
$$v|_{\partial S} = \varphi_1(x), (\Lambda-k)v|_{\partial S} = \varphi_2(x),\ldots,(\Lambda-k)^{[m-1]}v|_{\partial S} = \varphi_m(x),\ x \in \partial S.$$

By Lemma 3.22 this problem is unconditionally solvable. By Lemma 3.23 the solution $u(x)$ to the problem $\mathcal{N}_k$ can be obtained from the equation $v(x) = \Lambda^{[k]}u(x)$, which is solved in m-harmonic in S functions. The function $u(x)$ can be defined if and only if the m-harmonic in S function $v(x)$ has no terms up to the $(k-1)$th order of smallness, in its expansion in the origin. We represent the function $v(x)$ in the form

$$v(x) = v^{(0)}(x) + |x|^2 v^{(1)}(x) + \ldots + |x|^{2m-2} v^{(m-1)}(x),\ x \in S$$

and let the harmonic components $v(x)$ have the following expansion in a vicinity of the origin

$$v^{(i)}(x) = \sum_{j=0}^{\infty} v_j^{(i)}(x), \quad i = 0,1,\ldots,m-1.$$

Since the functions $\varphi_i(s)$ for $i = 1, 2, \dots, m$ satisfy the conditions (3.4.37) for $k \le m$, then from Theorems 3.17 and 3.18 it follows that the equalities (3.4.52) and (3.4.54) are true for the function $v(x)$. It follows from them that

$$v_l^{(0)} = \dots = v_l^{(k-l)/2-1} = 0,\ k-l \in 2\mathbb{N};$$
$$v_l^{(0)} = \dots = v_l^{[(k-l)/2]} = 0,\ k-l \in 2\mathbb{N}-1,$$

which means $v_l^{(i)} = 0$ if $i < (k-l)/2$, where $0 \le l \le k-1$. Hence $v_j^{(i)}(x) = 0$ if $2i + j < k$ and therefore in the expansion $v(x)$ in the vicinity of the origin

$$v(x) = \sum_{i=0}^{m-1} \sum_{j=0}^{\infty} |x|^{2i} v^{(i)}(x)$$

all homogeneous polynomials $|x|^{2i} v_j^{(i)}(x)$ having degrees $\deg |x|^{2i} v_j^{(i)}(x) = 2i + j < k$ become zero. Therefore, the conditions of Lemma 3.23 are satisfied, which means that the solution to the problem $\mathcal{N}_k$ exists and is unique up to m-harmonic polynomials of degree $k-1$. The sufficiency of the conditions (3.4.55)is proved.

The conditions of the theorem are necessary and sufficient conditions for the solvability of the problem $\mathcal{N}_k$ since, according to Remark 3.18 for $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1, \dots, m$ conditions (3.4.37) are also necessary conditions for the problem $\mathcal{N}_k$. The theorem is proved. □

Remark 3.19. Conditions (3.4.55) for $k = 1$ (in this case we get $N_1 = 1$ conditions) coincide with the condition for problem $\mathcal{N}_1$ obtained in Section 3.3, and for $k = 2$ (in this case we get $N_2 = 2$ conditions) coincide with the conditions for the problem $\mathcal{N}_2$ from [170].

Example 3.9. Let us write down the dynamics of the emergence of necessary conditions for the solvability of problems $\mathcal{N}_k$ for the 7-harmonic homogeneous equation for different $k \in \mathbb{N}$

$$\begin{gathered}\Delta^7 u = 0, \quad x \in S; \\ \left.\frac{\partial^k u}{\partial \nu^k}\right|_{\partial S} = \varphi_1(s),\ \left.\frac{\partial^{k+1} u}{\partial \nu^{k+1}}\right|_{\partial S} = \varphi_2(s),\ \left.\frac{\partial^{k+6} u}{\partial \nu^{k+6}}\right|_{\partial S} = \varphi_7(s),\ s \in \partial S.\end{gathered} \tag{3.4.56}$$

In our case $m = 7$. Taking into account Remark 3.17, consider the case $l = 0$. In accordance with (3.4.38) we have $N_{7,k,0} = \min\{k - \lambda_0, 7\}$ conditions (3.4.37) with polynomial coefficients $P_{7-\lambda}(t)$ from (3.4.28) for $\lambda_0 \le \lambda \le \min\{k, 7\} - 1$,

$\delta_\lambda+1 = 2\lambda-k+2$, $m-\sigma_\lambda = 8+\lambda-k$, $\lambda_0 = [k/2]$ and $\max\{2m+l-k,0\}+1 \le i \le m$, when $k > m$.

For $k = 1$ we obtain $\lambda_0 = 0$ and $N_{7,1,0} = 1$ condition for $0 \le \lambda \le 0$, $\delta_\lambda+1 = 2\lambda+1$, $m-\sigma_\lambda = 7+\lambda$,

$$\int_{\partial S}\left(p_1^{(7)}\varphi_1(s)+\ldots+p_7^{(7)}\varphi_7(s)\right)ds_x = 0. \tag{3.4.57}$$

For $k = 2$ we obtain $\lambda_0 = 1$ and $N_{7,2,0} = 1$ conditions for $1 \le \lambda \le 1$, $\delta_\lambda+1 = 2\lambda$, $m-\sigma_\lambda = 6+\lambda$,

$$\int_{\partial S}\left(p_1^{(6)}\varphi_2(s)+\ldots+p_6^{(6)}\varphi_7(s)\right)ds_x = 0. \tag{3.4.58}$$

For $k = 3$ we obtain $\lambda_0 = 1$ and $N_{7,3,0} = 2$ conditions for $1 \le \lambda \le 2$, $\delta_\lambda+1 = 2\lambda-1$, $m-\sigma_\lambda = 5+\lambda$,

$$\int_{\partial S}\left(p_1^{(6)}\varphi_1(s)+\ldots+p_6^{(6)}\varphi_6(s)\right)ds_x = 0,\ \int_{\partial S}\left(p_1^{(5)}\varphi_3(s)+\ldots+p_5^{(5)}\varphi_7(s)\right)ds_x = 0.$$

For $k = 4$ we obtain $\lambda_0 = 2$ and $N_{7,4,0} = 2$ conditions for $2 \le \lambda \le 3$, $\delta_\lambda+1 = 2\lambda-2$, $m-\sigma_\lambda = 4+\lambda$,

$$\int_{\partial S}\left(p_1^{(5)}\varphi_2(s)+\ldots+p_5^{(5)}\varphi_6(s)\right)ds_x = 0,$$
$$\int_{\partial S}\left(p_1^{(4)}\varphi_4(s)+\ldots+p_4^{(4)}\varphi_7(s)\right)ds_x = 0.$$

For $k = 5$ we obtain $\lambda_0 = 2$ and $N_{7,5,0} = 3$ conditions for $2 \le \lambda \le 4$, $\delta_\lambda+1 = 2\lambda-3$, $m-\sigma_\lambda = 3+\lambda$,

$$\int_{\partial S}\left(p_1^{(5)}\varphi_1(s)+\ldots+p_5^{(5)}\varphi_5(s)\right)ds_x = 0,$$
$$\int_{\partial S}\left(p_1^{(4)}\varphi_3(s)+\ldots+p_4^{(4)}\varphi_6(s)\right)ds_x = 0,$$
$$\int_{\partial S}\left(p_1^{(3)}\varphi_5(s)+\ldots+p_3^{(3)}\varphi_7(s)\right)ds_x = 0.$$

For $k = 6$ we obtain $\lambda_0 = 3$ and $N_{7,6,0} = 3$ conditions for $3 \le \lambda \le 5$, $\delta_\lambda+1 = 2\lambda-4$, $m-\sigma_\lambda = 2+\lambda$,

$$\int_{\partial S}\left(p_1^{(4)}\varphi_2(s)+\ldots+p_4^{(4)}\varphi_5(s)\right)ds_x = 0,$$

$$\int_{\partial S}\Big(p_1^{(3)}\varphi_4(s)+\ldots+p_3^{(3)}\varphi_6(s)\Big)\,ds_x=0,$$
$$\int_{\partial S}\Big(p_1^{(2)}\varphi_6(s)+p_2^{(2)}\varphi_7(s)\Big)\,ds_x=0.$$

For $k=7$ we obtain $\lambda_0=3$ and $N_{7,7,0}=4$ conditions for $3\le\lambda\le 6$, $\delta_\lambda+1=2\lambda-5$, $m-\sigma_\lambda=1+\lambda$,

$$\int_{\partial S}\Big(p_1^{(4)}\varphi_1(s)+\ldots+p_4^{(4)}\varphi_4(s)\Big)\,ds_x=0,$$
$$\int_{\partial S}\Big(p_1^{(3)}\varphi_3(s)+\ldots+p_3^{(3)}\varphi_5(s)\Big)\,ds_x=0,$$
$$\int_{\partial S}\Big(p_1^{(2)}\varphi_5(s)+p_2^{(2)}\varphi_6(s)\Big)\,ds_x=0,\quad \int_{\partial S}p_1^{(1)}\varphi_7(s)\,ds_x=0.$$

For $k=8$ we obtain $\lambda_0=4$ and $N_{7,8,0}=4$ conditions for $4\le\lambda\le 6$, $\delta_\lambda+1=2\lambda-6$, $m-\sigma_\lambda=\lambda$ and $7\le i\le 7$

$$\int_{\partial S}\Big(p_1^{(3)}\varphi_2(s)+\ldots+p_3^{(3)}\varphi_4(s)\Big)\,ds_x=0,\ \int_{\partial S}\Big(p_1^{(2)}\varphi_4(s)+p_2^{(2)}\varphi_5(s)\Big)\,ds_x=0,$$
$$\int_{\partial S}p_1^{(1)}\varphi_6(s)\,ds_x=0,\quad \int_{\partial S}\varphi_i(s)\,ds_x=0,\ (i=7).$$

For $k=9$ we obtain $\lambda_0=4$ and $N_{7,9,0}=5$ conditions for $4\le\lambda\le 6$, $\delta_\lambda+1=2\lambda-7$, $m-\sigma_\lambda=\lambda-1$ and $6\le i\le 7$

$$\int_{\partial S}\Big(p_1^{(3)}\varphi_1(s)+\ldots+p_3^{(3)}\varphi_3(s)\Big)\,ds_x=0,\ \int_{\partial S}\Big(p_1^{(2)}\varphi_3(s)+p_2^{(2)}\varphi_4(s)\Big)\,ds_x=0,$$
$$\int_{\partial S}p_1^{(1)}\varphi_5(s)\,ds_x=0,\quad \int_{\partial S}\varphi_i(s)\,ds_x=0,\ (i=6,7).$$

For $k=10$ we obtain $\lambda_0=5$ and $N_{7,10,0}=5$ conditions for $5\le\lambda\le 6$, $\delta_\lambda+1=2\lambda-8$, $m-\sigma_\lambda=\lambda-2$ and $5\le i\le 7$

$$\int_{\partial S}\Big(p_1^{(2)}\varphi_2(s)+p_2^{(2)}\varphi_3(s)\Big)\,ds_x=0,\quad \int_{\partial S}p_1^{(1)}\varphi_4(s)\,ds_x=0,$$
$$\int_{\partial S}\varphi_i(s)\,ds_x=0,\ (i=5,6,7).$$

For $k=11$ we obtain $\lambda_0=5$ and $N_{7,11,0}=6$ conditions for $5\le\lambda\le 6$, $\delta_\lambda+1=2\lambda-9$, $m-\sigma_\lambda=\lambda-3$ and $4\le i\le 7$

$$\int_{\partial S}\Big(p_1^{(2)}\varphi_1(s)+p_2^{(2)}\varphi_2(s)\Big)\,ds_x=0,\ \int_{\partial S}p_1^{(1)}\varphi_3(s)\,ds_x=0,$$

$$\int_{\partial S} \varphi_i(s)\, ds_x = 0,\ (i = 4, \ldots, 7).$$

For $k = 12$ we obtain $\lambda_0 = 6$ and $N_{7,12,0} = 6$ conditions for $6 \le \lambda \le 6$, $\delta_\lambda + 1 = 2\lambda - 10$, $m - \sigma_\lambda = \lambda - 4$ and $3 \le i \le 7$

$$\int_{\partial S} p_1^{(1)} \varphi_2(s)\, ds_x = 0, \quad \int_{\partial S} \varphi_i(s)\, ds_x = 0,\ (i = 3, \ldots, 7).$$

For $k = 13$ we obtain $\lambda_0 = 6$ and $N_{7,13,0} = 7$ conditions for $6 \le \lambda \le 6$, $\delta_\lambda + 1 = 2\lambda - 11$, $m - \sigma_\lambda = \lambda - 5$ and $2 \le i \le 7$

$$\int_{\partial S} p_1^{(1)} \varphi_1(s)\, ds_x = 0, \quad \int_{\partial S} \varphi_2(i)\, ds_x = 0,\ (i = 2, \ldots, 7).$$

For $k \ge 14$, according to the remark after the formulation of Theorem 3.15, since the condition $[k/2]+1 \le 7$ is not satisfied, then the necessary conditions for the problem $\mathcal{N}_k$ must be taken only from the second equality (3.4.37) and there are $N_{7,k,0} = 7$ conditions of the form

$$\int_{\partial S} \varphi_i(s)\, ds_x = 0, \quad i = 1, \ldots, 7.$$

Finally, from the above analysis, taking into account Remark 3.17, it is not difficult to obtain all the necessary and sufficient conditions for solvability, for example for the problem $\mathcal{N}_3$ for the homogeneous 7-harmonic equation, following from Theorem 3.15.

Due to the specificity of the conditions (3.4.55) the condition for the problem $\mathcal{N}_3$ for $l = 1$ is obtained from the condition (3.4.58) for the problem $\mathcal{N}_2$ for $l = 0$ by entering the factor $H_1(x)$ under the integral sign, and the condition for problem $\mathcal{N}_3$ for $l = 2$ is obtained from the condition (3.4.57) for problem $\mathcal{N}_1$ for $l = 0$ by entering the factor $H_2(x)$ under the integral sign. Using the values of the coefficients $p_i^{(5)}$, $p_i^{(6)}$ and $p_i^{(7)}$ from the fifth, sixth and seventh rows of the triangle $\mathbb{P}$

$$\begin{array}{ccccccc} 105 & -105 & 45 & -10 & 1 & & \\ -945 & 945 & -420 & 105 & -15 & 1 & \\ 10395 & -10395 & 4725 & -1260 & 210 & -21 & 1 \end{array}$$

we have four necessary and sufficient conditions for the solvability of the problem $\mathcal{N}_3$ (this is the problem (3.4.56) for $k = 3$) in the form

$$\int_{\partial S} \Big(-945\varphi_1 + 945\varphi_2 - 420\varphi_3 + 105\varphi_4 - 15\varphi_4 + \varphi_6 \Big)\, ds_x = 0,$$

$$\int_{\partial S}\Big(105\varphi_3-105\varphi_4+45\varphi_5-10\varphi_6+\varphi_7\Big)\,ds_x=0,$$
$$\int_{\partial S}H_1(x)\Big(-945\varphi_2+945\varphi_3-420\varphi_4+105\varphi_5-15\varphi_6+\varphi_7\Big)\,ds_x=0,$$
$$\int_{\partial S}H_2(x)\Big(10395\varphi_1-10395\varphi_2+4725\varphi_3-1260\varphi_4+210\varphi_4$$
$$-21\varphi_6+\varphi_7\Big)\,ds_x=0,$$

where $H_l(x)$ is an arbitrary homogeneous harmonic polynomial.

3.4.6 Inhomogeneous equation

Consider the problem $\mathcal{N}_k$ for the inhomogeneous polyharmonic equation

$$\Delta^m u=f(x),\quad x\in S; \tag{3.4.59}$$

$$\frac{\partial^k u}{\partial\nu^k}\Big|_{\partial S}=\varphi_1(x),\ldots,\frac{\partial^{k+m-1}u}{\partial\nu^{k+m-1}}\Big|_{\partial S}=\varphi_m(x),\ x\in\partial S. \tag{3.4.60}$$

Let $f\in C^1(\bar S)$ and $f(x)\neq 0$. Consider the volume potential $V[f](x)$ with density $f(x)$

$$V[f](x)=-\frac{1}{\omega_n}\int_S E(x,\xi)f(\xi)\,d\xi,$$

where $E(x,\xi)=(n-2)^{-1}|\xi-x|^{2-n}$ $(n>2)$ is the elementary solution to the Laplace equation, and ω_n is the area of the unit sphere in $\mathbb{R}^n$. Let us denote $V_m[f]=\underbrace{V[\ldots V}_{m}[f]]$. It is known that $f\in C^1(\bar S)\Rightarrow V[f]\in C^1(\bar S)\cap C^2(S)$ and $\Delta V[f]=f$ in S [28], and therefore $V_m[f]\in C^1(\bar S)$ and $\Delta^m V_m[f]=f$. Let us assume that a solution $u(x)$ to the problem (3.4.59)-(3.4.60) exists. Let us represent this solution as the sum $u(x)=V_m[f]+w(x)$. Then we have

$$\Delta^m u(x)-f=\Delta^k V_m[f]+\Delta^m w(x)-f=\Delta^m w(x)$$

and therefore with respect to $w(x)$ we obtain the following problem

$$\Delta^m w=0,\quad x\in S;$$
$$\frac{\partial^k w}{\partial\nu^k}\Big|_{\partial S}=\tilde\varphi_1,\quad \frac{\partial^{k+1}w}{\partial\nu^{k+1}}\Big|_{\partial S}=\tilde\varphi_2,\ldots,\frac{\partial^{k+m-1}w}{\partial\nu^{k+m-1}}\Big|_{\partial S}=\tilde\varphi_m,$$

where

$$\tilde{\varphi}_i(s) = \varphi_i(s) - \frac{\partial^{k+i-1}}{\partial \nu^{k+i-1}} V_m[f],$$

for $i = 1 \ldots, m$. The solvability conditions for the resulting problem $\mathcal{N}_k$ for a homogeneous equation, according to Theorem 3.15, for $l < k$ have the form

$$\begin{aligned}
0 = \int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\tilde{\varphi}_{\delta_\lambda+1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)}\tilde{\varphi}_{m-\sigma_\lambda}(s)\Big)\, ds_x \\
= \int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{\delta_\lambda+1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)}\varphi_{m-\sigma_\lambda}(s)\Big)\, ds_x \\
- \int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\frac{\partial^{k+\delta_\lambda}V_m[f]}{\partial \nu^{k+\delta_\lambda}} + \ldots + p_{m-\lambda}^{(m-\lambda)}\frac{\partial^{m+k-1-\sigma_\lambda}V_m[f]}{\partial \nu^{m+k-1-\sigma_\lambda}}\Big)\, ds_x
\end{aligned}$$

where $\lambda_0 \le \lambda \le \min\{k-l, m\} - 1$ and

$$0 = \int_{\partial S} H_l(x)\tilde{\varphi}_i(s)\, ds_x = \int_{\partial S} H_l(x)\varphi_i(s)\, ds_x - \int_{\partial S} H_l(x)\frac{\partial^{k+i-1}V_m[f]}{\partial \nu^{k+i-1}}\, ds_x,$$

for i such that $\max\{2m+l-k, 0\} + 1 \le i \le m$.

Due to the properties of the operator Λ, the equality $\frac{\partial^i w}{\partial \nu^i} = \Lambda^{[i]} w$ is true on ∂S and taking into account (3.4.38), the conditions obtained above for $\lambda_0 \equiv [(k-l)/2] \le \lambda \le \min\{k-l, m\} - 1$ can be rewritten in the form

$$\begin{aligned}
\int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{2\lambda+l-k+2}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)}\varphi_{m+l+\lambda-k+1}(s)\Big)\, ds_x \\
= \int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\Lambda^{[2\lambda+l+1]} + \ldots + p_{m-\lambda}^{(m-\lambda)}\Lambda^{[\lambda+m+l]}\Big)V_m[f]\, ds_x, \quad (3.4.61)
\end{aligned}$$

and if $k > m + l$, then to these conditions we add conditions of the form

$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = \int_{\partial S} H_l(x)\Lambda^{[k+i-1]}V_m[f]\, ds_x, \qquad (3.4.62)$$

where $\max\{2m+l-k, 0\} + 1 \le i \le m$. If we consider the mapping Φ from (3.4.12) and the polynomial

$$t^{l+2\lambda}P_{m-\lambda}(t) = \sum_{i=1}^{m-\lambda} p_i^{(m-\lambda)} t^{i+l+2\lambda},$$

which, as shown above (see, for example, (3.4.43)), belongs to $\mathcal{L}^{(l)}_{\mathcal{N}_k}$, then the condition (3.4.61) can be rewritten in the form

$$\int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{\delta_\lambda+1}(s)+\ldots+p_{m-\lambda}^{(m-\lambda)}\varphi_{m-\sigma_\lambda}(s)\Big)\,ds_x$$
$$=\int_{\partial S} H_l(x)\Phi[t^{l+2\lambda}P_{m-\lambda}(t)](\Lambda)V_m[f]\,ds_x. \quad (3.4.63)$$

Therefore, it is appropriate to introduce the following functional

$$I_\lambda(f)=\int_{\partial S} H_l(x)\Phi[\Lambda^{l+2\lambda}P_{m-\lambda}(\Lambda)]V_m[f]\,ds_x,$$

which is obviously linear in f.

Lemma 3.31. *Let $0\le\lambda\le m-1$ and $l\in\mathbb{N}_0$, then*

$$\Phi[t^{l+2\lambda}P_{m-\lambda}(t)]=(t,-1)_l(t-l-1,-2)_\lambda(t-l,-2)_m, \quad (3.4.64)$$

where the mapping Φ is defined in (3.4.12)*, and $(a,b)_m$ is the generalized Pochhammer symbol.*

Proof. Let us first prove the equality

$$\Phi[t^{l+2\lambda}P_{m-\lambda}(t)]$$
$$=\begin{cases}(t-1,-2)_{\lambda+[l/2]}(t,-2)_{m+[l/2]}, & l\in 2\mathbb{N}_0,\\ (t,-2)_{\lambda+[l/2]+1}(t-1,-2)_{m+[l/2]}, & l\in 2\mathbb{N}_0+1\end{cases}. \quad (3.4.65)$$

Suppose $l=0$. Let us carry out induction on λ. For $\lambda=0$, according to (3.4.29) we have $\Phi[P_m(t)]=(t,-2)_m$. Assuming that (3.4.65) is true for $0\le\lambda<m-1$, using (3.4.25) and (3.4.29) we find

$$\Phi[t^{2\lambda+2}P_{m-\lambda-1}(t)]=\Phi[t^{2\lambda}P_{m+1-\lambda}(t)+(2m-2\lambda-1)t^{2\lambda}P_{m-\lambda}(t)]$$
$$=\Phi[t^{2\lambda}P_{m+1-\lambda}(t)]+(2m-2\lambda-1)\Phi[t^{2\lambda}P_{m-\lambda}(t)]=(t-1,-2)_\lambda(t,-2)_{m+1}$$
$$+(2m-2\lambda-1)(t-1,-2)_\lambda(t,-2)_m=(t-1,-2)_\lambda(t,-2)_m(t-2m+2m-2\lambda-1)$$
$$=(t-1,-2)_\lambda(t-1-2\lambda)(t,-2)_m=(t-1,-2)_{\lambda+1}(t,-2)_m.$$

The induction step is proved, and with it the case $l=0$. Let $l=1$. We use induction on λ again. For $\lambda=0$, according to (3.4.25), we have

$$\Phi[tP_m(t)]=\Phi[t^2P_{m-1}^{(1)}(t)]=\Phi[P_{m+1}^{(1)}(t)]+(2m+1)\Phi[P_m^{(1)}(t)]$$
$$=(t-1,-2)_{m+1}+(2m+1)(t-1,-2)_m$$

$$= (t-1,-2)_m(t-1-2m+2m+1) = t(t-1,-2)_m,$$

which coincides with (3.4.65) for $l = 1$ and $\lambda = 0$. Assuming that (3.4.65) is true for $0 \le \lambda < m-1$, using (3.4.25) we find

$$\begin{aligned}\Phi[t^{1+2\lambda+2}P_{m-\lambda-1}(t)] &= \Phi[t^{1+2\lambda}P_{m+1-\lambda}(t) + (2m-2\lambda-1)t^{1+2\lambda}P_{m-\lambda}(t)]\\ &= \Phi[t^{1+2\lambda}P_{m+1-\lambda}(t)] + (2m-2\lambda-1)\Phi[t^{1+2\lambda}P_{m-\lambda}(t)]\\ &= (t,-2)_{\lambda+1}(t-1,-2)_{m+1} + (2m-2\lambda-1)(t,-2)_{\lambda+1}(t-1,-2)_m\\ &= (t,-2)_{\lambda+1}(t-1,-2)_m(t-1-2m+2m-2\lambda-1)\\ &= (t,-2)_{\lambda+1}(t-1,-2)_m(t-2\lambda-2) = (t,-2)_{\lambda+2}(t-1,-2)_m.\end{aligned}$$

Let now $l = 2l'$, $l' \in \mathbb{N}_0$. Then it is not difficult to obtain (3.4.65) in the even case

$$\Phi[t^{l+2\lambda}P_{m-\lambda}(t)] = \Phi[t^{2l'+2\lambda}P_{m+l'-\lambda-l'}(t)] = (t-1,-2)_{\lambda+l'}(t,-2)_{m+l'}.$$

Similarly, for $l = 2l'+1$ we have (3.4.65) in the odd case

$$\Phi[t^{l+2\lambda}P_{m-\lambda}(t)] = \Phi[t^{1+2l'+2\lambda}P_{m+l'-l'-\lambda}(t)] = (t,-2)_{\lambda+l'+1}(t-1,-2)_{m+l'}.$$

The equality (3.4.65) is proved. Let us transform it. It is easy to see that (3.4.65) can be rewritten in the same form

$$\begin{aligned}\Phi[t^{l+2\lambda}P_{m-\lambda}(t)] &= (t-1+\theta_l,-2)_{\lambda+[l/2]+\theta_l}(t-\theta_l,-2)_{m+[l/2]} =\\ &= (t-1+\theta_l,-2)_{\lambda+[l/2]+\theta_l}(t-\theta_l,-2)_{[l/2]}(t-l,-2)_m,\end{aligned}$$

where $\theta_l = 0$ if l is even and $\theta_l = 1$ if l is odd. Now let us prove that

$$A_{l,\lambda} \equiv (t-1+\theta_l,-2)_{\lambda+[l/2]+\theta_l}(t-\theta_l,-2)_{[l/2]} = (t,-1)_l(t-l-1,-2)_\lambda,$$

and therefote, taking into account the previous equality, (3.4.64) will be proved. Indeed, if $l = 2l'$, then

$$\begin{aligned}A_{l,\lambda} &= (t-1,-2)_{\lambda+l'}(t,-2)_{l'} = (t-1-2l',-2)_\lambda(t-1,-2)_{l'}(t,-2)_{l'}\\ &= (t-l-1,-2)_\lambda(t,-1)_{2l'} = (t,-1)_l(t-l-1,-2)_\lambda,\end{aligned}$$

and if $l = 2l'+1$, then

$$\begin{aligned}A_{l,\lambda} &= (t,-2)_{\lambda+l'+1}(t-1,-2)_{l'} = (t-2l'-2,-2)_\lambda(t,-2)_{l'+1}(t-1,-2)_{l'}\\ &= (t-l-1,-2)_\lambda(t,-1)_{2l'+1} = (t,-1)_l(t-l-1,-2)_\lambda.\end{aligned}$$

The lemma is proved. □

Using Lemma 3.31 the functional $I_\lambda(f)$ can be rewritten as

$$I_\lambda(f) = \int_{\partial S} H_l(x) S_{l,\lambda}(\Lambda)(\Lambda - l, -2)_m V_m[f]\, ds_x, \tag{3.4.66}$$

where

$$S_{l,\lambda}(t) = (t, -1)_l (t - l - 1, -2)_\lambda, \tag{3.4.67}$$

and the condition (3.4.63), in this case, takes the form

$$\int_{\partial S} H_l(x) \Big(p_1^{(m-\lambda)} \varphi_{\delta_\lambda+1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(s) \Big)\, ds_x$$
$$= \int_{\partial S} H_l(x) S_{l,\lambda}(\Lambda)(\Lambda - l, -2)_m V_m[f]\, ds_x. \tag{3.4.68}$$

This condition is valid when $\lambda_0 \equiv [(k-l)/2] \le \lambda \le m-1$.

Remark 3.20. Let $k > m + l$. In this case, in the orthogonality conditions (3.4.62) the polynomial $t^{[k+i-1]}$ arises, where $i = \max\{2m - k + l, 0\} + 1, \ldots, m$. Then since

$$t^{[k+i-1]} = (t, -1)_{k+i-1} = (t, -1)_l (t - l - 1, -2)_m$$
$$\times (t - l - 2m, -1)_{i-1-(2m-k+l)} (t - l, -2)_m = S'_{i+k-m-1}(t)(t - l, -2)_m,$$

where a notation is used

$$S'_{i+k-m-1}(t) = S_{l,m}(t)(t - l - 2m, -1)_{i-1-(2m-k+l)},$$

then the orthogonality conditions (3.4.62) can be written in the form

$$\int_{\partial S} H_l(x) \varphi_i(s)\, ds_x$$
$$= \int_{\partial S} H_l(x) S'_{i+k-m-1}(\Lambda)(\Lambda - l, -2)_m V_m[f]\, ds_x, \tag{3.4.69}$$

for $i = \max\{2m - k + l, 0\} + 1, \ldots, m$. From the equalities above it is clear that

$$S'_{i+k-m-1}(t) = \frac{(t, -1)_{i+k-1}}{(t - l, -2)_m}. \tag{3.4.70}$$

Thus the statement is proved.

Theorem 3.20. *Let the functions $f(x)$ and $\varphi_i(s)$ for $i = 1, 2, \ldots, m$ be such that the solution $u(x)$ to the problem $\mathcal{N}_k$ for the equation* (3.4.59) *exists and $u \in C^{k+m-1}(\bar{S})$. Then, for any $l \in \mathbb{N}_0$ such that $l < k$ the orthogonality conditions* (3.4.68) *must be satisfied when $\lambda_0 \equiv [(k-l)/2] \le \lambda \le m - 1$ and conditions* (3.4.69) *for $i =$* $\max\{2m - k + l, 0\} + 1, \ldots, m$.

3.4.7 Polynomial right-hand side of the equation

Let the function $f(x)$ in the equation (3.4.59) be a polynomial. According to the Almansi formula, the polynomial $f(x)$ can always be written in the form

$$f(x) = \sum_{r,s} |x|^{2r} H_s^{(r)}(x), \tag{3.4.71}$$

where $H_s^{(r)}(x)$ are homogeneous harmonic polynomials of degree s. As such polynomials, we can take polynomials from the orthogonal system of harmonic polynomials from Definition 1.5. To calculate the potential $V[f](x)$ we use Theorem 3.9 on page 195

$$V[f](x) = \sum_{r,s} \frac{|x|^{2r+2} H_s^{(r)}(x)}{(2r+2)(2r+2s+n)} - \sum_{r,s} \frac{H_s^{(r)}(x)}{(2r+2)(2s+n-2)}. \tag{3.4.72}$$

Based on this equality, we prove the following statement.

Theorem 3.21. *If the polynomial $f(x)$ is written in the form* (3.4.71) *and* $[(k-l)/2]+1 \le m$*, then the functional $I_\lambda(f)$ has the form*

$$I_\lambda(f) = \sum_{r} \frac{S_{l,\lambda}(2r+2m+l)}{(2r+2l+n,2)_m} (H_l, H_l^{(r)}), \tag{3.4.73}$$

where $[(k-l)/2] \le \lambda \le m-1$ *and* $(H_l, H_s) = \int_{\partial S} H_l(x) H_s(x)\, ds_x$.

Proof. Let us first assume that $f(x)$ is a polynomial of a special form $f(x) = H_{r,s}(x) = |x|^{2r} H_s^{(r)}(x)$, where $H_s^{(r)}(x)$ is a homogeneous harmonic polynomial of degree s. From the equality (3.4.72) it follows that for $H_{r,s}(x) = |x|^{2r} H_s^{(r)}(x)$ the equality holds

$$\begin{aligned} V_m[H_{r,s}] &= V[\dots V[H_{r,s}]] \\ &= \frac{|x|^{2r+2m} H_s^{(r)}(x)}{(2r+2,2)_m (2r+2s+n,2)_m} + Q_{m-1}(|x|^2) H_s^{(r)}(x), \end{aligned} \tag{3.4.74}$$

where $Q_{m-1}(|x|^2)$ is some polynomial of degree $m-1$ in $|x|^2$, the specific form of which does not matter. To shorten the entries, let us denote $R(t) = S_{l,\lambda}(t)(t-l,-2)_m$. Suppose $\alpha \in \mathbb{R}$. Since for the operator Λ the following equalities are true

$$\begin{aligned} (\Lambda-\alpha)\big(Q_{m-1}(|x|^2) H_s^{(r)}(x)\big) &= H_s^{(r)}(x)(\Lambda-\alpha) Q_{m-1}(|x|^2) \\ + Q_{m-1}(|x|^2)\Lambda H_s^{(r)}(x) &= \big((\Lambda-\alpha)Q_{m-1}(|x|^2) + sQ_{k-1}(|x|^2)\big) H_s^{(r)}(x) \end{aligned}$$

$$= H_s^{(r)}(x)(\Lambda + s - \alpha)Q_{m-1}(|x|^2),$$

and $R(t)$ is a product of prime factors, then

$$\begin{aligned}R(\Lambda)\big(Q_{m-1}(|x|^2)H_s^{(r)}(x)\big) &= S_{l,\lambda}(\Lambda)\prod_{i=0}^{m-1}(\Lambda - l - 2i)\big(Q_{m-1}(|x|^2)H_s^{(r)}(x)\big)\\ &= S_{l,\lambda}(\Lambda)H_s^{(r)}(x)\prod_{i=0}^{p-1}(\Lambda - l + s - 2i)Q_{m-1}(|x|^2)\\ &= S_{l,\lambda}(\Lambda)H_s^{(r)}(x)(\Lambda - l + s, -2)_m Q_{m-1}(|x|^2) = H_s^{(r)}(x)S_{l,\lambda}(\Lambda + s)\\ &\times (\Lambda + s - l, -2)_m Q_{m-1}(|x|^2) = H_s^{(r)}(x)R(\Lambda + s)Q_{m-1}(|x|^2)\end{aligned}$$

and similarly

$$R(\Lambda)\frac{|x|^{2r+2m}H_s^{(r)}(x)}{(2r+2,2)_m(2r+2s+n,2)_m} = H_s^{(r)}(x)\frac{R(2r+2m+s)}{(2r+2,2)_m(2r+2s+n,2)_m}|x|^{2r+2m}.$$

Therefore, using (3.4.74), we have

$$R(\Lambda)V_m[H_{r,s}] = H_s^{(r)}(x)\frac{R(2r+2m+s)}{(2r+2,2)_m(2r+2s+n,2)_m}|x|^{2r+2m} + H_s^{(r)}(x)R(\Lambda + s)Q_{m-1}(|x|^2).$$

Therefore, using (3.4.74), we have

$$I_\lambda(H_{r,s}) = \Big(\frac{R(2r+2m+s)}{(2r+2,2)_m(2r+2s+n,2)_m} + R(\Lambda + s)Q_{m-1}(|x|^2)\big|_{|x|=1}\Big)\int_{\partial S} H_l(x)H_s^{(r)}(x)\,ds_x. \quad (3.4.75)$$

Hence, for $s \neq l$, due to the orthogonality of harmonic polynomials of various degrees on ∂S, we obtain $I_\lambda(H_{r,s})(x) = 0$. If $s = l$, then consider the polynomial $R(\Lambda + s)Q_{m-1}(|x|^2)$. Let the monomial $q_i|x|^{2i}$, $0 \le i \le m-1$ be included in $Q_{m-1}(|x|^2)$. In accordance with the definition $R(t) = S_{l,\lambda}(t)(t-l,-2)_m$ we have

$$R(\Lambda + s)q_i|x|^{2i} = q_i S_{l,\lambda}(\Lambda + s)(\Lambda + s - l, -2)_m|x|^{2i}$$

$$= q_i S_{l,\lambda}(\Lambda + l)(\Lambda, -2)_m |x|^{2i} = q_i S_{l,\lambda}(2i + l)(2i, -2)_m |x|^{2i} = 0,$$

since $0 \le i \le m-1$. Therefore, $R(\Lambda + s)Q_{m-1}(|x|^2) = 0$, and therefore from (3.4.75) with $s = l$, similarly to the previous equality, we find

$$I_\lambda(H_{r,l}) = \frac{S_{l,\lambda}(2r + 2m + l)(2r + 2m, -2)_m}{(2r + 2, 2)_m(2r + 2l + n, 2)_m} \int_{\partial S} H_l(x) H_l^{(r)}(x)\, ds_x$$
$$= \frac{S_{l,\lambda}(2r + 2m + l)}{(2r + 2l + n, 2)_m} (H_l, H_l^{(r)}).$$

Since the polynomial $f(x)$ from (3.4.71) is the sum of polynomials of the form $|x|^{2r} H_s(x)$ and the functional $I_\lambda(f)$ is linear, then summing the functionals $I_\lambda(H_{r,s})$ by r and s, we get (3.4.73). The theorem is proved. □

Based on this theorem, we establish the necessary conditions for the existence of a solution to the Neumann problems $\mathcal{N}_k$ for the inhomogeneous polyharmonic equation with a polynomial right-hand side.

Theorem 3.22. *Let $f(x)$ be a polynomial and functions $\varphi_i(s)$ for $i = 1, 2, \ldots, m$ such that the solution $u(x)$ to the problem $\mathcal{N}_k$ for the equation* (3.4.59) *exists and $u \in C^{k+m-1}(\bar{S})$. Then, for any $l \in \mathbb{N}_0$ such that $l < k$ the following orthogonality conditions must be satisfied*

$$\begin{gathered} \int_{\partial S} H_l(x) \Big(p_1^{(m-\lambda)} \varphi_{\delta_\lambda + 1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(s) \Big)\, ds_x \\ = \int_S \frac{\big(1 - |x|^2\big)^{m-1}}{(2m-2)!!} H_l(x) S_{l,\lambda}(\Lambda + 2m) f(x)\, dx, \\ \lambda = \lambda_0, \ldots, \min\{k - l, m\} - 1, \\ \int_{\partial S} H_l(x) \varphi_i(s)\, ds_x = \int_S \frac{\big(1 - |x|^2\big)^{m-1}}{(2m-2)!!} H_l(x) S'_{i+k-m-1}(\Lambda) f(x)\, dx, \\ i = \max\{2m + l - k, 0\} + 1, \ldots, m, \end{gathered} \tag{3.4.76}$$

where $p_i^{(j)}$ are taken from the triangle $\mathbb{P}$ (3.4.27)*, $\lambda_0 = [(k-l)/2]$, $\delta_\lambda = 2\lambda - k + l + 1$, $\sigma_\lambda = k - l - \lambda - 1$ and*

$$S_{l,\lambda}(t) = (t, -1)_l (t - l - 1, -2)_\lambda, \quad S'_{i+k-m-1}(t) = (t, -1)_{k+i-1} / (t - l, -2)_m.$$

The number of conditions (3.4.76) *for $l < k$ is equal to*

$$N_{m,k} = \begin{cases} [(k+1)/2][(k+2)/2], & k \le 2m, \\ N_{m,k} = m(k - m + 1), & k \ge 2m \end{cases}.$$

Proof. Let $u \in C^{k+m-1}(\bar{S})$. We represent the polynomial $f(x)$ in the form (3.4.71). Consider the case $\lambda_0 = [(k-l)/2] \le m-1$. By virtue of Theorem 3.15, the necessary conditions for the solvability of the problem $\mathcal{N}_k$ have the form (3.4.68). By Theorem 3.21 the functional $I_\lambda(f)$ can be transformed to the form (3.4.73)

$$I_\lambda(f) = \sum_r \frac{b_r}{(2r+2l+n,2)_m}(H_l, H_l^{(r)}),$$

where for brevity we denote $b_r = S_{l,\lambda}(2r+2m+l)$. By Lemma 3.19 on page 198 we have

$$\frac{1}{(2r+2l+n,2)_m} = \frac{1}{(2m-2)!!}\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^i}{2r+2l+n+2i},$$

which means, using the orthogonality on ∂S of harmonic polynomials of various degrees, it is easy to obtain

$$\begin{aligned}
I_\lambda(f) &= \sum_r \frac{b_r}{(2m-2)!!}\sum_{i=0}^{m-1}\binom{m-1}{i}\frac{(-1)^i}{2r+2l+n+2i}(H_l, H_l^{(r)}) \\
&= \sum_{r,s}\frac{b_r}{(2m-2)!!}\int_0^1 \rho^{2r}\int_{|x|=1}\rho^{2l}H_l(x)H_s^{(r)}(x)\sum_{i=0}^{m-1}(-\rho^2)^i\binom{m-1}{i}\,ds_x \\
\times \rho^{n-1}\,d\rho &= \sum_{r,s}\frac{b_r}{(2m-2)!!}\int_0^1 \rho^{2r}\int_{|x|=\rho}H_l(x)H_s^{(r)}(x)(1-\rho^2)^{m-1}\,ds_x\,d\rho \\
&= \int_{|x|<1}\frac{\left(1-|x|^2\right)^{m-1}}{(2m-2)!!}H_l(x)\sum_{r,s}S_{l,\lambda}(2r+2m+l)|x|^{2r}H_s^{(r)}(x)\,dx \\
&= \int_{|x|<1}\frac{\left(1-|x|^2\right)^{m-1}}{(2m-2)!!}H_l(x)S_{l,\lambda}(\Lambda+2m)\sum_{r,s}|x|^{2r}H_s^{(r)}(x)\,dx \\
&= \int_{|x|<1}\frac{\left(1-|x|^2\right)^{m-1}}{(2m-2)!!}H_l(x)S_{l,\lambda}(\Lambda+2m)f(x)\,dx.
\end{aligned}$$

Substituting the value of the resulting functional $I_\lambda(f)$ into the right side of the equality (3.4.68) we have the first equality from (3.4.76)

$$\begin{aligned}
&\int_{\partial S} H_l(x)\Big(p_1^{(m-\lambda)}\varphi_{\delta_\lambda+1}(s) + \ldots + p_{m-\lambda}^{(m-\lambda)}\varphi_{m-\sigma_\lambda}(s)\Big)\,ds_x \\
&\qquad = \int_S \frac{\left(1-|x|^2\right)^{m-1}}{(2m-2)!!}H_l(x)S_{l,\lambda}(\Lambda+2m)f(x)\,dx,
\end{aligned}$$

where $\lambda = \lambda_0, \dots, \min\{k-l, m\} - 1$ and $S_{l,\lambda}(t)$ is defined in (3.4.67).

Further, for $k > m$, according to Remark 3.19, orthogonality conditions arise (3.4.69)

$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = \int_{\partial S} H_l(x) S'_{i+k-m-1}(\Lambda)(\Lambda - l, -2)_m V_m[f]\, ds_x,$$

where $i = \max\{2m-k+l, 0\}+1, \dots, m$ and $S'_{i+k-m-1}(t)$ is defined in (3.4.70). Operating with the integral on the right similarly to the previous case with the replacement of the polynomial $S_{l,\lambda}(t)$ by the polynomial $S'_{i+k-m-1}(t)$ (when calculating the specific form of $S_{l,\lambda}(t)$ is not used anywhere) we get

$$\int_{\partial S} H_l(x)\varphi_i(s)\, ds_x = \int_S H_l(x) S'_{i+k-m-1}(\Lambda + 2m) f(x)\, dx,$$

where $i = \max\{2m - k + l, 0\} + 1, \dots, m$.

The total number of orthogonality conditions for a given k (see Theorem 3.15 on page 231) is equal to

$$N_{m,k} = \sum_{l=0}^{k-1} N_{m,k,l} = \sum_{l=0}^{k-1} \min\{[(k-l+1)/2], m\} = \sum_{l=1}^{k} \min\{[(l+1)/2], m\},$$

where the equality $k - l - \lambda_0 = [(k-l+1)/2]$ and the change of summation index $k - l \to l$ are used. If $k \le 2m$, then we have

$$N_{m,k} = \sum_{l=1}^{k} [(l+1)/2] = 1 + 1 + \ldots + [(k+1)/2] = [(k+1)/2][(k+2)/2].$$

Indeed, for $k = 2k'$ we obtain $N_{m,k} = 1 + 1 + \ldots + k' + k' = k'(k'+1) = [(k+1)/2][(k+2)/2]$, and for $k = 2k'+1$ we find $N_{m,k} = 1 + 1 + \ldots + k' + k' + (k'+1) = (k'+1)(k'+1) = [(k+1)/2][(k+2)/2]$. If $k \ge 2m$, then we have $N_{m,k} = N_{m,2m} + m(k-2m) = m(k-m+1)$. The theorem is proved. □

Remark 3.21. The solvability conditions for the problem $\mathcal{N}_k$ for the inhomogeneous equation (3.4.59) from Theorem 3.20 can probably be written in a simpler form (3.4.76). This assumption is based on the results of Sections 3.3.1 and 3.3.3, where a similar transition is made for the problem $\mathcal{N}_1$: from polynomials (Theorem 3.10) to a general function (Theorem 3.11).

Example 3.10. Let us consider the problem $\mathcal{N}_3$ studied in Example 3.9, but for an inhomogeneous 7-harmonic equation with a polynomial right-hand side

$$\begin{gathered}\Delta^7 u = f(x), \quad x \in S; \\ \frac{\partial^3 u}{\partial \nu^3}\Big|_{\partial S} = \varphi_1(s),\ \frac{\partial^4 u}{\partial \nu^4}\Big|_{\partial S} = \varphi_2(s), \dots, \frac{\partial^9 u}{\partial \nu^9}\Big|_{\partial S} = \varphi_7(s),\ s \in \partial S.\end{gathered} \tag{3.4.77}$$

Obviously, $m = 7$, $k = 3$ and $l = 0, 1, 2$. Let us use the results of calculations from Example 3.9 taking into account Remark 3.17. The number of conditions is $N_{7,3} = [(3+1)/2][(3+2)/2] = 4$.

For $l = 0$ we have $\lambda_0 = 1$ and $N_{7,3,0} = 2$ conditions for $1 \le \lambda \le 2$, where $\delta_\lambda + 1 = 2\lambda - 1$, $m - \sigma_\lambda = 5 + \lambda$, $S_{0,1}(t) = (t, -1)_0(t-1, -2)_1 = t - 1$, $S_{0,2}(t) = (t-1, -2)_2 = (t-1)(t-3)$

$$\int_{\partial S} \left(p_1^{(6)} \varphi_1(s) + \ldots + p_6^{(6)} \varphi_6(s)\right) ds_x = \int_S \frac{\left(1 - |x|^2\right)^6}{12!!} (\Lambda + 13) f(x)\, dx,$$

$$\int_{\partial S} \left(p_1^{(5)} \varphi_3(s) + \ldots + p_5^{(5)} \varphi_7(s)\right) ds_x = \int_S \frac{\left(1 - |x|^2\right)^6}{12!!} \times (\Lambda + 13)(\Lambda + 11) f(x)\, dx.$$

For $l = 1$ we have $\lambda_0 = 1$ and $N_{7,3,1} = 1$ conditions for $1 \le \lambda \le 1$, where $\delta_\lambda + 1 = 2\lambda$, $m - \sigma_\lambda = 6 + \lambda$, $S_{1,1}(t) = (t, -1)_1(t-2, -2)_1 = t(t-2)$

$$\int_{\partial S} H_2(x)\left(p_1^{(7)} \varphi_1(s) + \ldots + p_7^{(7)} \varphi_7(s)\right) ds_x = \int_S \frac{\left(1 - |x|^2\right)^6}{12!!} H_2(x)(\Lambda + 14)(\Lambda + 13) f(x)\, dx.$$

For $l = 2$ we obtain $\lambda_0 = 0$ and $N_{7,3,2} = 1$ conditions for $0 \le \lambda \le 0$, where $\delta_\lambda + 1 = 2\lambda + 1$, $m - \sigma_\lambda = 7 + \lambda$, $S_{2,0}(t) = (t, -1)_2(t-2, -2)_0 = t(t-1)$

$$\int_{\partial S} H_2(x)\left(p_1^{(7)} \varphi_1(s) + \ldots + p_7^{(7)} \varphi_7(s)\right) ds_x = \int_S \frac{\left(1 - |x|^2\right)^6}{12!!} H_2(x)(\Lambda + 14)(\Lambda + 13) f(x)\, dx.$$

Since $k = 3 < 7 = m$, then by virtue of Theorem 3.19 for the solvability of the problem (3.4.77) the conditions written above are necessary and sufficient solvability conditions.

3.5 Generalized Dirichlet-Riquier problem for the biharmonic equation

Steady-state processes of various physical natures are described by elliptic-type equations. One of the important special cases of fourth-order elliptic equations is the biharmonic equation $\Delta^2 u(x) = f(x)$. Solving plane deformation problems in elasticity theory in many cases comes down to integrating the biharmonic equation under appropriate boundary conditions.

In addition, many problems in continuum mechanics are reduced to solving harmonic and biharmonic equations, but convenient analytical expressions for solutions have been obtained only for some domains of a particular type. Numerous scientific studies have been devoted to the application of biharmonic problems in mechanics and physics (see [7, 8, 31, 43–45, 138]).

Various applications of boundary value problems for the biharmonic equation in mechanics and physics stimulate the study of various formulations of boundary value problems for the biharmonic equation. One of the first important works on biharmonic equations is Almansi's paper [5]. In this work, Almansi proved his remarkable representation for polyharmonic functions (see also [162]). Following this work, great interest arose in the study of boundary value problems for the biharmonic equation (see [55, 141, 144, 192]). In [71] the Almansi representation is extended to analytic functions.

For the biharmonic equation, the Dirichlet problem is well known. Recently, other types of boundary value problems for the biharmonic equation have begun to be actively studied, such as the Navier problem, the Neumann problem, the Robin problem [49], etc. In spectral theory, the Steklov spectral problem [48] is relevant. The conditions for the solvability of these problems are studied in the classical theory of boundary value problems for elliptic equations and systems of equations satisfying the so-called complementarity condition. It has been established that all problems of this type are Fredholm and therefore their solvability for homogeneous boundary conditions is guaranteed by the orthogonality of the right-hand sides to all solutions of the homogeneous adjoint equation. In each specific case, it is possible to obtain more detailed results.

This section is organized as follows. In Section 3.5.1 for the biharmonic equation in the unit ball, a boundary value problem with boundary conditions of the general form (3.5.1)-(3.5.2) is formulated, which we call the Dirichlet-Riquier problem. Special cases of the problem under consideration are the classical Dirichlet, Navier, Robin problems listed above, the Steklov spectral problem (see Section 3.5.4), as well as many other boundary value problems generated by these boundary conditions. Theorem 3.23 from Section 3.5.1 formulates conditions for the uniqueness of a solution to the problem under study. Then, in Theorem 3.24, conditions for the unconditional solvability of the problem for the homogeneous biharmonic equation are found, which coincided with the uniqueness conditions from Theorem 3.23. Theorem 3.25 of Section 3.5.3 considers the case when the conditions of Theorem 3.24 are not satisfied, but a solution to the problem under consideration still exists. Finally, Section 3.5.4 considers some special cases of the general problem. The main results of Section 3.5 are published in papers [92, 93, 96, 97, 102].

3.5.1 Statement of the problem

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, and $\partial S = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. In S we consider the following boundary value problem for the biharmonic equation

$$\Delta^2 u = f(x), \quad x \in S, \tag{3.5.1}$$

$$\begin{aligned} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u\Big|_{\partial S} &= \varphi_1(s), \quad s \in \partial S, \\ a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u\Big|_{\partial S} &= \varphi_2(s), \quad s \in \partial S, \end{aligned} \tag{3.5.2}$$

where $\frac{\partial}{\partial\nu}$ is the outer normal derivative to ∂S, the coefficients a_{0j} and a_{1j} for $j = 1, 2, 3$ are real and constant, and $f(x), \varphi_1(x), \varphi_2(x)$ are given functions, the smoothness of which will be indicated below. The solution to the problem (3.5.1)-(3.5.2) is the function $u(x)$ from the class $u \in C^4(S) \cap C^3(\overline{S})$, satisfying the equation (3.5.1) in S and the conditions (3.5.2) on ∂S.

The problem (3.5.1)-(3.5.2) generalizes the well-known Dirichlet problem ($a_{00} \neq 0$, $a_{11} \neq 0$, and all other coefficients are zero), the Navier problem from Section 1.2 ($a_{00} \neq 0$, $a_{12} \neq 0$, and all other coefficients are zero), but does not generalize the Neumann problem from Section 3.3

$$\Delta^2 u(x) = f(x), \quad x \in S,$$

$$\frac{\partial u}{\partial\nu}\Big|_{\partial S} = \varphi_1(x), \quad \frac{\partial^2 u}{\partial\nu^2}\Big|_{\partial S} = \varphi_2(x), \quad x \in \partial S.$$

Let $a_{00} \neq 0$, $a_{12} > 0$, $a_{11} < 0$, and all other coefficients be zero. Then the conditions (3.5.2) coincide with the conditions of the Steklov problem from the work [48].

The paper [131] considered an extended version of the problem (3.5.1)-(3.5.2). Instead of boundary conditions (3.5.2), conditions of the form are taken there

$$\begin{aligned} a_{00}u + a_{01}\frac{\partial}{\partial \nu}u + a_{02}\Delta u + a_{03}\frac{\partial}{\partial \nu}\Delta u\Big|_{\partial S} &= \varphi_1(s), \quad s \in \partial S, \\ a_{10}u + a_{11}\frac{\partial}{\partial \nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial \nu}\Delta u\Big|_{\partial S} &= \varphi_2(s), \quad s \in \partial S, \end{aligned} \tag{3.5.3}$$

which can easily be reduced to (3.5.2). In [131] a special case of uniqueness conditions is obtained, namely: if

$$\begin{vmatrix} a_{00} & a_{01} + na_{02} \\ a_{10} & a_{11} + na_{02} \end{vmatrix} \neq 0,$$

then $u = const$ is not a solution to the homogeneous problem (3.5.1)-(3.5.3). In this section, we establish a criterion for the existence of a solution to the problem (3.5.1)-(3.5.2) in the general case.

Note that for different values of the coefficients a_{0j} and a_{1j} and $j = 1, 2, 3$ the problem (3.5.1)-(3.5.2) coincides with problems considered in [36,43,48,49,54,131]. In [27, 171] the existence of a solution to the Neumann problem and other boundary value problems for the biharmonic and harmonic equation with fractional operators in boundary conditions is investigated.

Theorem 3.23. *The solution to the problem* (3.5.1)-(3.5.2) *is unique if and only if the following polynomial*

$$\Delta(\lambda) = \begin{vmatrix} a_{00} + a_{01}\lambda & 2a_{01} + (2n + 4\lambda)a_{02} \\ \lambda a_{11} & 2a_{11} + (2n + 4\lambda)a_{12} + \lambda(2n + 4\lambda)a_{13} \end{vmatrix}$$

has no roots in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *If* $\Delta(m) = 0$, $m \in \mathbb{N}_0$, *then the homogeneous problem* (3.5.1)-(3.5.2) *has a solution*

$$u(x) = \left(C_2|x|^2 + C_1 - C_2\right) H_m(x),$$

where $H_m(x)$ *is an arbitrary homogeneous harmonic polynomial of degree* m, *and the constants* C_1 *and* C_2 *are found from the system algebraic equations*

$$\begin{pmatrix} a_{00} + ma_{01} & 2a_{01} + (2n + 4m)a_{02} \\ ma_{11} & 2a_{11} + (2n + 4m)a_{12} + m(2n + 4m)a_{13} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{3.5.4}$$

Proof. Let us prove that the homogeneous problem (3.5.1)-(3.5.2)

$$\Delta^2 u = 0, \quad x \in S, \tag{3.5.5}$$

$$\begin{aligned} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u = 0, \quad s \in \partial S, \\ a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u = 0, \quad s \in \partial S \end{aligned} \tag{3.5.6}$$

has only a zero solution when the polynomial $\Delta(\lambda)$ has no roots in $\mathbb{N}_0$. It is known that every biharmonic function in S is such that $u(x) \in C^2(\overline{S})$ can be expanded into a power series (see Theorem 1.8 on page 50 and Lemma 3.32 below), and therefore the solution to the problem (3.5.5)-(3.5.6) can be written in the form

$$u(x) = u_0(x) + |x|^2 v_0(x) = \sum_{m=0}^{\infty}\sum_{i=1}^{h_n}\left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x), \tag{3.5.7}$$

where $x \in S$, $h_m = \frac{2m+n-2}{n-2}\binom{m+n-3}{n-3}$, and $\{H_m^{(i)}(x), m \in \mathbb{N}_0, i = \overline{1, h_k}\}$ is a complete system of harmonic polynomials orthogonal on ∂S from the definition refd31d0. The series (3.5.7) converges uniformly for $|x| \le \varepsilon < 1$. Let us look at the operators

$$L_1 = a_{00} + a_{01}\Lambda + a_{02}\Delta, \quad L_2 = a_{11}\Lambda + a_{12}\Delta + a_{13}\Lambda\Delta,$$

where as usual $\Lambda = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Since we assume that $u \in C^3(\overline{S})$, then from properties of the operator Λ it follows that

$$\begin{aligned} L_1 u(x) \underset{s\in\partial S}{\rightrightarrows} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta\, u = 0 \\ L_2 u(x) \underset{s\in\partial S}{\rightrightarrows} a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta\, u + a_{13}\frac{\partial}{\partial\nu}\Delta u = 0, \end{aligned} \tag{3.5.8}$$

for $x \to s \in \partial S$. Calculations show (see below) that

$$\begin{aligned} &L_1\left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\ &\quad = \left(u_m^{(i)}(a_{00} + ma_{01}) + v_m^{(i)}(a_{00} + (m+2)a_{01} + (2n+4m)a_{02})\right) H_m^{(i)}(x), \end{aligned}$$

and

$$\begin{aligned} &L_2\left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\ &\quad \left(u_m^{(i)} m a_{11} + v_m^{(i)}((m+2)a_{11} + (2n+4m)a_{12} + m(2n+4m)a_{13})\right) H_m^{(i)}(x), \end{aligned}$$

and therefore, for a fixed $j = 1,2$, the polynomials $L_j\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big)H_m^{(i)}(x)$ are orthogonal on the sphere $\{x : |x| = \varepsilon < 1\}$ to the polynomials $H_p^{(k)}(x)$ for $(m,i) \neq (p,k)$ since such is the system of harmonic polynomials $H_m^{(i)}(x)$.

Let for some $m \in \mathbb{N}_0$ and $i = \overline{1, h_m}$ or $u_m^{(i)} \neq 0$, or $v_m^{(i)} \neq 0$ in the expansion (3.5.7). Then, due to the uniform convergence of the series from (3.5.7) for $|x| \leq \varepsilon < 1$ and the orthogonality of the polynomials, we have

$$\int_{|x|=\varepsilon} H_m^{(i)}(x) L_j u(x) ds_x = \int_{|x|=\varepsilon} H_m^{(i)}(x) L_j \sum_{p=0}^{\infty} \sum_{k=1}^{h_p} \left(u_p^{(k)} + |x|^2 v_p^{(k)}\right) \times H_p^{(k)}(x) ds_x = \int_{|x|=\varepsilon} H_m^{(i)}(x) L_j \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) ds_x.$$

Letting $\varepsilon \to 1$ in the resulting equality and using (3.5.8) we obtain for $j = 1,2$

$$\int_{|x|=1} H_m^{(i)}(x) L_j \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) ds_x = 0. \tag{3.5.9}$$

Let us prove the calculations written above for the integrand expression. To do this, note that the equality $\Lambda(uw) = w\Lambda u + u\Lambda w$ is true and

$$\Delta(|x|^2 P_m(x)) = 2nP_m(x) + 4mP_m(x) = (2n+4m)P_m(x).$$

Then on the unit sphere ∂S we have the equalities

$$\begin{aligned} L_1\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big)H_m^{(i)}(x) &= \big(a_{00} + a_{01}\Lambda + a_{02}\Delta\big)\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big)H_m^{(i)}(x) \\ &= \Big(a_{00}\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big)H_m^{(i)}(x) + a_{01}\Big(2|x|^2 v_m^{(i)} + m u_m^{(i)} + m|x|^2 v_m^{(i)}\Big) \\ &\quad + a_{02} v_m^{(i)}(2n+4m)\Big)H_m^{(i)}(x) = \Big(u_m^{(i)}(a_{00} + m a_{01}) + v_m^{(i)}(a_{00} \\ &\quad + (m+2)a_{01} + (2n+4m)a_{02})\Big)H_m^{(i)}(x) \end{aligned}$$

and

$$\begin{aligned} L_2\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big)H_m^{(i)}(x) &= \big(a_{11}\Lambda + a_{12}\Delta + a_{13}\Lambda\Delta\big)\Big(u_m^{(i)} + |x|^2 v_m^{(i)}\Big) \\ &\quad \times H_m^{(i)}(x) = (a_{11}\Big(m u_m^{(i)} + (m+2)|x|^2 v_m^{(i)}\Big)H_m^{(i)}(x) + a_{12} v_m^{(i)}(2n+4m) \\ &\quad + a_{13} v_m^{(i)} m(2n+4m))H_m^{(i)}(x) = \Big(u_m^{(i)} m a_{11} + v_m^{(i)}((m+2)a_{11} + (2n+4m)a_{12} \end{aligned}$$

$$+ m(2n+4m)a_{13})\Big)H_m^{(i)}(x).$$

Therefore (3.5.9) can be rewritten in the form

$$\Big(u_m^{(i)}(a_{00}+ma_{01}) + v_m^{(i)}(a_{00}+(m+2)a_{01}+(2n+4m)a_{02})\Big)$$
$$\times\big\|H_m^{(i)}(x)\big\|^2_{L_2(\partial S)} = 0,$$
$$\Big(u_m^{(i)}ma_{11} + v_m^{(i)}((m+2)a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13})\Big)$$
$$\times\big\|H_m^{(i)}(x)\big\|^2_{L_2(\partial S)} = 0.$$

Since $\|H_m^{(i)}(x)\|^2_{L_2(\partial S)} \neq 0$, then we get

$$u_m^{(i)}\,(a_{00}+ma_{01}) + v_m^{(i)}\,(a_{00}+(m+2)a_{01}+(2n+4m)a_{02}) = 0,$$
$$u_m^{(i)}ma_{11} + v_m^{(i)}\,((m+2)a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13}) = 0,$$

or in the matrix form

$$\begin{pmatrix} a_{00}+ma_{01} & a_{00}+(m+2)a_{01}+(2n+4m)a_{02} \\ ma_{11} & (m+2)a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13} \end{pmatrix}\begin{pmatrix} u_m^{(i)} \\ v_m^{(i)} \end{pmatrix} = 0. \quad (3.5.10)$$

Let us calculate the determinant Δ of this system of equations

$$\Delta = \begin{vmatrix} a_{00}+ma_{01} & a_{00}+(m+2)a_{01}+(2n+4m)a_{02} \\ ma_{11} & (m+2)a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13} \end{vmatrix}.$$

If we subtract the first from the second column and use the assumptions of the theorem, we get

$$\Delta = \begin{vmatrix} a_{00}+ma_{01} & 2a_{01}+(2n+4m)a_{02} \\ ma_{11} & 2a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13} \end{vmatrix} = \Delta(m) \neq 0.$$

Therefore, the system (3.5.10) has only the zero solution $\binom{u_m^{(i)}}{v_m^{(i)}} = 0$. This contradicts the assumption that either $u_m^{(i)} \neq 0$ or $v_m^{(i)} \neq 0$. Thus, the problem (3.5.5)-(3.5.6) has only a zero solution.

If for some $m \in \mathbb{N}_0$ the equality $\Delta(m) = 0$ holds, then the system of algebraic equations (3.5.4) has a non-zero solution $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \neq 0$, which means

$$\begin{pmatrix} a_{00}+ma_{01} & a_{00}+(m+2)a_{01}+(2n+4m)a_{02} \\ ma_{11} & (m+2)a_{11}+(2n+4m)a_{12}+m(2n+4m)a_{13} \end{pmatrix}\begin{pmatrix} C_1-C_2 \\ C_2 \end{pmatrix} = 0.$$

Therefore, on the unit sphere ∂S the following equalities are true

$$L_1\left(C_1 - C_2 + |x|^2 C_2\right) H_m(x) = 0, \quad L_2\left(C_1 - C_2 + |x|^2 C_2\right) H_m(x) = 0$$

and therefore the biharmonic polynomial

$$u(x) = C_1 H_m(x) + C_2(|x|^2 - 1)H_m(x)$$

is a solution to the homogeneous problem (3.5.5)-(3.5.6). The theorem is proved. □

3.5.2 Unconditional existence of a solution

Let us prove one auxiliary statement in which the harmonic functions from the Almansi representation are explicitly defined.

Lemma 3.32. *Let the function $u(x)$ be biharmonic in the star like domain D, then the Almansi representation is valid for it*

$$u(x) = u_0(x) + |x|^2 u_1(x), \quad x \in D, \tag{3.5.11}$$

where $u_0(x)$ and $u_1(x)$ are functions harmonic in D, defined by the equalities

$$u_0(x) = u(x) - \frac{|x|^2}{2}\int_0^1 t^{n-1}\Delta u(t^2x)\,dt, \quad u_1(x) = \frac{1}{2}\int_0^1 t^{n-1}\Delta u(t^2x)\,dt. \tag{3.5.12}$$

Proof. It is easy to see that since the domain D is star like, the functions $u_0(x)$ and $u_1(x)$, defined by the equalities (3.5.12) are defined in D, and the representation (3.5.11) is correct. Let us prove that the functions $u_0(x)$ and $u_1(x)$ from (3.5.12) are harmonic in D. It is obvious that the function $u_1(x)$ is harmonic in D if the function $u(x)$ is biharmonic in D. Further, since the equality $\Delta(|x|^2 v(x)) = (2n+4\Lambda)v(x)$ is true for the harmonic function $v(x)$, and also for the differentiable function $w(x)$ the chain of equalities holds

$$\begin{aligned}(2n + 4\Lambda)\frac{1}{2}\int_0^1 t^{n-1}w(t^2x)dt &= n\int_0^1 t^{n-1}w(t^2x)dt \\ + 2\int_0^1 t^{n+1}\sum_{i=1}^n x_i w_{x_i}(t^2)\,dt &= n\int_0^1 t^{n-1}w(t^2x)dt + \int_0^1 t^n w_t(t^2x)\,dt \\ &= n\int_0^1 t^{n-1}w(t^2x)dt + t^n w(t^2x)|_0^1 - n\int_0^1 t^{n-1}w(t^2x)dt = w(x),\end{aligned}$$

then the equalities are valid

$$\Delta u_0(x) = \Delta u(x) - \frac{1}{2}\Delta\left(|x|^2\int_0^1 t^{n-1}\Delta u(t^2x)dt\right)$$
$$= \Delta u(x) - (2n+4\Lambda)\frac{1}{2}\int_0^1 t^{n-1}\Delta u(t^2x)dt = \Delta u(x) - \Delta u(x) = 0.$$

Therefore the function $u_0(x)$ is harmonic in D. The lemma is proved. □

Let us prove the first existence theorem. Let $\varepsilon > 0$ be small.

Theorem 3.24. *Solution of the boundary value problem*

$$\Delta^2 u = 0, \quad x \in S, \tag{3.5.13}$$

$$\begin{aligned} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u\Big|_{\partial S} &= \varphi_1(s), \quad x \in \partial S, \\ a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u\Big|_{\partial S} &= \varphi_2(s), \quad x \in \partial S \end{aligned} \tag{3.5.14}$$

from the class $u \in C^3(\overline{S})$ for arbitrary functions $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C^{1+\varepsilon}(\partial S)$ exists if and only if the next third degree polynomial

$$\Delta(\lambda) = a_{00}(a_{11} + a_{12}n) + 2(2a_{00}a_{12} - a_{02}a_{11}n + a_{01}a_{12}n + a_{00}a_{13}n)\lambda +$$
$$+ (2a_{01}a_{12} - 2a_{02}a_{11} + 2a_{00}a_{13} + a_{01}a_{13}n)\lambda^2 + 2a_{01}a_{13}\lambda^3$$

has no roots in $\mathbb{N}_0$.

Proof. Consider functions $v_1(x)$ and $v_2(x)$ harmonic in S such that for them $v_1(x)|_{\partial S} = \varphi_1(s)$ and $v_2(x)|_{\partial S} = \varphi_2(s)$, i.e. $v_1(x)$ and $v_2(x)$ are solutions to the corresponding Dirichlet problems in S. By Lemma 3.32, the biharmonic function $u(x)$ satisfies the equality (3.5.11). Let us indicate the functions $u_0(x)$ and $u_1(x)$ for which the biharmonic function $u(x) = u_0(x)+|x|^2u_1(x)$ is a solution to the problem (3.5.13)-(3.5.14). Consider the operators L_1 and L_2

$$L_1 = a_{00} + a_{01}\Lambda + a_{02}\Delta, \quad L_2 = a_{11}\Lambda + a_{12}\Delta + a_{13}\Lambda\Delta.$$

Then, due to the fact that $\Delta\left(|x|^2 v(x)\right) = (2n+4\Lambda)v(x)$ we have the equalities

$$L_1 u = L_1\left(u_0(x) + |x|^2u_1(x)\right) = a_{00}u + a_{01}\Lambda u + a_{02}(2n+4\Lambda)u_1,$$

$$L_2u = L_2\left(u_0(x) + |x|^2u_1(x)\right) = a_{11}\Lambda u + a_{12}(2n+4\Lambda)u_1 + a_{13}\Lambda(2n+4\Lambda)u_1.$$

Passing here to the limit at $x \to s \in \partial S$, as in Theorem 3.23, we obtain

$$\begin{aligned} a_{00}(u_0+u_1) + a_{01}\left(\Lambda u_0 + (\Lambda+2)u_1\right) + a_{02}(2n+4\Lambda)\,u_1 &= \varphi_1(s), \\ a_{11}\left(\Lambda u_0 + (\Lambda+2)u_1\right) + a_{12}(2n+4\Lambda)u_1 + a_{13}\Lambda(2n+4\Lambda)u_1 &= \varphi_2(s). \end{aligned} \tag{3.5.15}$$

From the uniqueness of the solution to the Dirichlet problem for the Laplace equation in S we conclude that the equalities (3.5.15) can be continued in S

$$\begin{aligned} a_{00}(u_0+u_1) + a_{01}\left(\Lambda u_0 + (\Lambda+2)u_1\right) + a_{02}(2n+4\Lambda)\,u_1 &= v_1(x), \\ a_{11}\left(\Lambda u_0 + (\Lambda+2)u_1\right) + a_{12}(2n+4\Lambda)u_1 + a_{13}\Lambda(2n+4\Lambda)u_1 &= v_2(x), \end{aligned}$$

since by the property of the operator Λ the functions on the left in this equality are harmonic in S. Let us rewrite the resulting equations in matrix form

$$\begin{pmatrix} a_{00}+a_{01}\Lambda & a_{00}+(\Lambda+2)a_{01}+(2n+4\Lambda)a_{02} \\ a_{11}\Lambda & (\Lambda+2)a_{11}+(2n+4\Lambda)a_{12}+\Lambda(2n+4\Lambda)a_{13} \end{pmatrix} \times \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.5.16}$$

Equation (3.5.16) is a system of differential equations in functions harmonic in S. Let us show that the equation (3.5.16) is also satisfied in $\overline{S}$. Since by condition $u(x) \in C^3(\overline{S})$, then by the equalities (3.5.12) we have $u_0(x), u_1(x) \in C^1(\overline{S})$. Taking into account that $\Lambda u_1 = \frac{1}{4}\Delta u - \frac{n}{2}u_1$ (see Lemma 3.32) we obtain $\Lambda u_1(x) \in C^1(\overline{S})$. Therefore, for $u_0(x) \in C^1(\overline{S})$ and $\Lambda u_1(x) \in C^1(\overline{S})$ equation (3.5.16) is satisfied in $\overline{S}$. So, from the above reasoning it follows that if the function $u(x) = u_0(x) + |x|^2u_1(x) \in C^3(\overline{S})$ is a solution to the problem (3.5.13)-(3.5.14), then the harmonic functions $u_0(x)$ and $u_1(x)$ satisfy the equation (3.5.16) in $\overline{S}$.

The converse is also true, i.e. if the harmonic functions $u_0(x) \in C^1(\overline{S})$ and $u_1(x) \in C^1(\overline{S})$, $\Lambda u_1(x) \in C^1(\overline{S})$ satisfy the equation (3.5.16) in $\overline{S}$, then they also satisfy the equality (3.5.15), which means the function $u(x) = u_0(x) + |x|^2u_1(x)$ is a solution to the problem (3.5.13)-(3.5.14). Since $\Delta u(x) = (2n+4\Lambda)u_1(x)$, then $u(x) \in C^3(\overline{S})$. So the problem (3.5.13)-(3.5.14) and the equation (3.5.16) in $\overline{S}$ are equivalent.

Consider the matrix

$$A(\lambda) = \begin{pmatrix} a_{00}+a_{01}\lambda & a_{00}+(\lambda+2)a_{01}+(2n+4\lambda)a_{02} \\ a_{11}\lambda & (\lambda+2)a_{11}+(2n+4\lambda)a_{12}+\lambda(2n+4\lambda)a_{13} \end{pmatrix},$$

depending on the parameter $\lambda \in \mathbb{R}$. It is easy to see that $\det A(\lambda) = \Delta(\lambda)$ and $\deg \det A(\lambda) = 3$. Therefore, under the conditions of the theorem, the matrix $A(\lambda)$ is nonsingular for $\lambda \in \mathbb{N}_0$. The system (3.5.16) can be written in the form

$$A(\Lambda)U(x) = V(x), \quad x \in \overline{S}, \tag{3.5.17}$$

where $U(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}$ and $V(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$. Let us factorize the polynomial $\det A(\lambda)$

$$\det A(\lambda) = \Delta(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

where $\lambda_i \in \mathbb{C}$ are the roots of the polynomial $\Delta(\lambda)$. Let us first assume that in the case of complex roots $\lambda_1 = \overline{\lambda}_2$ the harmonic function $V(x)$ on the right does not contain monomials of degree $\operatorname{Re} \lambda_1$ in its expansion at the origin.

1) Let all roots λ_i of the polynomial $\Delta(\lambda)$ be different. It is known that in this case

$$\frac{1}{\Delta(\lambda)} = \frac{b_1}{\lambda - \lambda_1} + \frac{b_2}{\lambda - \lambda_2} + \frac{b_3}{\lambda - \lambda_3},$$

where $b_i \in \mathbb{R}$. The matrix $A^{-1}(\lambda)$, inverse of $A(\lambda)$, has the form

$$A^{-1}(\lambda) = \frac{1}{\Delta(\lambda)} A^*(\lambda),$$

where

$$A^*(\lambda) = \begin{pmatrix} [(\lambda+2)a_{11} + (2n+4\lambda)a_{12} & [-a_{00} - (\lambda+2)a_{01} \\ +\lambda(2n+4\lambda)a_{13}] & -(2n+4\lambda)a_{02}] \\ -a_{11}\lambda & a_{00} + a_{01}\lambda \end{pmatrix}.$$

Let us consider operators depending on the roots of λ_i (see Section 1.1)

$$M_i(\Lambda)v(x) = \int_0^1 t^{-\lambda_i - 1} v(tx)\, dt \tag{3.5.18}$$

for $i = 1, 2, 3$. It is easy to see that for a differentiable function $w(x)$

$$\begin{aligned} (\Lambda - \lambda_i) M_i(\Lambda) w &= (\Lambda - \lambda_i) \int_0^1 t^{-\lambda_i - 1} w(tx) dt \\ = -\lambda_i \int_0^1 t^{-\lambda_i - 1} w(tx) dt &+ \int_0^1 t^{-\lambda_i} \sum_{i=1}^n x_i w_{x_i}(tx) dt = -\lambda_i \int_0^1 t^{-\lambda_i - 1} w(tx) dt \\ &+ \int_0^1 t^{-\lambda_i} w_t(tx)\, dt - \lambda_i \int_0^1 t^{-\lambda_i - 1} w(tx)\, dt + t^{-\lambda_i} w(tx) \Big|_0^1 \end{aligned}$$

$$+\lambda_i \int_0^1 t^{-\lambda_i-1} w(tx)\,dt = t^{-\lambda_i} w(tx)\big|_0^1 = w(x), \quad (3.5.19)$$

if $\lim\limits_{t\to 0} t^{-\lambda_i} w(tx) = 0$. Since the equalities are true

$$\begin{aligned}1 &= \Delta(\lambda)\left(\frac{b_1}{\lambda-\lambda_1} + \frac{b_2}{\lambda-\lambda_2} + \frac{b_3}{\lambda-\lambda_3}\right) = \frac{b_1\Delta(\lambda)}{\lambda-\lambda_1} + \frac{b_2\Delta(\lambda)}{\lambda-\lambda_2} + \frac{b_3\Delta(\lambda)}{\lambda-\lambda_3}\\ &= cb_1(\lambda-\lambda_2)(\lambda-\lambda_3) + cb_2(\lambda-\lambda_1)(\lambda-\lambda_3) + cb_3(\lambda-\lambda_1)(\lambda-\lambda_2),\end{aligned}$$

then for a function $w \in C^2(\overline{S})$ such that $\lim\limits_{t\to 0} t^{-\lambda_i} w(tx) = 0$ where $i = 1, 2, 3$ we have

$$\begin{aligned}\Delta(\Lambda)\left(b_1M_1(\Lambda) + b_2M_2(\Lambda) + b_3M_3(\Lambda)\right) w &= \big(cb_1(\Lambda-\lambda_2)(\Lambda-\lambda_3)\\ &+ cb_2(\Lambda-\lambda_1)(\Lambda-\lambda_3) + cb_3(\Lambda-\lambda_1)(\Lambda-\lambda_2)\big)w = 1\cdot w. \quad (3.5.20)\end{aligned}$$

Therefore, if the function $V(x)$ satisfies the conditions $\lim\limits_{t\to 0} t^{-\lambda_i} V(tx) = 0$ for $i = 1, 2, 3$, then the solution to the equation (3.5.17) can be written in the form

$$U(x) = A^*(\Lambda)\left(b_1M_1(\Lambda) + b_2M_2(\Lambda) + b_3M_3(\Lambda)\right) V(x). \quad (3.5.21)$$

Indeed, the function $U(x)$ found from (3.5.21) by virtue of the equality (3.5.20) satisfies the equation (3.5.17) in $\overline{S}$

$$\begin{aligned}A(\Lambda)U(x) &= A(\Lambda)A^*(\Lambda)\left(b_1M_1(\Lambda) + b_2M_2(\Lambda) + b_3M_3(\Lambda)\right) V(x)\\ &= \Delta(\Lambda)\left(b_1M_1(\Lambda) + b_2M_2(\Lambda) + b_3M_3(\Lambda)\right) V(x) = V(x).\end{aligned}$$

2) Let two roots of the polynomial $\Delta(\lambda)$ coincide $\lambda_1 = \lambda_2$, i.e. $\Delta(\lambda) = c(\lambda-\lambda_1)^2(\lambda-\lambda_3)$. It is known that in this case

$$\frac{1}{\Delta(\lambda)} = \frac{b_1}{\lambda-\lambda_1} + \frac{b_2}{(\lambda-\lambda_1)^2} + \frac{b_3}{\lambda-\lambda_3},$$

and therefore the equalities hold

$$\begin{aligned}1 &= \Delta(\lambda)\left(\frac{b_1}{\lambda-\lambda_1} + \frac{b_2}{(\lambda-\lambda_1)^2} + \frac{b_3}{\lambda-\lambda_3}\right) = \frac{b_1\Delta(\lambda)}{\lambda-\lambda_1} + \frac{b_2\Delta(\lambda)}{(\lambda-\lambda_1)^2}\\ &+ \frac{b_3\Delta(\lambda)}{\lambda-\lambda_3} = cb_1(\lambda-\lambda_1)(\lambda-\lambda_3) + cb_2(\lambda-\lambda_3) + cb_3(\lambda-\lambda_1)^2. \quad (3.5.22)\end{aligned}$$

Consider the operator

$$M_1^{(2)}(\Lambda)v(x) = -\int_0^1 \ln t\, t^{-\lambda_1-1}v(tx)\,dt.$$

It is easy to see that if the function $w(x)$ is differentiable, then the equalities hold true

$$\begin{aligned}(\Lambda-\lambda_1)M_1^{(2)}(\Lambda)w &= -(\Lambda-\lambda_1)\int_0^1 \ln t\, t^{-\lambda_1-1}w(tx)dt\\ &= \lambda_1\int_0^1 \ln t\, t^{-\lambda_1-1}w(tx)dt - \int_0^1 \ln t\, t^{-\lambda_1}w_t(tx)dt = \lambda_1\int_0^1 \ln t\, t^{-\lambda_1-1}w(tx)dt\\ &\quad - \ln t\, t^{-\lambda_1}w(tx)|_0^1 - \lambda_1\int_0^1 \ln t\, t^{-\lambda_1-1}w(tx)dt + \int_0^1 t^{-\lambda_1-1}w(tx)dt\\ &= M_1(\Lambda)w + \ln t\, t^{-\lambda_1}w(tx)|_0^1 = M_1(\Lambda)w,\end{aligned}$$

which means $(\Lambda-\lambda_1)^2M_1^{(2)}(\Lambda)w = w$ if $\lim\limits_{t\to0}\ln t\, t^{-\lambda_1}w(tx) = 0$. Since the equalities (3.5.22) are true, then for a function $w\in C^2(\overline{S})$ satisfying the condition $\lim\limits_{t\to0}\ln t\, t^{-\lambda_1}w(tx) = 0$, we have

$$\begin{aligned}&\Delta(\Lambda)\left(b_1M_1(\Lambda)+b_2M_1^{(2)}(\Lambda)+b_3M_3(\Lambda)\right)w\\ &\qquad = \left(cb_1(\Lambda-\lambda_1)(\Lambda-\lambda_3)+cb_2(\Lambda-\lambda_3)+cb_3(\Lambda-\lambda_1)^2\right)w = 1\cdot w.\end{aligned}$$

So, if $\lambda_1=\lambda_2$ and the harmonic function $V(x)$ satisfies the conditions $\lim\limits_{t\to0}\ln t\, t^{-\lambda_1}V(tx) = 0$ and $\lim\limits_{t\to0} t^{-\lambda_3}V(tx) = 0$, then the solution to the equation (3.5.17) can be written in the form

$$U(x) = A^*(\Lambda)\left(b_1M_1(\Lambda)+b_2M_1^{(2)}(\Lambda)+b_3M_3(\Lambda)\right)V(x). \qquad (3.5.23)$$

3) Let the three roots of the polynomial $\Delta(\lambda)$ be equal, i.e. $\Delta(\lambda) = c(\lambda-\lambda_1)^3$. Consider the operator

$$M_1^{(3)}(\Lambda)v(x) = \frac{1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}v(tx)\,dt.$$

It is easy to see that if the function $w(x)$ is differentiable, then for it the equalities hold true

$$
\begin{aligned}
(\Lambda-\lambda_1)M_1^{(3)}(\Lambda)w &= (\Lambda-\lambda_1)\frac{1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}w(tx)\,dt \\
&= -\frac{\lambda_1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}w(tx)\,dt + \frac{1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1}w_t(tx)\,dt \\
&= -\frac{\lambda_1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}w(tx)\,dt + \frac{1}{2}\ln^2 t\, t^{-\lambda_1}w(tx)|_0^1 \\
&+ \frac{\lambda_1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}w(tx)\,dt - \int_0^1 \ln t\, t^{-\lambda_1-1}w(tx)\,dt \\
&= M_1^{(2)}(\Lambda)w + \frac{1}{2}\ln^2 t\, t^{-\lambda_1}w(tx)|_0^1 = M_1^{(2)}(\Lambda)w,
\end{aligned}
$$

provided that $\lim\limits_{t\to 0}\ln^2 t\, t^{-\lambda_1}w(tx) = 0$. By virtue of what is proved above, we have $(\Lambda-\lambda_1)^3 M_1^{(3)}(\Lambda)w = w$, and therefore in this case the solution to the equation (3.5.17) can be written in the form

$$
U(x) = \frac{1}{c}A^*(\Lambda)M_1^{(3)}(\Lambda)V(x). \tag{3.5.24}
$$

Now let us find out the conditions that need to be imposed on the function $V(x)$ so that we can use the equalities (3.5.21), (3.5.23) and (3.5.24). First, the limits $\lim\limits_{t\to 0} t^{-\lambda_i}V(tx) = 0$ must be valid for different roots, where $i = 1,2,3$, or $\lim\limits_{t\to 0}\ln^k t\, t^{-\lambda_1}V(tx) = 0$ $(k = 1,2)$ in the case of multiple roots. Second, in order for the function $U(x)$ to have the smoothness necessary to satisfy the boundary conditions of the problem, it is necessary that the operator $A^*(\Lambda)$ be applied to the function $V(x)$ in $\overline{S}$. Due to the structure of the operator $A^*(\Lambda)$ this is possible if $v_1 \in C^2(\overline{S})$ and $v_2 \in C^1(\overline{S})$. Such smoothness of the functions $v_1(x)$ and $v_2(x)$ is ensured if $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C^{1+\varepsilon}(\partial S)$, which are satisfied.

4) Let us return to the conditions

$$
\lim_{t\to 0} t^{-\lambda_i}V(tx) = 0, \quad i = 1,2,3, \tag{3.5.25}
$$

which are actually imposed on the functions $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C^{1+\varepsilon}(\partial S)$. Note that due to the smoothness of the function $V(x)$ and $\operatorname{Re}\lambda_i \notin \mathbb{N}_0$ the conditions $\lim\limits_{t\ to 0} t^{-\lambda_i}V(tx) = 0$ and $\lim\limits_{t\to 0}\ln^k t\, t^{-\lambda_i}V(tx) = 0$ are equivalent. If $\operatorname{Re}\lambda_i < 0$, then the conditions (3.5.25) are met. Otherwise, i.e. if for some i we have $\operatorname{Re}\lambda_i > 0$, then the conditions (3.5.25) may not be satisfied for some functions φ_1 and φ_2.

Let all roots λ_i be different. Let us slightly change the form of the solution (3.5.21) in the case of $\operatorname{Re}\lambda_i > 0$. First, note that if $\lim\limits_{t\to\infty} t^{-\lambda_i} w(tx) = 0$, then for the operator

$$\widehat{M}_i(\Lambda)v(x) = -\int_1^\infty t^{-\lambda_i-1} v(tx)\,dt,$$

for $w(x) \in C^1(\mathbb{R}^n)$, equalities similar to (3.5.19) are true

$$\begin{aligned}(\Lambda-\lambda_i)\widehat{M}_i(\Lambda)w &= \lambda_i \int_1^\infty t^{-\lambda_i-1} w(tx)\,dt - \int_1^\infty t^{-\lambda_i} w_t(tx)\,dt \\ &= \lambda_i \int_1^\infty t^{-\lambda_i-1} w(tx)\,dt - -t^{-\lambda_i} w(tx)|_1^\infty - \lambda_i \int_1^\infty t^{-\lambda_i-1} w(tx)\,dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -t^{-\lambda_i} w(tx)|_1^\infty = w(x).\end{aligned}$$

For example, for $|\alpha| < \operatorname{Re}\lambda_i$ we have

$$\widehat{M}_i(\Lambda)x^\alpha = -x^\alpha \int_1^\infty t^{|\alpha|-\lambda_i-1} dt = -x^\alpha \left.\frac{t^{|\alpha|-\lambda_i}}{|\alpha|-\lambda_i}\right|_1^\infty = \frac{x^\alpha}{|\alpha|-\lambda_i}, \tag{3.5.26}$$

where $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, i.e. the operator $\widehat{M}_i(\Lambda)$ is applicable to the polynomial x^α. Let us change the equality (3.5.21). We represent the function $V(x)$ in the form

$$\begin{aligned}V(x) = \sum_{k<\operatorname{Re}\lambda_i} \sum_{j=1}^{h_k} V_k^{(j)} H_k^{(j)}(x) + \Big(V(x) - \sum_{k<\operatorname{Re}\lambda_i} \sum_{j=1}^{h_k} V_k^{(j)} H_k^{(j)}(x)\Big) \\ \equiv V_1^{(i)}(x) + V_2^{(i)}(x),\end{aligned}$$

where $\{H_m^{(i)}(x), m \in \mathbb{N}_0,\ i = \overline{1, h_m}\}$ is a complete system of harmonic polynomials from Definition 1.5 orthogonal on ∂S, and $V_k^{(j)}$ are the coefficients of the expansion of the function $V(x)$ in this system. It is clear that for $\operatorname{Re}\lambda_i < 0$ we have $V_1^{(i)}(x) = 0$ and therefore $V_2^{(i)}(x) = V(x)$. Consider the expression

$$U_i(x) = b_i A^*(\Lambda)\left(\widehat{M}_i(\Lambda)V_1^{(i)}(x) + M_i(\Lambda)V_2^{(i)}(x)\right). \tag{3.5.27}$$

Since $V_1^{(i)}(x)$ is a polynomial of degree $n_0 \le [\operatorname{Re}\lambda_i]$ and and by assumption $n_0 \ne \operatorname{Re}\lambda_i$, then for each polynomial $V_k^{(j)} H_k^{(j)}(x)$ included in it, the inequality $\operatorname{Re}(k-\lambda_i) < \operatorname{Re}\lambda_i - \operatorname{Re}\lambda_i = 0$ holds true and therefore, by (3.5.26), we can apply the operator

$\widehat{M}_i(\Lambda)$ to the polynomial $V_1^{(i)}(x)$. Since $\mathrm{Re}(-\lambda_i + n_0 + 1) > 0$, then $\left|t^{-\lambda_i} V_2^{(i)}(tx)\right| \le C|t|^{-\mathrm{Re}\,\lambda_i + n_0 + 1} \to 0$ at $t \to 0$. This means that the operator $M_i(\Lambda)$ can be applied to the function $V_2^{(i)}(x)$. So the expression (3.5.27) is legal.

Due to the properties of the operators $\widehat{M}_i(\Lambda)$ and $M_i(\Lambda)$ the equalities hold

$$\begin{aligned} A(\Lambda)U_i(x) &= b_i A(\Lambda)A^*(\Lambda)\left(\widehat{M}_i(\Lambda)V_1^{(i)}(x) + M_i(\Lambda)V_2^{(i)}(x)\right) \\ &= b_i \Delta(\Lambda)\Big(\widehat{M}_i(\Lambda)V_1^{(i)}(x) + M_i(\Lambda)V_2^{(i)}(x)\Big) \\ &= cb_i(\Lambda-\lambda_2)(\Lambda-\lambda_3)\left(V_1^{(i)}(x) + V_2^{(i)}(x)\right) = cb_i(\Lambda-\lambda_2)(\Lambda-\lambda_3)V, \end{aligned}$$

which means for the next function

$$U(x) = U_1(x) + U_2(x) + U_3(x) \tag{3.5.28}$$

the equalities are true

$$\begin{aligned} A(\Lambda)U = \sum_{i=1}^{3} A(\Lambda)U_i(x) &= \big(cb_1(\Lambda-\lambda_2)(\Lambda-\lambda_3) \\ &+ cb_2(\Lambda-\lambda_1)(\Lambda-\lambda_3) + cb_3(\Lambda-\lambda_1)(\Lambda-\lambda_2)\big)V = 1\cdot V = V. \end{aligned}$$

This means that the function $U(x)$ from (3.5.28) is a solution to the equation (3.5.17) in $\overline{S}$, and therefore the function $u(x) = u_0(x) + |x|^2 u_1(x)$ is a solution to the problem (3.5.13)-(3.5.14).

In the case of multiple roots $\lambda_1 = \lambda_2 > 0$ (real roots), together with the operator $\widehat{M}_1(\Lambda)$ it is necessary to consider the operator

$$\widehat{M}_1^{(2)}(\Lambda)v(x) = \int_1^\infty \ln t\; t^{-\lambda_i-1} v(tx)\,dt,$$

for which, in the case of a differentiable function $w(x) \in C^1(\mathbb{R}^n)$, the equalities hold true

$$\begin{aligned} (\Lambda-\lambda_1)\widehat{M}_1^{(2)}(\Lambda)w &= -\lambda_1 \int_1^\infty \ln t\; t^{-\lambda_1-1} w(tx)dt + \int_1^\infty \ln t\; t^{-\lambda_1} w_t(tx)dt \\ = -\lambda_1 \int_1^\infty \ln t\; t^{-\lambda_1-1} w(tx)dt &+ \ln t\; t^{-\lambda_1} w(tx)\big|_1^\infty + \lambda_1 \int_1^\infty \ln t\; t^{-\lambda_1-1} w(tx)dt \\ &- \int_1^\infty t^{-\lambda_1-1} w(tx)dt = -t^{-\lambda_i} w(tx)\big|_1^\infty + \widehat{M}_1(\Lambda)w = \widehat{M}_1(\Lambda)w(x) \end{aligned}$$

and that means $(\Lambda - \lambda_1)^2\widehat{M}_1^{(2)}(\Lambda)w = w$. For example, for $|\alpha| < \lambda_1$ we have

$$
\begin{aligned}
\widehat{M}_1^{(2)}(\Lambda)x^\alpha &= x^\alpha \int_1^\infty \ln t\, t^{|\alpha|-\lambda_1-1}dt = \frac{x^\alpha}{|\alpha|-\lambda_1}\int_1^\infty \ln t\, dt^{|\alpha|-\lambda_1} \\
&= x^\alpha \left.\frac{t^{|\alpha|-\lambda_1}\ln t}{|\alpha|-\lambda_1}\right|_1^\infty - \frac{x^\alpha}{|\alpha|-\lambda_1}\int_1^\infty t^{|\alpha|-\lambda_1-1}dt = \frac{1}{|\alpha|-\lambda_1}\widehat{M}_1(\Lambda)x^\alpha \\
&= \frac{1}{(|\alpha|-\lambda_1)^2}
\end{aligned}
$$

and therefore the operator $\widehat{M}_1^{(2)}(\Lambda)$ is applicable to the polynomial $V_1^{(1)}(x)$. The solution to the equation (3.5.17) can be represented as

$$
U(x) = U_1(x) + U_1^{(2)}(x) + U_3(x), \tag{3.5.29}
$$

where the functions $U_1(x)$ and $U_3(x)$ are defined in (3.5.27), and the function $U_1^{(2)}(x)$ is written in the form

$$
U_1^{(2)}(x) = b_2A^*(\Lambda)\left(\widehat{M}_1^{(2)}(\Lambda)V_1^{(1)}(x) + M_1^{(2)}(\Lambda)V_2^{(1)}(x)\right).
$$

In the case of triple roots $\lambda_1 = \lambda_2 = \lambda_3 > 0$ the solution to the equation (3.5.17) has the form

$$
U(x) = \frac{1}{c}A^*(\Lambda)\left(\widehat{M}_1^{(3)}(\Lambda)V_1^{(1)}(x) + M_1^{(3)}(\Lambda)V_2^{(1)}(x)\right), \tag{3.5.30}
$$

where

$$
\widehat{M}_1^{(3)}(\Lambda)v(x) = -\frac{1}{2}\int_0^1 \ln^2 t\, t^{-\lambda_1-1}v(tx)\,dt,
$$

and the operator $M_1^{(3)}(\Lambda)$ and the functions $V_1^{(1)}(x)$ and $V_2^{(1)}(x)$ are defined as before.

Let us consider the last case, when for complex roots $\lambda_1 = \overline{\lambda}_2$ the harmonic function $V(x)$ in the equation (3.5.17) contains in its expansion at the origin homogeneous terms of degree $m_0 = \operatorname{Re}\lambda_1$. We denote them as $V_{m_0}(x)$ and solve the equation $A(\Lambda)U_{m_0}(x) = V_{m_0}(x)$. Since $\det A(m_0) \neq 0$, this solution has the form $U_{m_0}(x) = A^{-1}(m_0)V_{m_0}(x)$. Let us add this solution to the one constructed in case 4).

So, the solutions to the equation (3.5.17) in $\overline{S}$ are constructed in all cases of the location of the roots of the polynomial $\Delta(\lambda)$. After finding these solutions – the harmonic functions $u_0(x)$ and $u_1(x)$, the solution to the problem (3.5.13)-(3.5.14) is written in the form $u(x) = u_0(x) + |x|^2u_1(x)$. The theorem is proved. □

3.5.3 Conditional existence of a solution

Let us consider the case of the problem (3.5.13)-(3.5.14) not studied in Theorem 3.24, when for some $m \in \mathbb{N}_0$ the equality $\Delta(m) = 0$ holds.

Theorem 3.25. *Solution of the problem* (3.5.13)-(3.5.14) *from the class* $u \in C^3(\overline{S})$ *in the case, when for some* $m \in \mathbb{N}_0$ *the equality* $\Delta(m) = 0$ *holds, exists if and only if the functions* $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ *and* $\varphi_2 \in C^{1+\varepsilon}(\partial S)$ *satisfy the equalities*

$$\int_{\partial S} H_m(x)\left(q_1(m)\varphi_1(x) + q_2(m)\varphi_2(x)\right)\, ds_x = 0, \tag{3.5.31}$$

where $H_m(x)$ *is an arbitrary homogeneous harmonic polynomial of degree* m*, and a vector of the form* $\binom{q_1(m)}{q_2(m)}$ *is an arbitrary solution to a system of algebraic equations*

$$\begin{gathered}(a_{00} + a_{01}m)q_1(m) + a_{11}mq_2(m) = 0,\\ (a_{00} + (m+2)a_{01} + (2n+4m)a_{02})q_1(m) +\\ +((m+2)a_{11} + (2n+4m)a_{12} + m(2n+4m)a_{13})q_2(m) = 0.\end{gathered} \tag{3.5.32}$$

Proof. The solution to the problem (3.5.13)-(3.5.14) constructed in Theorem 3.24 is not suitable under the conditions of this theorem. It needs to be tweaked a little. Consider the equation (3.5.17). Let us decompose the function $V(x)$ into two terms $V(x) = \widehat{V}(x) + V_m(x)$, where

$$V_m(x) = \sum_{i=1}^{h_m} P_m^{(i)} H_m^{(i)}(x). \tag{3.5.33}$$

Let us find the conditions under which the equation (3.5.17) has a solution in the special case when the right-hand side of the equation is a monomial of the form $V_m(x) = P_m^{(i)} H_m^{(i)}(x)$, where

$$P_m^{(i)} = \frac{1}{\|H_m^{(i)}\|^2_{L_2(\partial S)}} \begin{pmatrix} \int_{\partial S} H_m^{(i)}(x)\varphi_1(x)\, ds_x \\ \int_{\partial S} H_m^{(i)}(x)\varphi_2(x)\, ds_x \end{pmatrix}.$$

Due to the homogeneity of the operator $A(\Lambda)$, the solution to the equation (3.5.17), in this case, can only be of the form $U_m(x) = Q_m^{(i)} H_m^{(i)}(x)$. Substituting the polynomial $U_m(x) = Q_m^{(i)} H_m^{(i)}(x)$ into (3.5.17) we get

$$A(\Lambda) Q_m^{(i)} H_m^{(i)}(x) = H_m^{(i)}(x) A(m) Q_m^{(i)} = H_m^{(i)}(x) P_m^{(i)},$$

which means the equality must be satisfied

$$A(m)Q_m^{(i)} = P_m^{(i)}. \tag{3.5.34}$$

The resulting system of algebraic equations has a solution only if the right-hand side of the system $P_m^{(i)}$ is orthogonal to the zeros of the conjugate system $A^T(m)Q = 0$, that is to all vectors $\binom{q_1(m)}{q_2(m)}$ from the conditions of the theorem

$$\begin{aligned}
P_m^{(i)} \cdot \begin{pmatrix} q_1(m) \\ q_2(m) \end{pmatrix} &= \begin{pmatrix} \int_{\partial S} H_m^{(i)}(x)\varphi_1(x)\,ds_x \\ \int_{\partial S} H_m^{(i)}(x)\varphi_2(x)\,ds_x \end{pmatrix} \cdot \begin{pmatrix} q_1(m) \\ q_2(m) \end{pmatrix} \\
&= \int_{\partial S} q_1(m)H_m^{(i)}(x)\varphi_1(x)\,ds_x + \int_{\partial S} q_2(m)H_m^{(i)}(x)\varphi_2(x)\,ds_x \\
&= \int_{\partial S} H_m^{(i)}(x)\left(q_1(m)\varphi_1(x) + q_2(m)\varphi_2(x)\right)\,ds_x = 0.
\end{aligned}$$

By virtue of (3.5.31) this condition is satisfied, which means a solution to the system (3.5.34) exists. Going through all the terms in the sum (3.5.33) and using the conditions of the theorem, we conclude that a solution to the equation (3.5.17) for $V(x) = V_m(x)$ exists. Let us denote it $U_m(x)$. After this, if the polynomial $\Delta(\lambda)$ has no other roots in $\mathbb{N}_0$, then according to Theorem 3.24 we solve the equation (3.5.17) for $V(x) = \widehat{V}(x)$ and add the resulting solutions $U(x) = U_m(x) + \widehat{U}(x)$. The function $U(x)$ is the solution to the equation (3.5.17) in $\overline{S}$, which means the problem (3.5.13)-(3.5.14) has a solution. If the polynomial $\Delta(\lambda)$, in addition to the number m, also has other roots in $\mathbb{N}_0$, then we proceed with the function $\widehat{V}(x)$ in the same way as we did above with the function $V(x)$. Thus, for each root of the polynomial $\Delta(\lambda)$ in $\mathbb{N}_0$ the conditions (3.5.31) must be satisfied. The theorem is proved. □

3.5.4 Special cases of the problem

1. Dirichlet boundary value problem: for $a_{00} \neq 0$, $a_{11} \neq 0$, and all other coefficients are equal to zero, we have

$$\Delta(\lambda) = \begin{vmatrix} a_{00} & 0 \\ a_{11}\lambda & 2a_{11} \end{vmatrix} = 2a_{00}a_{11} \neq 0,$$

which means, by Theorem 3.24, this problem is unconditionally solvable, which is true (see Section 2.4).

2. Navier boundary value problem (see Section 1.2): for $a_{00} \neq 0$, $a_{12} \neq 0$ and all other coefficients are equal to zero we have

$$\Delta(\lambda) = \begin{vmatrix} a_{00} & 0 \\ 0 & (2n+4\lambda)a_{12} \end{vmatrix} = (2n+4\lambda)a_{00}a_{12} \neq 0,$$

which means, according to Theorem 3.24, this problem is unconditionally solvable, which is also true.

3. Riquier-Neumann boundary value problem: for $a_{01} \neq 0$, $a_{13} \neq 0$, and all other coefficients are equal to zero, we have

$$\begin{gathered} \Delta^2 u = 0,\ x \in S; \\ a_{01}\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_1(s),\ a_{13}\frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = \varphi_2(s), \quad s \in \partial S. \end{gathered} \tag{3.5.35}$$

To study this problem, let us find the polynomial $\Delta(\lambda)$

$$\Delta(\lambda) = \begin{vmatrix} a_{01}\lambda & 2a_{01} \\ 0 & \lambda(2n+4\lambda)a_{13} \end{vmatrix} = \lambda^2(2n+4\lambda)a_{01}a_{13}.$$

It can be seen that $\Delta(0) = 0$. The system of algebraic equations (3.5.4) from Theorem 3.23 has the form

$$\begin{pmatrix} 0 & 2a_{01} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0,$$

and its solution is written in the form $C_2 = 0$ and C_1 is arbitrary. By Theorem 3.23 the solution to problem (3.5.35) is unique up to the constant $u(x) = C_1 H_0(x) = C_1$. For the existence of a solution to the Riquier-Neumann problem, consider the system (3.5.32) for the Riquier-Neumann problem for $m = 0$

$$\begin{pmatrix} 0 & 0 \\ 2a_{01} & 0 \end{pmatrix}\begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = 0,$$

which means $q_1(0) = 0$ and the space of zeros of the system (3.5.32) in this case is one-dimensional and spanned by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, the condition (3.5.31) for the existence of a solution to the problem (3.5.35) has the form $\int_{\partial S} \varphi_2(x)\, ds_x = 0$.

For example, with $a_{01} = a_{13} = 1$ and $\varphi_1 = 0$, $\varphi_2 = 1$ the resulting condition for the existence of a solution to the problem

$$\Delta^2 u = 0,\ x \in S; \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0,\ \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = 1$$

is not satisfied. Let us make sure that there is no solution to this problem. From the second boundary condition it follows that for the solution $u(x) = u_0(x) + |x|^2 u_1(x)$ (see Lemma 3.32) of this problem the equalities must be satisfied

$$\omega_n = \int_{\partial S} ds_x = \int_{\partial S} \frac{\partial}{\partial \nu}(2n + 4\Lambda)u_1(x)\, ds_x = 0,$$

which is not true. Here it is taken into account that the integral of the normal derivatives of a function harmonic in S over ∂S is equal to zero. Therefore this problem has no solution.

4. For the problem (3.5.13)-(3.5.14) when $a_{02} = 0$, $a_{11} = 0$ we have

$$\Delta(\lambda) = \begin{vmatrix} a_{00} + a_{01}\lambda & 2a_{01} \\ 0 & (2n + 4\lambda)a_{12} + \lambda(2n + 4\lambda)a_{13} \end{vmatrix}$$
$$= (2n + 4\lambda)(a_{00} + a_{01})(a_{12}\lambda + a_{13}\lambda).$$

Let us put in (3.5.2) $a_{00} = -2$, $a_{01} = 1$, $a_{12} = -3$, $a_{13} = 1$, that is consider the problem

$$\begin{gathered} \Delta^2 u = 0,\ x \in S; \\ -2u + \frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_1(s),\ \ -3\Delta\, u + \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = \varphi_2(s),\ s \in \partial S. \end{gathered} \tag{3.5.36}$$

We have $\Delta(\lambda) = (2n + 4\lambda)(\lambda - 2)(\lambda - 3)$ and therefore $\Delta(2) = 0$ and $\Delta(3) = 0$. For $m = 2$ the system (3.5.4) has the form

$$\begin{pmatrix} 0 & 2 \\ 0 & -(2n+8) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

and that means $C_2 = 0$ and C_1 is arbitrary. Therefore, the polynomial $u_2(x) = C_1 H_2(x)$ is a solution to the homogeneous problem (3.5.36). For $m = 3$ the system (3.5.4) has the form

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

and that means $C_1 = -2C_2$ and C_2 is arbitrary. Therefore, the polynomial $u_3(x) = C_2(|x|^2 - 3)H_3(x)$ is also a solution to the homogeneous problem (3.5.36). Let us check the solutions obtained. It is clear that $u_2(x)$ and $u_3(x)$ are biharmonic polynomials. Further it is easy to calculate that

$$L_1 u_2 = -2u_2 + \Lambda u_2 = C_1(-2H_2 + \Lambda H_2) = 0,\ L_2 u_2 = -3\Delta\, u_2 + \Lambda \Delta u_2 = 0,$$

and also

$$
\begin{aligned}
L_1 u_3 = -2u_3 + \Lambda u_3 &= C_2\left(-2(|x|^2-3) + (5|x|^2-9)\right) H_3(x) \\
&= C_2\left(3|x|^2-3\right) H_3(x)|_{\partial S} = 0, \\
L_2 u_3 = \ -3\Delta u_3 + \Lambda\Delta u_3 &= C_2\left(-3(2n+12) + 3(2n+12)\right) H_3(x) = 0,
\end{aligned}
$$

that is the boundary conditions of the homogeneous problem (3.5.36) are satisfied. So, if a solution to the problem (3.5.36) exists, then it is unique up to polynomials of the form $u(x) = C_1 H_2(x) + C_2(|x|^2-3)H_3(x)$.

For the existence of a solution to the problem (3.5.36), consider the system (3.5.32)

$$
\begin{pmatrix} m-2 & 0 \\ m & (m-3)(2n+4m) \end{pmatrix}\begin{pmatrix} q_1(m) \\ q_2(m) \end{pmatrix} = 0
$$

for $m=2$ and $m=3$. Let $m=2$, then we have

$$
\begin{pmatrix} 0 & 0 \\ 2 & -(2n+8) \end{pmatrix}\begin{pmatrix} q_1(2) \\ q_2(2) \end{pmatrix} = 0,
$$

which means $q_1(2) - (n+4)q_2(2) = 0$ and the space of zeros of the system (3.5.32) in this case is one-dimensional and spanned by the vector $\begin{pmatrix} n+4 \\ 1 \end{pmatrix}$. Therefore, the first condition for the existence of a solution to problem (3.5.36) has the form

$$
\int_{\partial S} H_2(x)((n+4)\varphi_1(x) + \varphi_2(x))\, ds_x = 0.
$$

Let now $m=3$, then we have

$$
\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} q_1(3) \\ q_2(3) \end{pmatrix} = 0,
$$

which means $q_1(3) = 0$ and the space of zeros of the system (3.5.32) in this case is one-dimensional and spanned by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore the second condition for the existence of a solution to the problem (3.5.36) has the form $\int_{\partial S} H_3(x)\varphi_2(x)\, ds_x = 0$. So, the problem (3.5.36) has a solution if

$$
\int_{\partial S} H_2(x)((n+4)\varphi_1(x) + \varphi_2(x))\, ds_x = 0, \int_{\partial S} H_3(x)\varphi_2(x)\, ds_x = 0, \qquad (3.5.37)
$$

where $H_2(x)$ and $H_3(x)$ are arbitrary homogeneous harmonic polynomials of degrees 2 and 3, respectively.

For example, for $n \geq 2$, $\varphi_1 = x_1x_2$ and $\varphi_2 = 0$ solution to the problem (3.5.36), if it exists, then it has the form $u(x) = u_0(x) + |x|^2u_1(x)$ and due to the second boundary condition $(2n + 4\Lambda)(\Lambda - 3)u_1(x) = 0$, whence $(\Lambda - 3)u_1(x) = 0$ and therefore $u_1(x) = H_3(x)$. Due to the first boundary condition, $u_0(x)$ must be such that

$$\left(\frac{\partial}{\partial\nu} - 2\right)(u_0(x) + |x|^2H_3(x))\Big|_{\partial S} = x_1x_2$$

and that means $(\Lambda - 2)u_0(x) = x_1x_2 - 3H_3(x)$ in S. This equation has no solution in functions harmonic in S since the harmonic function on the left $(\Lambda - 2)u_0(x)$ does not contain second-order terms in its expansion at the origin, but on the right there are such terms. Note that in this case, the first condition from (3.5.37) is not satisfied for $H_2(x) = x_1x_2$ since $\int_{\partial S} x_1^2x_2^2\, ds_x \neq 0$.

5. Robin problem [49]: with $a_{02} = a_{11} = 0$ and all other coefficients are positive we get

$$\Delta^2 u = f(x), \quad x \in S,$$
$$a_{00}u + a_{01}\frac{\partial}{\partial\nu}u\Big|_{\partial S} = \varphi_1(s),\; a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u\Big|_{\partial S} = \varphi_2(s), \quad s \in \partial S.$$

For this problem we have

$$\Delta(\lambda) = \begin{vmatrix} a_{00} + a_{01}\lambda & 2a_{01} \\ 0 & (2n+4\lambda)a_{12} + \lambda(2n+4\lambda)a_{13} \end{vmatrix}$$
$$= (a_{00} + a_{01}\lambda)(2n+4\lambda)(a_{12} + a_{13}\lambda) \neq 0,$$

for $\lambda \in \mathbb{N}_0$ and therefore the Robin problem is unconditionally solvable and its solution is unique.

3.6 Neumann type problem for the biharmonic equation

The boundary value problem of this section (3.6.1)-(3.6.2) is not included in the class of problems for the biharmonic equation studied in Sections 3.4 and 3.5, but is included in the problems of general form $\mathcal{H}_2$ studied in Section 1.1. First, in Theorem 3.26, the condition for the existence of a solution to the problem (3.6.1)-(3.6.2) for a homogeneous equation is formulated and this solution is found in (3.6.15). Then, in

Theorem 3.27, the uniqueness up to a constant of the solution is established. Finally, in the Theorem 3.28, with the help of Lemmas 3.34-3.35, necessary and sufficient conditions for the solvability of the problem (3.6.1)-(3.6.2) for an inhomogeneous equation are found. Example 3.11 shows the solvability conditions for the problem with the right-hand side of the equation of the form $f(x) = |x|^m h(x)$. The main results of Section 3.6 are published in [95].

3.6.1 Statement of the problem

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be, as usual, the n-dimensional unit ball in the Euclidean space $\mathbb{R}^n$, and ∂S be the unit sphere. In S we consider the following boundary value problem of Neumann type for the biharmonic equation

$$\Delta^2 u = f(x), \quad x \in S, \tag{3.6.1}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_1(s), \quad \frac{\partial^3 u}{\partial \nu^3}\Big|_{\partial S} = \varphi_2(s), \quad s \in \partial S, \tag{3.6.2}$$

where $\dfrac{\partial}{\partial \nu}$ is the outer normal derivative on ∂S, and $f(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are given functions, smoothness which is indicated below. We use, as before, the operator $\Lambda u = \sum_{k=1}^n x_k u'_{x_k}$ and the factorial power of the variable t according to the equality $t^{[k]} = t(t-1)\ldots(t-k+1)$ with the agreement $t^{[0]} = 1$. For example, using the Stirling triangle (1.1.2) on page 13, we easily obtain $t^{[3]} = t^3 - 3t^2 + 2t$.

The following simple statement is needed in what follows.

Lemma 3.33. *Let $\varphi, \psi \in C^k$, then the equality holds true for the operator $\Lambda^{[k]}$*

$$\Lambda^{[k]}(\varphi\psi) = \sum_{i=0}^{k} \binom{k}{i} \Lambda^{[i]}\varphi \Lambda^{[k-i]}\psi. \tag{3.6.3}$$

Proof. Let us prove the equality (3.6.3) by induction. Since $\Lambda^{[1]} = \Lambda$, then for $k = 1$ the equality (3.6.3) holds: $\Lambda(\varphi\psi) = \psi\Lambda\varphi + \varphi\Lambda\psi$. Let the equality (3.6.3) be true for $k-1$. Then given that $\Lambda^{[k]} = \Lambda \cdots (\Lambda - k + 1)$ we get

$$\Lambda^{[k]}(\varphi\psi) = (\Lambda - k + 1)\Lambda^{[k-1]}(\varphi\psi) = \sum_{i=0}^{k-1} (\Lambda - k + 1)\binom{k-1}{i}\Lambda^{[i]}\varphi\Lambda^{[k-i-1]}\psi.$$

Since for $a, b \in \mathbb{R}$ we have

$$(\Lambda - a - b)(\varphi\psi) = \psi\Lambda\varphi + \varphi\Lambda\psi - (a+b)\varphi\psi = \psi(\Lambda - a)\varphi + \varphi(\Lambda - b)\psi,$$

then it follows that

$$\Lambda^{[k]}(\varphi\psi) = \sum_{i=0}^{k-1}\binom{k-1}{i}\left(\Lambda^{[i+1]}\varphi\Lambda^{[k-i-1]}\psi + \Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi\right)$$
$$= \sum_{i=1}^{k}\binom{k-1}{i-1}\Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi + \sum_{i=0}^{k-1}\binom{k-1}{i}\Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi$$
$$= \binom{k-1}{k-1}\Lambda^{[k]}\varphi\Lambda^{[0]}\psi + \sum_{i=1}^{k-1}\left(\binom{k-1}{i-1} + \binom{k-1}{i}\right)\Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi$$
$$+ \binom{k-1}{0}\Lambda^{[0]}\varphi\Lambda^{[k]}\psi = \binom{k}{k}\Lambda^{[k]}\varphi\Lambda^{[0]}\psi + \sum_{i=1}^{k-1}\binom{k}{i}\Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi$$
$$+ \binom{k}{0}\Lambda^{[0]}\varphi\Lambda^{[k]}\psi = \sum_{i=0}^{k}\binom{k}{i}\Lambda^{[i]}\varphi\Lambda^{[k-i]}\psi.$$

The lemma is proved. □

For example, by virtue of Lemma 3.33 the equalities

$$\Lambda^{[3]}(u(x)|x|^2) = |x|^2\Lambda^{[3]}u(x) + 3\Lambda^{[2]}u(x)\Lambda^{[1]}|x|^2 + 3\Lambda^{[1]}u(x)\Lambda^{[2]}|x|^2$$
$$+ u(x)\Lambda^{[3]}|x|^2 = |x|^2(\Lambda^{[3]} + 6\Lambda^{[2]} + 6\Lambda^{[1]})u(x)$$

hold true, since $\Lambda|x|^2 = \Lambda^{[2]}|x|^2 = 2|x|^2$ and $\Lambda^{[3]}|x|^2 = 0$. By virtue of the binomial theorem for factorial powers, we find

$$\Lambda^{[3]}(u(x)|x|^2) = |x|^2(\Lambda + 2)^{[3]}u(x). \tag{3.6.4}$$

To represent a biharmonic in S function, we also use Lemma 3.32.

3.6.2 Existence of a solution to the problem for homogeneous equation

Let us prove the existence theorem for the problem (3.6.1)-(3.6.2) with a homogeneous equation. As a preliminary analysis, we note that Corollary 3.2 on page 223 shows that the sphere integrals of normal derivatives of orders $m \ge 2\cdot 2 - 1 = 3$ of a function biharmonic in S are equal to zero, i.e. if $u(x)$ is a solution to the problem under consideration, then the equality $\int_{\partial S}\varphi_2(x)\,ds_x = 0$ must hold. This necessary condition for the existence of a solution to the problem turns out to be a sufficient condition for existence of a solution.

Theorem 3.26. *The solution of the problem*

$$\Delta^2 u = 0, \quad x \in S, \tag{3.6.5}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_1(s), \quad \frac{\partial^3 u}{\partial \nu^3}\Big|_{\partial S} = \varphi_2(s), \quad s \in \partial S, \tag{3.6.6}$$

from the class $u \in C(\overline{S})$, $\Lambda u \in C(\overline{S})$, $\Lambda^{[3]} u \in C(\overline{S})$ *exists if* $\varphi_1 \in C^{2+\varepsilon}(\partial S)$, $\varphi_2 \in C(\partial S)$ *and the condition is satisfied*

$$\int_{\partial S} \varphi_2(x)\, ds_x = 0. \tag{3.6.7}$$

Remark 3.22. If we require the smoothness of $u \in C^4(S) \cap C^3(\overline{S})$ from the solution of the problem (3.6.1)-(3.6.2), then the following smoothness of the boundary functions $\varphi_1 \in C^{5+\varepsilon}(\partial S)$ and $\varphi_2 \in C^{3+\varepsilon}(\partial S)$ are sufficient.

Proof. Consider functions $v_1(x)$ and $v_2(x)$ that are harmonic in S and continuous in $\overline{S}$ and such that for them $v_1(x)|_{\partial S} = \varphi_1(s)$ and $v_2(x)|_{\partial S} = \varphi_2(s)$, i.e. $v_1(x)$ and $v_2(x)$ are solutions to the corresponding Dirichlet problems in S. Let the biharmonic function $u(x) = u_0(x) + |x|^2 u_1(x)$ be a solution to the problem (3.6.5)-(3.6.6). By Lemma 3.32 (S is a ar like domain), the biharmonic function $u(x)$ has a representation (3.5.11) in S. Let us indicate the functions $u_0(x)$ and $u_1(x)$ from (3.5.12) for which this is the case. Consider the operators L_1 and L_2 of the form

$$L_1 = \Lambda, \quad L_2 = \Lambda^{[3]} \equiv \Lambda(\Lambda - 1)(\Lambda - 2).$$

Due to (3.6.4) the equalities hold true

$$L_1 u = \Lambda\left(u_0(x) + |x|^2 u_1(x)\right) = \Lambda u_0(x) + |x|^2(\Lambda + 2)u_1(x),$$

$$L_2 u = \Lambda^{[3]}\left(u_0(x) + |x|^2 u_1(x)\right) = \Lambda^{[3]} u_0(x) + |x|^2(\Lambda + 2)^{[3]} u_1(x).$$

Passing here to the limit at $x \to s \in \partial S$ and taking into account that $L_1 u|_{\partial S} = \frac{\partial u}{\partial \nu}|_{\partial S} = \varphi_1(s)$, $L_2 u|_{\partial S} = \frac{\partial^3 u}{\partial \nu^3}|_{\partial S} = \varphi_2(s)$ we get

$$\begin{aligned} \Lambda u_0(s) + (\Lambda + 2)u_1(s) &= \varphi_1(s), \\ \Lambda^{[3]} u_0(s) + (\Lambda + 2)^{[3]} u_1(s) &= \varphi_2(s). \end{aligned} \tag{3.6.8}$$

Let $v_1(x)$, $v_2(x)$ be solutions to the Dirichlet problems for the Laplace equation in S with boundary values $\varphi_1(s)$ and $\varphi_2(s)$, respectively. From the uniqueness of the solution to the Dirichlet problem for the Laplace equation in S we conclude that the equalities (3.6.8) can be continued in $\overline{S}$

$$\Lambda u_0(x) + (\Lambda + 2)u_1(x) = v_1(x),$$
$$\Lambda^{[3]}u_0(x) + (\Lambda + 2)^{[3]}u_1(x) = v_2(x),$$

since by the property of the operator Λ the functions on the left in this equality are harmonic in S. Let us rewrite the resulting equations in matrix form

$$\begin{pmatrix} \Lambda & \Lambda + 2 \\ \Lambda^{[3]} & (\Lambda + 2)^{[3]} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} (x) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (x), \quad x \in \overline{S}. \tag{3.6.9}$$

Equation (3.6.9) is a system of differential equations in functions that are harmonic in S. The smoothness of the solution needs to be examined only in $\overline{S}$, since in S the harmonic functions are infinitely differentiable. Let the harmonic functions $u_0, u_1 \in C^3(\overline{S})$ satisfy the equation (3.6.9) in $\overline{S}$ and therefore they satisfy the equality (3.6.8). Then for the biharmonic function $u(x) = u_0(x) + |x|^2 u_1(x)$ we have $\Lambda u \in C(\overline{S})$, $\Lambda^{[3]}u \in C(\overline{S})$. Therefore, the function $u(x) = u_0(x) + |x|^2 u_1(x)$ is the desired solution to the problem (3.6.5)-(3.6.6). So, the problem (3.6.5)-(3.6.6) has been reduced to solving the equation (3.6.9) in functions harmonic in S.

Consider the matrix $A(\lambda)$ of the symbol of the operator from (3.6.9). By virtue of the equalities (3.6.4) we have

$$A(\lambda) = \begin{pmatrix} \lambda & \lambda + 2 \\ \lambda^{[3]} & (\lambda + 2)^{[3]} \end{pmatrix} = \begin{pmatrix} \lambda & \lambda + 2 \\ \lambda^{[3]} & \lambda^{[3]} + 6\lambda^{[2]} + 6\lambda \end{pmatrix}, \tag{3.6.10}$$

where $\lambda \in \mathbb{R}$. It is easy to calculate that

$$\det A(\lambda) = \lambda(\lambda + 2) \begin{vmatrix} 1 & 1 \\ (\lambda - 1)^{[2]} & (\lambda + 1)^{[2]} \end{vmatrix} = 2\lambda(2\lambda - 1)(\lambda + 2) \tag{3.6.11}$$

and this means that $\lambda_1 = 0$, $\lambda_2 = 1/2$, $\lambda_3 = -2$ are zeros of $A(\lambda)$. Therefore the matrix $A(\lambda)$ is nonsingular for $\lambda \in \mathbb{N}$. The system (3.6.9) can be written in the form

$$A(\Lambda)U(x) = V(x), \quad x \in \overline{S}, \tag{3.6.12}$$

where $U(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}$ and $V(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$. It is easy to see that the equality holds true

$$\frac{1}{\det A(\lambda)} = \frac{b_1}{\lambda} + \frac{b_2}{\lambda - 1/2} + \frac{b_3}{\lambda + 2},$$

where $b_i = 1/(\det A)'_\lambda(\lambda_i)$. It is clear that the inverse matrix $A^{-1}(\lambda)$ has the form

$$A^{-1}(\lambda) = \frac{1}{\Delta(\lambda)} A^*(\lambda),$$

where

$$A^*(\lambda) = \begin{pmatrix} (\lambda+2)^{[3]} & -\lambda-2 \\ -\lambda^{[3]} & \lambda \end{pmatrix}.$$

Consider the following operators depending on the roots of λ_i, similar to those used in the previous Section (3.5.18)

$$M_i(\Lambda)v(x) = \int_0^1 t^{-\lambda_i-1} v(tx)\,dt,$$

where $i = 1,2,3$. It is easy to see that if $\lim_{t\to 0} t^{-\lambda_i} w(tx) = 0$, then the differentiable function $w(x)$ satisfies the equalities

$$\begin{gathered}(\Lambda-\lambda_i)M_i(\Lambda)w = (\Lambda-\lambda_i)\int_0^1 t^{-\lambda_i-1} w(tx)\,dt = -\lambda_i \int_0^1 t^{-\lambda_i-1} w(tx)\,dt \\ + \int_0^1 t^{-\lambda_i} \sum_{i=1}^n x_i w_{x_i}(tx)\,dt = -\lambda_i \int_0^1 t^{-\lambda_i-1} w(tx)\,dt + \int_0^1 t^{-\lambda_i} w_t(tx)\,dt \\ -\lambda_i \int_0^1 t^{-\lambda_i-1} w(tx)\,dt + t^{-\lambda_i} w(tx)\Big|_0^1 + \lambda_i \int_0^1 t^{-\lambda_i-1} w(tx)\,dt \\ = t^{-\lambda_i} w(tx)\Big|_0^1 = w(x).\end{gathered} \tag{3.6.13}$$

Since the identities

$$1 = \det A(\lambda)\left(\frac{b_1}{\lambda} + \frac{b_2}{\lambda-1/2} + \frac{b_3}{\lambda+2}\right) = \frac{b_1 \det A(\lambda)}{\lambda} + \frac{b_2 \det A(\lambda)}{\lambda-1/2}$$
$$+\frac{b_3 \det A(\lambda)}{\lambda+2} = 4b_1(\lambda-1/2)(\lambda+2) + 4b_2\lambda(\lambda+2) + 4b_3\lambda(\lambda-1/2),$$

hold true then for a function $w \in C^2(S)$ and such that $\lim_{t\to 0} t^{-\lambda_i} w(tx) = 0$ for $i = 1,2,3$ we have

$$\begin{gathered}\det A(\Lambda)\,(b_1M_1(\Lambda) + b_2M_2(\Lambda) + b_3M_3(\Lambda))\,w = \big(4b_1(\Lambda-1/2) \\ \times(\Lambda+2) + 4b_2\Lambda(\Lambda+2) + 4b_3\Lambda(\Lambda-1/2)\big)w = 1\cdot w.\end{gathered} \tag{3.6.14}$$

Therefore, if the harmonic function $V(x)$ satisfies the following conditions $\lim_{t\to 0} t^{-\lambda_i} V(tx) = 0$ for $i = 1, 2, 3$, then the solution to the equation (3.6.12) can be written in the form

$$U(x) = A^*(\Lambda)\,(b_1 M_1(\Lambda) + b_2 M_2(\Lambda) + b_3 M_3(\Lambda))\,V(x). \tag{3.6.15}$$

Indeed, the harmonic function $U(x)$ (it is infinitely differentiable in S), found from (3.6.15), due to the equality (3.6.14), turns the equation (3.6.12) into an identity in S

$$\begin{aligned} A(\Lambda)U(x) &= A(\Lambda)A^*(\Lambda)\,(b_1 M_1(\Lambda) + b_2 M_2(\Lambda) + b_3 M_3(\Lambda))\,V(x) \\ &= \det A(\Lambda)\,(b_1 M_1(\Lambda) + b_2 M_2(\Lambda) + b_3 M_3(\Lambda))\,V(x) = V(x), \end{aligned}$$

which, by continuity of $V(x)$, extends into $\overline{S}$. The solution to the equation (3.6.9) has been found. A similar method for constructing solutions to partial differential equations is applied in [89].

Now let us find out the conditions that need to be imposed on the function $V(x)$ so that we can use the equality (3.6.15). First, the limits $\lim_{t\to 0} t^{-\lambda_i} V(tx) = 0$ must be true, where $i = 1, 2, 3$. Second, in order for the harmonic function $U(x)$ to be continuous in $\overline{S}$, due to the presence of operators of the form $A^*(\Lambda)M_i(\Lambda)$ in the structure of the operator $A^*(\Lambda)$ and according to the equality (3.6.13), it is sufficient that $v_1 \in C^2(\overline{S})$ and $v_2 \in C(\overline{S})$. This holds if $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C(\partial S)$. From the fulfillment of the equation (3.6.9) it follows that $\Lambda u \in C(\overline{S})$, $\Lambda^{[3]} u \in C(\overline{S})$.

Note that if we require smoothness of a solution of the form $U \in C^3(\overline{S})$, or $u \in C^3(\overline{S})$, then it is necessary that the operator $A(\Lambda)$ be applicable to the function $U(x)$ in $\overline{S}$. This is possible if $v_1 \in C^5(\overline{S})$ and $v_2 \in C^3(\overline{S})$. Such smoothness of the functions $v_1(x)$ and $v_2(x)$ is ensured if $\varphi_1 \in C^{5+\varepsilon}(\partial S)$ and $\varphi_2 \in C^{3+\varepsilon}(\partial S)$ (see Remark 3.22).

Let us return to the conditions

$$\lim_{t\to 0} t^{-\lambda_i} V(tx) = 0, \quad i = 1, 2, 3, \tag{3.6.16}$$

which are actually imposed on the functions $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C(\partial S)$. Since $t^{-\lambda_3} = t^2$, then for $\lambda_3 = -2$ the condition (3.6.16) is satisfied

$$\lim_{t\to 0} t^{-\lambda_3} V(tx) = \lim_{t\to 0} t^2 V(tx) = 0.$$

However, the operators $M_1(\Lambda)$ and $M_2(\Lambda)$ are not applicable to all functions $V(x)$ since $t^{-\lambda_1 - 1} = t^{-1}$ and $t^{-\lambda_2 - 1} = t^{-3/2}$. Therefore, let us decompose the function $V(x)$ from (3.6.12) into two terms $V(x) = V_0(x) + V_1(x)$, where $V_0(x) = V_0 \equiv V(0)$ is homogeneous harmonic polynomial of zero degree. Then $V_1(0) = 0$. Let us solve two equations separately

$$A(\Lambda)U_0(x) = V_0(x), \quad A(\Lambda)U_1(x) = V_1(x), \quad x \in \overline{S} \tag{3.6.17}$$

and then the sum of their solutions $U(x) = U_0(x) + U_1(x)$ is a solution to the equation (3.6.12). Let us look at the first equation. It is clear that $U_0(x) = U_0$. Let us denote $U_0 = \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \end{pmatrix}$ and $V_0 = \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix}$. It is easy to see that if $P(\lambda)$ is a polynomial, then $P(\Lambda)x^\alpha = P(\alpha)x^\alpha$. That is why

$$A(\Lambda)U_0 = A(0)U_0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \end{pmatrix} = \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix}.$$

According to the Fredholm alternatives, a solution to this equation exists if its right-hand side is orthogonal to all solutions of the conjugate homogeneous equation

$$A^*(0) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0.$$

It can be seen that the solution space of the homogeneous adjoint equation is spanned by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and therefore a solution to the first equation (3.6.17) exists if $\begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2^{(0)} = 0$. It is known that $v_2^{(0)} = \int_{\partial S} \varphi_2(x)\, ds_x$ (see, for example, Theorem 1.8 on page 50) and therefore the solution to the first equation (3.6.17) exists if the condition (3.6.7) is met.

So, the solution $U(x) = U_0(x) + U_1(x)$ to the equation (3.6.12) in $\overline{S}$ is constructed. After finding the harmonic functions $u_0(x)$ and $u_1(x)$, the solution to the problem (3.6.5)-(3.6.6) is written in the form $u(x) = u_0(x) + |x|^2 u_1(x)$. The theorem is proved. □

3.6.3 Uniqueness of solution to the problem

Theorem 3.27. *The solution to the problem* (3.6.1)-(3.6.2) *is unique up to a constant.*

Proof. Similar to Theorem 3.26, we reduce the solution of the homogeneous problem (3.6.1)-(3.6.2) to the homogeneous equation (3.6.12)

$$A(\Lambda)U(x) = 0, \quad x \in \overline{S}, \tag{3.6.18}$$

where the matrix $A(\lambda)$ has the form (3.6.10). Let $U(x) = \binom{u_0(x)}{u_1(x)}$ be the solution to (3.6.18). It is known that in a certain neighborhood of the origin the harmonic function $U(x)$ expands into a uniformly convergent power series (Theorem 1.8 on page 50)

$$U(x) = \sum_{m=0}^{\infty} U_m(x),$$

where $U_m(x)$ are homogeneous harmonic polynomials of degree $m \in \mathbb{N}_0$. This series can be differentiated indefinitely under the sum sign. Therefore, since, due to the properties of the operator Λ, the equality $A(\Lambda)U_m(x) = A(m)U_m(x)$ holds, then

$$\sum_{m=0}^{\infty} A(m)U_m(x) = 0.$$

This means $A(m)U_m(x) = 0$ and therefore either $\det A(m) = 0$ or $U_m(x) \equiv 0$. However, according to (3.6.11) $\det A(\lambda) = 2\lambda(2\lambda-1)(\lambda+2)$ and therefore the matrix $A(\lambda)$ is nonsingular for $\lambda\, in \mathbb{N}$. Thus, the only possible option is $m = 0$, when $U_m(x) \not\equiv 0$, since $\det A(0) = 0$. In this case, the vector $U_0 = \binom{p_1}{p_2}$ must be a solution to the equation

$$A(0)U_0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0,$$

which means the solutions to the equation (3.6.18) have the form $U(x) = U_0 = \binom{C}{0}$. Therefore, the solution to the problem (3.6.1)-(3.6.2) is unique up to functions $u_0(x) = C + 0 \cdot |x|^2 = \text{const}$. The theorem is proved. □

3.6.4 Existence of a solution to the problem for inhomogeneous equation

Let us consider the case of the inhomogeneous equation (3.6.1). Let $n \geq 3$. Let us prove a statement generalizing Theorem 3.26.

Theorem 3.28. *Let $f \in C^2(\bar{S})$, $\varphi_1 \in C^{2+\varepsilon}(\partial S)$ and $\varphi_2 \in C(\partial S)$. Then a solution to the problem* (3.6.1)-(3.6.2) *from the class $u \in C^3(\overline{S})$ exists if and only if the equality holds*

$$\int_{\partial S} \varphi_2(x)\, ds_x = \frac{1}{2} \int_S (1 - |\tau|^2)(\Lambda + 3) f(\tau)\, d\tau. \tag{3.6.19}$$

Proof. Let $V[f](x)$ be a volume potential with density $f(x)$, i.e.

$$V[f](x) = -\frac{1}{\omega_n}\int_S E(x,\xi)f(\xi)\,d\xi, \tag{3.6.20}$$

where $E(x,\xi) = (n-2)^{-1}|\xi - x|^{2-n}$ $(n > 2)$ is an elementary solution to the Laplace equation. We also denote

$$V_2[f](x) = V[V[f]](x).$$

It is known [28] that if $f \in C^1(\bar{S})$, then $V[f] \in C^1(\bar{S})$ and $\Delta V[f] = f$ in S, and therefore $V_2[f]$, $\Delta V_2[f] = V_1[f] \in C^1(\bar{S})$ and $\Delta^2 V_2[f] = f$ in S. Therefore, in the equation (3.6.1) we can replace the variable $u = w + V_2[f]$. Since the following equalities hold $\Lambda V_2[f]\big|_{\partial S} = \frac{\partial}{\partial\nu}V_2[f](s)$, $\Lambda^{[3]}V_2[f]\big|_{\partial S} = \frac{\partial^3}{\partial\nu^3}V_2[f](s)$, $s \in \partial S$, then the problem (3.6.1)-(3.6.2) is equivalent to the following boundary value problem

$$\Delta^2 w = 0, \quad x \in S, \tag{3.6.21}$$

$$\frac{\partial w}{\partial\nu} = \hat{\varphi}_1(s), \quad \frac{\partial^3 w}{\partial\nu^3} = \hat{\varphi}_2(s), \quad s \in \partial S, \tag{3.6.22}$$

where

$$\hat{\varphi}_1(s) = \varphi_1(s) - \Lambda V_2[f]\Big|_{\partial S}, \quad \hat{\varphi}_2(s) = \varphi_2(s) - \Lambda^{[3]}V_2[f]\Big|_{\partial S}.$$

The solvability condition for the problem (3.6.21)-(3.6.22), and therefore for the problem (3.6.1)-(3.6.2), is found in Theorem 3.26 and has the form

$$\int_{\partial S}\hat{\varphi}_2(x)\,ds_x = 0.$$

This condition can be rewritten as

$$\int_{\partial S}\varphi_2(x)\,ds_x = \int_{\partial S}\Lambda^{[3]}V_2[f](x)\,ds_x. \tag{3.6.23}$$

Let us calculate the integral included in (3.6.23) on the right. To do this, we examine the expression

$$\begin{aligned}\Lambda^{[3]}V_2[f](x) = -\frac{1}{\omega_n}\Lambda^{[3]}\int_S E(x,\xi)V[f](\xi)\,d\xi \\ = \frac{1}{\omega_n^2}\Lambda^{[3]}\int_S E(x,\xi)\int_S E(\xi,\tau)f(\tau)\,d\tau\,d\xi.\end{aligned}$$

Recall that $f \in C^2(\overline{S})$. Let us prove two equalities necessary in what follows

$$\Lambda_x \int_S E(x,\xi) f(\xi)\, d\xi$$
$$= \int_S E(x,\xi)(\Lambda_\xi + 2) f(\xi)\, d\xi - \int_{\partial S} E(x,\xi) f(\xi)\, ds_\xi \quad (3.6.24)$$

and

$$\Lambda_x \int_{\partial S} E(x,\xi) f(\xi)\, ds_\xi = -\int_{\partial S} E(x,\xi)(\Lambda_\xi + n - 2) f(\xi)\, ds_\xi$$
$$+ \omega_n f(x) + \int_S E(x,\xi) \Delta_\xi f(\xi)\, d\xi. \quad (3.6.25)$$

According to divergence theorem we can write

$$\int_S u \Lambda v\, d\xi = \int_S \sum_{i=1}^n (\xi_i u v)_{\xi_i}\, d\xi - n \int_S uv\, d\xi - \int_S v \Lambda u\, d\xi$$
$$= \int_{\partial S} \sum_{i=1}^n \xi_i^2 uv\, ds_\xi - \int_S v(\Lambda + n) u\, d\xi$$
$$= \int_{\partial S} uv\, ds_\xi - \int_S v(\Lambda + n) u\, d\xi, \quad (3.6.26)$$

even if Λv has an integrable singularity in S of order $E(x,\xi)$. Therefore the equality (3.6.24) is correct

$$\Lambda_x \int_S E(x,\xi) f(\xi)\, d\xi = \int_S \sum_{i=1}^n (x_i - \xi_i) E'_{x_i}(x,\xi) f(\xi)\, d\xi - \int_S f(\xi) \Lambda_\xi E(x,\xi)\, d\xi$$
$$= (2-n) \int_S E(x,\xi) f(\xi)\, d\xi - \int_{\partial S} f(\xi) E(x,\xi)\, ds_\xi + \int_S E(x,\xi)(\Lambda + n) f(\xi)\, d\xi$$
$$= \int_S E(x,\xi)(\Lambda + 2) f(\xi)\, d\xi - \int_{\partial S} E(x,\xi) f(\xi)\, ds_\xi.$$

Using the integral representation of functions from C^2 [143] we can write

$$\omega_n f(x) = \int_{\partial S} (E(x,\xi) \Lambda_\xi f(\xi) - f(\xi) \Lambda_\xi E(x,\xi))\, ds_\xi - \int_S E(x,\xi) \Delta_\xi f(\xi)\, d\xi,$$

from where, considering that

$$\int_{\partial S} f(\xi)\Lambda_\xi E(x,\xi)\,ds_\xi = \int_{\partial S}\sum_{i=1}^{n}(\xi_i - x_i)E'_{\xi_i}(x,\xi)f(\xi)\,ds_\xi - \Lambda_x\int_{\partial S} E(x,\xi)$$
$$\times f(\xi)\,ds_\xi = -\Lambda_x\int_{\partial S} E(x,\xi)f(\xi)\,ds_\xi - (n-2)\int_{\partial S} f(\xi)E(x,\xi)\,ds_\xi,$$

we get (3.6.25).

First, note that since $\Delta V[f](\xi) = f(\xi)$, then from (3.6.25) for $f(\xi) = V[f](\xi)$ it follows

$$\Lambda_x\int_{\partial S} E(x,\xi)V[f](\xi)\,ds_\xi - \int_{\partial S} E(x,\xi)(\Lambda_\xi + n - 2)V[f](\xi)\,ds_\xi + \omega_n V[f] + \int_S E(x,\xi)f(\xi)\,d\xi = -\int_{\partial S} E(x,\xi)(\Lambda_\xi + n - 2)V[f](\xi)\,ds_\xi. \quad (3.6.27)$$

Now let us use the equality (3.6.24) also for $f(\xi) = V[f](\xi)$. We have

$$(\Lambda_x - 1)\int_S E(x,\xi)V[f](\xi)\,d\xi = \int_S E(x,\xi)(\Lambda_\xi + 1)V[f](\xi)\,d\xi - \int_{\partial S} E(x,\xi)V[f](\xi)\,ds_\xi,$$

from where, using (3.6.27) and again (3.6.24), we find

$$(\Lambda_x - 2)(\Lambda_x - 1)\int_S E(x,\xi)V[f](\xi)\,d\xi = (\Lambda_x - 2)\int_S E(x,\xi)(\Lambda_\xi + 1)V[f](\xi)\,d\xi$$
$$+ \int_{\partial S} E(x,\xi)(\Lambda_\xi + n)V[f](\xi)\,ds_\xi = \int_S E(x,\xi)\Lambda_\xi(\Lambda_\xi + 1)V[f](\xi)\,d\xi$$
$$- \int_{\partial S} E(x,\xi)(\Lambda_\xi + 1)V[f](\xi)\,d\xi + \int_{\partial S} E(x,\xi)(\Lambda_\xi + n)V[f](\xi)\,ds_\xi$$
$$= \int_S E(x,\xi)\Lambda_\xi(\Lambda_\xi + 1)V[f](\xi)\,d\xi + (n-1)\int_{\partial S} E(x,\xi)V[f](\xi)\,ds_\xi.$$

Finally, we calculate $\Lambda^{[3]}V_2[f](x) = \Lambda_x(\Lambda_x - 1)(\Lambda_x - 2)V_2[f](x)$. Again using (3.6.27) we write

$$\Lambda^{[3]}V_2[f](x) = -\frac{1}{\omega_n}\Lambda_x\int_S E(x,\xi)\Lambda_\xi(\Lambda_\xi + 1)V[f](\xi)\,d\xi$$

$$+\frac{n-1}{\omega_n}\int_{\partial S}E(x,\xi)(\Lambda_\xi+n-2)V[f](\xi)\,ds_\xi=-\frac{1}{\omega_n}\int_S\Lambda_x E(x,\xi)\Lambda_\xi(\Lambda_\xi+1)$$
$$\times V[f](\xi)\,d\xi-\frac{n-1}{\omega_n^2}\int_{\partial S}E(x,\xi)\int_S(\Lambda_\xi+n-2)E(\xi,\tau)f(\tau)\,d\tau\,ds_\xi.$$

From here we get

$$\int_{\partial S}\Lambda^{[3]}V_2[f](x)ds_\xi=-\frac{1}{\omega_n}\int_{\partial S}ds_x\int_S\Lambda_x E(x,\xi)\Lambda_\xi(\Lambda_\xi+1)V[f](\xi)\,d\xi$$
$$-\frac{n-1}{\omega_n^2}\int_{\partial S}ds_x\int_{\partial S}E(x,\xi)\int_S(\Lambda_\xi+n-2)E(\xi,\tau)f(\tau)\,d\tau\,ds_\xi$$
$$\equiv I_1+I_2. \quad (3.6.28)$$

Here $V[f]\in C^2(\bar S)$, since $f\in C^2(\bar S)$. Let us first consider the integral I_2. It has integrable singularities and therefore, according to Fubini's theorem, the order of integration in it can be reversed

$$I_2=-\frac{n-1}{\omega_n^2}\int_S f(\tau)\,d\tau\int_{\partial S}(\Lambda_\xi+n-2)E(\xi,\tau)\,ds_\xi\int_{\partial S}E(x,\xi)\,ds_x.$$

For further proof, auxiliary statements are needed. Let $\{H_k^{(i)}(x),\ i=1,\dots,h_k\}$ be a complete system of homogeneous degree $k\in\mathbb{N}_0$ orthogonal harmonics polynomials (see Definition 1.5) normalized so that $\int_{\partial S}(H_k^{(i)}(\xi))^2\,ds_\xi=\omega_n$, where ω_n is the area of the unit sphere ∂S and $h_k=\frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}$.

Lemma 3.34. *If $|\xi|<|x|$, then the representation is valid*

$$E(x,\xi)=\sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)H_k^{(i)}(\xi). \quad (3.6.29)$$

The series in (3.6.29)*: 1) converges uniformly in ξ for $|\xi|<a<|x|$ and fixed x; 2) converges uniformly in x for $|\xi|<a<|x|$ and fixed ξ.*

Proof. In Lemma 2.7 on page 88 the representation (3.6.29) is established. Let us prove the uniform convergence of the series in (3.6.29). From the equality (3.6.29) it follows that for any ξ such that $|\xi|<a<|x|$

$$|E(x,\xi)|\le|x|^{-(n-2)}\sum_{k=0}^{\infty}\frac{(a/|x|)^k}{2k+n-2}\sum_{i=1}^{h_k}\left|H_k^{(i)}(x/|x|)H_k^{(i)}(\xi/|\xi|)\right|.$$

In [73, Lemma 12] it is established that the inequality holds true

$$\sum_{i=1}^{h_k} \left| H_k^{(i)}(x/|x|) H_k^{(i)}(\xi/|\xi|) \right| \le h_k$$

and therefore since $h_k = \mathrm{O}(k^{n-2})$ for $k \to \infty$, we have the estimate

$$|E(x,\xi)| \le \frac{1}{|x|^{n-2}} \sum_{k=0}^{\infty} \frac{h_k}{2k+n-2} \Big(\frac{a}{|x|}\Big)^k \le C \sum_{k=0}^{\infty} k^{n-3} \Big(\frac{a}{|x|}\Big)^k < \infty$$

for some $C > 0$. Therefore, the series in (3.6.29) converges uniformly in ξ for $|\xi| < a < |x|$ and fixed x. The first case is proved. Let us consider the second case. From the equality (3.6.29) it follows that for any x such that $|\xi| < a < |x|$

$$|E(x,\xi)| \le a^{-(n-2)} \sum_{k=0}^{\infty} \frac{(|\xi|/a)^k}{2k+n-2} \sum_{i=1}^{h_k} \left| H_k^{(i)}(x/|x|) H_k^{(i)}(\xi/|\xi|) \right|.$$

Again, using [73, Lemma 12], similarly to the previous case, we find

$$|E(x,\xi)| \le \frac{1}{a^{n-2}} \sum_{k=0}^{\infty} \frac{h_k}{2k+n-2} \Big(\frac{|\xi|}{a}\Big)^k \le C_1 \sum_{k=0}^{\infty} k^{n-3} \Big(\frac{|\xi|}{a}\Big)^k < \infty,$$

where $C_1 > 0$ and therefore the series in (3.6.29) converges uniformly in x for $|\xi| < a < |x|$ and fixed ξ. The lemma is proved. □

Lemma 3.35. *Let $H_m(x)$ be a homogeneous harmonic polynomial of degree m and $|\xi| < 1$, $|\tau| < 1$. Then the equalities hold*

$$\frac{1}{\omega_n} \int_{\partial S} H_m(x) E(x,\xi)\, ds_x = \frac{1}{2m+n-2} H_m(\xi), \tag{3.6.30}$$

$$\frac{1}{\omega_n} \int_{\partial S} H_m(x) \Lambda_x E(x,\xi)\, ds_x = -\frac{m+n-2}{2m+n-2} H_m(\xi), \tag{3.6.31}$$

$$\frac{1}{\omega_n} \int_S E(\xi,\tau) H_m(\xi)\, d\xi = \Big(\frac{|\tau|^2}{2m+n} + \frac{1-|\tau|^2}{2}\Big) \frac{H_m(\tau)}{2m+n-2}, \tag{3.6.32}$$

and

$$\frac{1}{\omega_n} \int_S H_m(\xi) \Lambda_\xi E(\tau,\xi)\, d\xi$$
$$= \Big(\frac{m|\tau|^2}{2m+n} - (m+n-2)\frac{1-|\tau|^2}{2}\Big) \frac{H_m(\tau)}{2m+n-2}. \tag{3.6.33}$$

Proof. 1^0. Let us use the representation (3.6.29) for $|\xi| < |x| = 1$ in the equality

$$\int_{\partial S} E(x,\xi)H_m(x)\,\frac{ds_x}{\omega_n} = \int_{\partial S} H_m(x)\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi)\,\frac{ds_x}{\omega_n}.$$

Since, by Lemma 3.34, the series under the integral converges uniformly in $x \in \partial S$ for a fixed $|\xi| < 1$, we can rearrange the order of integration and summation. If we then use the orthogonality of the system of polynomials $\{H_k^{(i)}(\xi),\ i = 1,\dots,h_k, k \in \mathbb{N}_0\}$ on ∂S, then we get

$$\frac{1}{\omega_n}\int_{\partial S} E(x,\xi)H_m(x)\,ds_x = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k}\frac{1}{\omega_n}\int_{\partial S} H_m(x)H_k^{(i)}(x)\,ds_x \\ \times H_k^{(i)}(\xi) = \frac{1}{2m+n-2}\sum_{i=1}^{h_m} H_m^{(i)}(\xi)\frac{1}{\omega_n}\int_{\partial S} H_m(x)H_m^{(i)}(x)\,ds_x.$$

Due to the completeness of the system $\{H_m^{(i)}(x),\ i = 1,\dots,h_m, m \in \mathbb{N}_0\}$, the expansion holds

$$H_m(x) = \sum_{j=1}^{h_m} c_j H_m^{(j)}(x).$$

Substituting this expansion into the previous equality and using the orthogonality of the polynomials $H_m^{(i)}(\xi)$ for $i = 1,\dots,h_m$ on ∂S and their normalization we obtain (3.6.30)

$$\frac{1}{\omega_n}\int_{\partial S} E(x,\xi)H_m(x)\,ds_x \\ = \frac{1}{2m+n-2}\sum_{i=1}^{h_m} H_m^{(i)}(\xi)\sum_{j=1}^{h_m} c_j\frac{1}{\omega_n}\int_{\partial S} H_m^{(j)}(x)H_m^{(i)}(x)\,ds_x \\ = \frac{1}{2m+n-2}\sum_{i=1}^{h_m} c_i H_m^{(i)}(\xi)\frac{1}{\omega_n}\int_{\partial S}(H_m^{(i)}(x))^2\,ds_x = \frac{1}{2m+n-2}H_m(\xi).$$

2^0. From (3.6.29) for $|\xi| < |x| = 1$ we find

$$\frac{1}{\omega_n}\int_{\partial S} H_m(\xi)\Lambda_x E(x,\xi)\,ds_x$$

$$= \frac{1}{\omega_n} \int_{\partial S} H_m(x) \sum_{k=0}^{\infty} \frac{-(k+n-2)}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi)\, ds_x.$$

From here, similarly to the first case, we derive (3.6.31).

3^0. It is easy to see that since the function $E(\tau,\xi)H_m(\xi)$ is integrable with respect to $\xi \in S$ for $|\tau| < 1$, we can write

$$\frac{1}{\omega_n} \int_S E(\tau,\xi) H_m(\xi)\, d\xi = \frac{1}{\omega_n} \lim_{\varepsilon\to+0} \left(\int_{|\xi|<|\tau|-\varepsilon} + \int_{|\tau|+\varepsilon<|\xi|<1} \right) \\ \times E(\tau,\xi) H_m(\xi)\, d\xi \equiv \lim_{\varepsilon\to+0} (I_1^\varepsilon(\tau) + I_2^\varepsilon(\tau)).$$

Let $\varepsilon > 0$ be small. Let us calculate the integral $I_1^\varepsilon(\tau)$. We use Lemma 3.34 to represent the function $E(\xi,\tau)$. Taking into account the uniform convergence of the series from (3.6.29) under the integral over $|\xi| < |\tau| - \varepsilon$ for $|\tau| < 1$ (Lemma 3.34 is applicable for $a = |\tau| - \varepsilon$ since $|\xi| < a < |\tau|$) we can change the order of integration and summation. That is why

$$I_1^\varepsilon(\tau) = \frac{1}{\omega_n} \int_{|\xi|<|\tau|-\varepsilon} E(\tau,\xi) H_m(\xi)\, d\xi \\ = \frac{1}{\omega_n} \int_{|\xi|<|\tau|-\varepsilon} \sum_{k=0}^{\infty} \frac{|\tau|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_m(\xi) H_k^{(i)}(\xi)\, d\xi H_k^{(i)}(\tau) \\ = \sum_{k=0}^{\infty} \frac{|\tau|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(\tau) \frac{1}{\omega_n} \int_{|\xi|<|\tau|-\varepsilon} H_m(\xi) H_k^{(i)}(\xi)\, d\xi.$$

Further, it is easy to verify the validity of the equality

$$\int_{a<|\xi|<b} f(\xi)\, d\xi = \int_a^b dr \int_{|\xi|=r} f(\xi)\, ds_\xi = \int_a^b r^{n-1}\, dr \int_{|\eta|=1} f(r\eta)\, ds_\eta. \quad (3.6.34)$$

Applying this equality, using the completeness and orthogonality of the system of harmonic polynomials $\{H_k^{(i)}(\xi), k \in \mathbb{N}_0\}$ on ∂S we transform the integral

$$I_1^\varepsilon(\tau) \\ = \sum_{k=0}^{\infty} \frac{|\tau|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(\tau) \int_0^{|\tau|-\varepsilon} r^{n-1}\, dr \frac{1}{\omega_n} \int_{|\eta|=1} r^{2k} H_m(\eta) H_k^{(i)}(\eta)\, ds_\eta$$

$$= |\tau|^{-(2m+n-2)}\frac{(|\tau|-\varepsilon)^{2m+n}}{(2m+n-2)(2m+n)}\sum_{i=1}^{h_m} H_m^{(i)}(\tau)\frac{1}{\omega_n}\int_{|\eta|=1} H_m(\eta)H_m^{(i)}(\eta)\,ds_\eta$$
$$= |\tau|^{-(2m+n-2)}\frac{(|\tau|-\varepsilon)^{2m+n}}{(2m+n-2)(2m+n)}H_m(\tau).$$

Now let us calculate the second integral $I_2^\varepsilon(\tau)$. Again we use Lemma 3.34 to represent the function $E(\tau,\xi)=E(\xi,\tau)$. Due to the uniform convergence of the series under the integral over ξ: $|\tau|+\varepsilon<|\xi|<1$ for $|\tau|<1$ (Lemma 3.34 is applicable for $a=|\tau|+\varepsilon$ since $|\tau|<a<|\xi|$) we can change the order of integration and summation. Therefore

$$I_2^\varepsilon(\tau) = \frac{1}{\omega_n}\int_{|\tau|+\varepsilon<|\xi|<1} E(\xi,\tau)H_m(\xi)\,d\xi$$
$$= \frac{1}{\omega_n}\int_{|\tau|+\varepsilon<|\xi|<1}\sum_{k=0}^{\infty}\frac{|\xi|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k} H_m(\xi)H_k^{(i)}(\xi)\,d\xi H_k^{(i)}(\tau)$$
$$= \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(\tau)\frac{1}{\omega_n}\int_{|\tau|+\varepsilon<|\xi|<1}|\xi|^{-(2k+n-2)}H_m(\xi)H_k^{(i)}(\xi)\,d\xi.$$

Applying the equality (3.6.34), using the completeness and orthogonality of the system of harmonic polynomials $\{H_k^{(i)}(\xi),\ i=1,\dots,h_k,\ k\in\mathbb{N}_0\}$ on ∂S we get

$$I_2^\varepsilon(\tau) = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(\tau)\int_{|\tau|+\varepsilon}^{1} r^{-(2k+n-2)}r^{n-1}\,dr$$
$$\times\frac{1}{\omega_n}\int_{|\eta|=1} r^{2k}H_m(\eta)H_k^{(i)}(\eta)\,ds_\eta = \frac{1-(|\tau|+\varepsilon)^2}{2(2m+n-2)}$$
$$\times\sum_{i=1}^{h_m}H_m^{(i)}(\tau)\frac{1}{\omega_n}\int_{|\eta|=1}H_m(\eta)H_m^{(i)}(\eta)\,ds_\eta = \frac{1-(|\tau|+\varepsilon)^2}{2(2m+n-2)}H_m(\tau).$$

Thus, taking into account the calculated values of the integrals $I_1^\varepsilon(\tau)$ and $I_2^\varepsilon(\tau)$ we obtain (3.6.32)

$$\frac{1}{\omega_n}\int_S E(\xi,\tau)H_m(\xi)\,d\xi = \lim_{\varepsilon\to+0}(I_1^\varepsilon(\tau)+I_2^\varepsilon(\tau))$$
$$= H_m(\tau)\lim_{\varepsilon\to+0}\left(|\tau|^{-(2m+n-2)}\frac{(|\tau|-\varepsilon)^{2m+n}}{(2m+n-2)(2m+n)}+\frac{1-(|\tau|+\varepsilon)^2}{2(2m+n-2)}\right)$$

$$= \Big(\frac{|\tau|^2}{2m+n} + \frac{1-|\tau|^2}{2}\Big)\frac{H_m(\tau)}{2m+n-2}.$$

A more general result is obtained in the equality (3.7.12) below.

4^0. Similar to case 3^0, due to the integrability of the function $H_m(\xi)\Lambda_\xi E(\tau,\xi)$ with respect to $\xi \in S$ for $|\tau| < 1$, we can write

$$\frac{1}{\omega_n}\int_S H_m(\xi)\Lambda_\xi E(\tau,\xi)\,d\xi = \frac{1}{\omega_n}\lim_{\varepsilon\to+0}\Big(\int_{|\xi|<|\tau|-\varepsilon} + \int_{|\tau|+\varepsilon<|\xi|<1}\Big)$$
$$\times H_m(\xi)\Lambda_\xi E(\tau,\xi)\,d\xi \equiv \lim_{\varepsilon\to+0}(J_1^\varepsilon(\tau) + J_2^\varepsilon(\tau)).$$

Proceeding in the same way as in the case of 3^0 we find

$$J_1^\varepsilon(\tau) = |\tau|^{-(2m+n-2)}\frac{m(|\tau|-\varepsilon)^{2m+n}}{(2m+n-2)(2m+n)}H_m(\tau)$$

and

$$J_2^\varepsilon(\tau) = -(m+n-2)\frac{1-(|\tau|+\varepsilon)^2}{2(2m+n-2)}H_m(\tau).$$

From here we derive (3.6.33)

$$\frac{1}{\omega_n}\int_S H_m(\xi)\Lambda_\xi E(\tau,\xi)\,d\xi = \Big(\frac{m|\tau|^2}{2m+n} - (m+n-2)\frac{1-|\tau|^2}{2}\Big)\frac{H_m(\tau)}{2m+n-2}.$$

The lemma is proved. □

Let us return to calculating the integrals I_1 and I_2 from (3.6.28). Using the equality (3.6.30) for $m=0$ we find the inner integral in I_2

$$\frac{1}{\omega_n}\int_{\partial S} E(x,\xi)\,ds_x = \frac{1}{n-2}.$$

That is why

$$I_2 = -\frac{n-1}{(n-2)\omega_n}\int_S f(\tau)\,d\tau\int_{\partial S}(\Lambda_\xi + n-2)E(\xi,\tau)\,ds_\xi.$$

Next, using the equalities (3.6.30) and (3.6.31) for $m=0$ we find

$$\frac{1}{\omega_n}\int_{\partial S}(\Lambda_\xi + n-2)E(\xi,\tau)\,ds_\xi = -1+1 = 0,$$

and that means $I_2 = 0$. Let us calculate the integral I_1. Since this integral has an integrable singularity, using Fubini's theorem we rearrange the order of integration

$$I_1 = -\frac{1}{\omega_n}\int_S \Lambda_\xi(\Lambda_\xi + 1)V[f](\xi)\,d\xi \int_{\partial S} \Lambda_x E(x,\xi)\,ds_x.$$

Using the equality (3.6.31) for $m = 0$ we find

$$\frac{1}{\omega_n}\int_{\partial S} \Lambda_\xi E(\xi,\tau)\,ds_\xi = -\frac{n-2}{n-2} = -1$$

and that means

$$\int_{\partial S} \Lambda^{[3]} V_2[f](x) ds_\xi = I_1 = \int_S \Lambda_\xi(\Lambda_\xi + 1)V[f](\xi)\,d\xi.$$

Using the equality (3.6.24) we get

$$(\Lambda_\xi + 1)V[f](\xi) = -\frac{1}{\omega_n}\int_S E(\xi,\tau)(\Lambda_\tau + 3)f(\tau)\,d\tau + \frac{1}{\omega_n}\int_{\partial S} E(\xi,\tau)f(\tau)\,ds_\tau$$

and therefore

$$\begin{aligned}\int_{\partial S} \Lambda^{[3]} V_2[f](x) ds_\xi = -\frac{1}{\omega_n}\int_S d\xi \int_S \Lambda_\xi E(\xi,\tau)(\Lambda_\tau + 3)f(\tau)\,d\tau \\ + \frac{1}{\omega_n}\int_S \Lambda_\xi\,d\xi \int_{\partial S} E(\xi,\tau)f(\tau)\,ds_\tau.\end{aligned}$$

Again, using Fubini's theorem, we rearrange the order of integration

$$\begin{aligned}\int_{\partial S} \Lambda^{[3]} V_2[f](x) ds_\xi = -\frac{1}{\omega_n}\int_S (\Lambda_\tau + 3)f(\tau)\,d\tau \int_S \Lambda_\xi E(\xi,\tau)\,d\xi \\ + \frac{1}{\omega_n}\int_{\partial S} f(\tau)\,ds_\tau \int_S \Lambda_\xi E(\xi,\tau)\,d\xi.\end{aligned}$$

Using the equality (3.6.33) for $m = 0$ we find

$$\frac{1}{\omega_n}\int_S \Lambda_\xi E(\tau,\xi)\,d\xi = -\frac{1-|\tau|^2}{2}.$$

Noting that $(1-|\tau|^2)\big|_{\partial S} = 0$ we get

$$\int_{\partial S} \Lambda^{[3]} V_2[f](x) ds_\xi = \int_S \frac{1-|\tau|^2}{2}(\Lambda_\tau + 3)f(\tau)\,d\tau.$$

Substituting the calculated value of the integral into the equality (3.6.23) we obtain the condition (3.6.19). When it is satisfied, a solution to the problem (3.6.21)-(3.6.22), and therefore to the problem (3.6.1)-(3.6.2), exists. The theorem is proved. □

Remark 3.23. Using the equality (3.6.26) we can write

$$\int_S (1-|\tau|^2)(\Lambda_\tau+3)f(\tau)\,d\tau = \int_{\partial S} 0\cdot f(\tau)\,ds_\tau - \int_S f(\tau)(\Lambda_\tau+n-3)(1-|\tau|^2)\,d\tau,$$

which means the condition (3.6.19) can be rewritten in the form

$$\int_{\partial S}\varphi_2(x)\,ds_x = \frac{1}{2}\int_S \big((n-1)|\tau|^2-n+3\big)f(\tau)\,d\tau.$$

Remark 3.24. The condition (3.6.19) differs in appearance from the corresponding condition for the solvability of the problem (3.6.1)-(3.6.2), obtained in [58, Theorem 3.2] and is more convenient for use. However, [58] requires less restrictive data smoothness.

Example 3.11. Let in the problem (3.6.1)-(3.6.2) $f(x) = |x|^m h(x)$, where $m \in \mathbb{N}_0$ and $h \in C^2(\bar{S})$ is a function harmonic in S. Then the existence condition (3.6.19) for a solution to the problem is rewritten in the form

$$\frac{1}{2}\int_S (1-|\tau|^2)(\Lambda_\tau+3)f(\tau)\,d\tau = \frac{1}{2}\int_S (1-|\tau|^2)|\tau|^m(\Lambda_\tau+m+3)h(\tau)\,d\tau.$$

If we denote $h_1(x) = \Lambda h(x)$, then using the equality (3.6.34) we can write

$$\frac{1}{2}\int_S (1-|\tau|^2)(\Lambda_\tau+3)f(\tau)\,d\tau$$
$$= \frac{1}{2}\int_0^1 (1-r^2)r^{m+n-1}\,dr\int_{|\xi|=1}(h_1(r\xi)+(m+3)h(r\xi))\,ds_\xi.$$

Since the functions $h(rx)$ and $h_1(rx)$ are harmonic in S for $r\in(0,1)$ and $h_1(0) = 0$, then by the mean value theorem

$$\frac{1}{\omega_n}\int_{|\xi|=1} h_1(r\xi)\,ds_\xi = h_1(0) = 0,\ \frac{1}{\omega_n}\int_{|\xi|=1}(m+3)h(r\xi)\,ds_\xi = (m+3)h(0).$$

Taking into account the found values of the integrals, we write

$$\frac{1}{2}\int_S (1-|\tau|^2)(\Lambda_\tau+3)f(\tau)\,d\tau = \frac{m+3}{2}\int_0^1 (1-r^2)r^{m+n-1}\,dr\,\omega_n h(0).$$

It is easy to calculate that

$$\int_0^1 (1-r^2)r^{m+n-1}\,dr = \frac{1}{m+n} - \frac{1}{m+n+2} = \frac{2}{(m+n)(m+n+2)},$$

which means that for $f(x) = |x|^m u(x)$ the condition for the existence of a solution to the problem (3.6.1)-(3.6.2) has the form

$$\frac{1}{\omega_n}\int_{\partial S} \varphi_2(x)\,ds_x = \frac{m+3}{(m+n)(m+n+2)}\,h(0).$$

In special cases $f(x) = 1$ or $f(x) = x_i$ we have

$$\frac{1}{\omega_n}\int_{\partial S} \varphi_2(x)\,ds_x = \frac{3}{n(n+2)}, \qquad \frac{1}{\omega_n}\int_{\partial S} \varphi_2(x)\,ds_x = 0,$$

respectively.

3.7 Green's function of the Robin problem for the Laplace equation

This section is devoted to the construction of the Green's function of the Robin problem for the Laplace equation. The results of this section are used in next Chapter 4.

The Green's function of the Neumann problem for the Poisson equation in the half-space $\mathbb{R}^n_+$ is explicitly constructed in [41], and the Green's function for the Robin problem in a circle in [22, 107, 156]. In [157, 158] a representation of the Green's function is given for the classical external and internal Neumann problems for the Poisson equation in the unit ball. The representation of the Green's function obtained in this work differs from that found in [157] (see Remark 3.25) and is a natural continuation of the Green's function of the Robin problem from (3.7.17) for the case $\lambda = 0$.

This section is organized as follows. In Lemma 3.36 of Section 3.7.1 a polynomial solution to the Neumann problem is found under zero boundary conditions and a polynomial right-hand side of equation. Then in Theorem 3.29 the resulting solution is written through an integral over S with the kernel as a function $\mathcal{N}(x,\xi)$ from (3.7.7). Theorem 3.30 of section 3.7.2 shows that the found function $\mathcal{N}(x,\xi)$ is the Green (Neumann) function of the Neumann problem and through it an explicit representation of the solution to the Neumann problem is given. Section 3.7.3 studies the Robin problem. In Lemma 3.37 the case of the polynomial right-hand side of the equation is studied. Further, based on it, the case of the right-hand side of general form is investigated. Equality (3.7.17) from Theorem 3.31 gives the Green's function $\mathcal{N}_\lambda(x,\xi)$ of the Robin problem and provides an explicit representation of the solution to this problem in integral form. The main results of Section 3.7 are published in papers [104, 107].

3.7.1 Auxiliary statements

Recall that the Green's function of the Dirichlet problem for the Poisson equation in the unit ball for $n > 2$ has the form

$$G(x,\xi) = E(x,\xi) - E\Big(\frac{x}{|x|}, |x|\xi\Big),$$

where $E(x,\xi) = (n-2)^{-1}|x-\xi|^{2-n}$ is an elementary solution to the Laplace equation (see Section 2.2). We assume throughout this section that $n > 2$.

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$ ($n > 2$) and $f \in C^1(\overline{S})$. Consider the following Neumann problem in the unit ball S

$$\Delta u(x) = f(x),\ x \in S;\quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = 0, \tag{3.7.1}$$

where ν is the outer unit normal to ∂S.

Let first the function $f(x)$ be a polynomial of a special form $f(x) = |x|^{2k}H_s(x)$, where $s, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $H_s(x)$ is some homogeneous harmonic polynomial of degree s and $|x|^2 = x_1^2 + \ldots + x_n^2$.

Lemma 3.36. *Let $f(x) = |x|^{2k}H_s(x)$ and $s > 0$, then the solution to problem* (3.7.1) *has the form*

$$u(x) = \frac{|x|^{2k+2} - (2k+2+s)/s}{(2k+2)(2k+2s+n)} H_s(x) \tag{3.7.2}$$

and this solution is unique up to an arbitrary constant. If $s = 0$, then a solution to problem (3.7.1) *does not exist.*

Proof. It is known that a necessary and sufficient condition for the solvability of the Neumann problem (3.7.1) with a homogeneous condition is the condition $\int_S f(x)\,dx = 0$ (see Remark 3.8 on page 205). Due to the orthogonality of harmonic polynomials $H_s(x)$ of various degrees on the sphere ∂S, for $s > 0$ the equalities hold true

$$\int_S f(x)\,dx = \int_S |x|^{2k} H_s(x)\,dx = \int_0^1 d\rho \int_{|x|=\rho} |x|^{2k} H_s(x)\,ds_x$$
$$= \int_0^1 \rho^{n-1}\,d\rho \int_{|\xi|=1} \rho^{2k} H_s(\rho\xi)\,ds_\xi = \int_0^1 \rho^{2k+s+n-1}\,d\rho \int_{\partial S} 1\cdot H_s(\xi)\,ds_\xi = 0.$$

Therefore, the problem under consideration is solvable for $s > 0$. If $s = 0$, then, in accordance with the calculations done above, we have

$$\int_S f(x)\,dx = \omega_n \int_0^1 \rho^{2k+n-1}\,d\rho = \frac{\omega_n}{2k+n} \neq 0,$$

where ω_n is the area of the unit sphere ∂S, which means the problem (3.7.1) for $s = 0$ is not solvable.

Let us find a solution to the problem (3.7.1). Let us act with the operator $\Lambda + 2$ on both sides of the equation (3.7.1) and use the equality $\Lambda\Delta u = \Delta(\Lambda - 2)u$ (see Theorem 3.12 on page 206). We have

$$\Delta\Lambda u(x) = (\Lambda + 2) f(x), \quad x \in S.$$

Since $\frac{\partial u}{\partial \nu}\big|_{\partial S} = \Lambda u\big|_{\partial S}$, then denoting $v = \Lambda u$ we reduce the problem Neumann (3.7.1) to the next Dirichlet problem

$$\Delta v(x) = (\Lambda + 2) f(x),\ x \in S; \quad v\big|_{\partial S} = 0. \tag{3.7.3}$$

Of course, in the general case, we need to more accurately justify the connection between these two problems, but we simply find the solution to the Dirichlet problem (3.7.3), after that we find the function $u(x)$ from the equation $v = \Lambda u$, and then we check the resulting solution. It is easy to see that, due to the homogeneity of the operator Λ, we have

$$(\Lambda + 2) f(x) = (\Lambda + 2)|x|^{2k} H_s(x) = (2k + s + 2)|x|^{2k} H_s(x).$$

Therefore, if we denote by $w(x)$ the solution to the Dirichlet problem

$$\Delta w(x) = |x|^{2k} H_s(x),\ x \in S; \quad w\big|_{\partial S} = 0,$$

then $v(x) = (2k + s + 2)w(x)$. From Theorem 2.10 on page 127 the solution $w(x)$ to this problem has the form

$$w(x) = \frac{|x|^{2k+2} - 1}{(2k+2)(2k+2s+n)} H_s(x), \tag{3.7.4}$$

which means

$$v(x) = (2k+s+2)w(x) = \frac{(|x|^{2k+2}-1)(2k+s+2)}{(2k+2)(2k+2s+n)}H_s(x).$$

Therefore, to find the function $u(x)$ it is necessary to solve the equation

$$\Lambda u = f_1(x) \equiv \frac{(|x|^{2k+2}-1)(2k+s+2)}{(2k+2)(2k+2s+n)}H_s(x)$$

in functions from $C^1(\overline{S})$. Theorem 3.12 on page 206 establishes that the solution to this equation under the condition $f_1(0)=0$ has the form

$$\begin{aligned} u(x) = \int_0^1 f_1(tx)\frac{dt}{t} + C &= \frac{(2k+s+2)|x|^{2k+2}H_s(x)}{(2k+2)(2k+2s+n)}\int_0^1 t^{2k+s+1}\,dt \\ &\quad - \frac{(2k+s+2)H_s(x)}{(2k+2)(2k+2s+n)}\int_0^1 t^{s-1}\,dt + C \\ &= \frac{|x|^{2k+2}-(2k+2+s)/s}{(2k+2)(2k+2s+n)}H_s(x) + C. \end{aligned} \tag{3.7.5}$$

It is obvious that for $s>0$ the solvability condition $f_1(0)=0$ is satisfied. Let us check the solution (3.7.2). It is not hard to see that

$$\Delta u(x) = \frac{\Delta(|x|^{2k+2}H_s(x))}{(2k+2)(2k+2s+n)} = |x|^{2k}H_s(x) = f(x)$$

and

$$\Lambda u(x) = \frac{(2k+s+2)|x|^{2k+2} - s(2k+s+2)/s}{(2k+2)(2k+2s+n)}H_s(x),$$

whence it follows that $\Lambda u(x)\big|_{\partial S} = 0$. The lemma is proved. □

Let us transform the solution (3.7.2) to another form.

Theorem 3.29. *Let, similarly to Lemma 3.36 $f(x) = |x|^{2m}H_s(x)$ and $s>0$, then the solution* (3.7.2) *of the Neumann problem* (3.7.1) *can be written in the following form*

$$u(x) = -\frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi)f(\xi)\,d\xi, \tag{3.7.6}$$

where the Green's function (or Neumann's function) $\mathcal{N}(x,\xi)$ has the form

$$\mathcal{N}(x,\xi) = E(x,\xi) - E_0(x,\xi), \tag{3.7.7}$$

the function $E_0(x,\xi)$ is written as

$$E_0(x,\xi) = \int_0^1 \Big(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big) + 1\Big)\frac{dt}{t} \tag{3.7.8}$$

and $\hat{E}(x,\xi) = \Lambda_x E(x,\xi)$, where Λ_x denotes the operator Λ on x.

Proof. Let us substitute the function $\mathcal{N}(x,\xi)$ into the equality (3.7.6). Then we get the sum of two integrals

$$u(x) = -\frac{1}{\omega_n}\int_S E(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi + \frac{1}{\omega_n}\int_S E_0(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi \\ \equiv I(x) + J(x).$$

First, let us calculate the first integral $I(x)$. A special case of this integral is calculated in Lemma 3.35 on page 300. Let $\{H_k^{(i)}(x),\ i = 1,\dots,h_k\}$ be a complete system of homogeneous degree $k \in \mathbb{N}_0$ orthogonal harmonic polynomials from Definition 1.5, $\int_{\partial S}(H_k^{(i)}(\xi))^2\,ds_\xi = \omega_n$, $h_k = \frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}$ is the dimension of the basis of homogeneous harmonic polynomials of degree k. Then the homogeneous harmonic polynomial $H_s(x)$ satisfies the equality

$$H_s(x) = \sum_{i=1}^{h_s} H_s^{(i)}(x)\frac{1}{\omega_n}\int_{|\eta|=1} H_s(\eta)H_s^{(i)}(\eta)\,ds_\eta. \tag{3.7.9}$$

Lemma 3.34 on page 299 for $|\xi| < |x|$ proves the representation

$$E(x,\xi) = \sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi), \tag{3.7.10}$$

where series: 1) converges uniformly in ξ for $|\xi| < a < |x|$ and fixed x; 2) converges uniformly in x for $|\xi| < a < |x|$ and fixed ξ. Since the function $|\xi|^{2m}H_s(\xi)E(x,\xi)$ is integrable on S with respect to ξ, then the equality holds true

$$\frac{1}{\omega_n}\int_S E(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi = \frac{1}{\omega_n}\lim_{\varepsilon\to+0}\Big(\int_{|\xi|<|x|-\varepsilon} + \int_{|x|+\varepsilon<|\xi|<1}\Big) \\ \times E(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi \equiv \lim_{\varepsilon\to+0}(I_1^\varepsilon(x) + I_2^\varepsilon(x)).$$

Let $\varepsilon > 0$ be small. Let us calculate the integral $I_1^\varepsilon(x)$. We use the equality (3.7.10) to represent the function $E(x,\xi)$. Taking into account the uniform convergence of the series from (3.7.10) under the integral over $|\xi| < |x| - \varepsilon$ for $|x| < 1$ (the equality is applicable for $a = |x| - \varepsilon$ since $|\xi| < a < |x|$) we can change the order of integration and summation. That is why

$$\begin{aligned} I_1^\varepsilon(x) &= \frac{1}{\omega_n}\int_{|\xi|<|x|-\varepsilon} E(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi \\ &= \frac{1}{\omega_n}\int_{|\xi|<|x|-\varepsilon}\sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k}|\xi|^{2m}H_s(\xi)H_k^{(i)}(\xi)\,d\xi H_k^{(i)}(x) \\ &= \sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)\frac{1}{\omega_n}\int_{|\xi|<|x|-\varepsilon}|\xi|^{2m}H_s(\xi)H_k^{(i)}(\xi)\,d\xi. \end{aligned}$$

Further, similarly to (3.6.34), for the integrable function $f(x)$, the equality holds

$$\int_{a<|\xi|<b} f(\xi)\,d\xi = \int_a^b r^{n-1}\,dr\int_{|\eta|=1} f(r\eta)\,ds_\eta. \tag{3.7.11}$$

Applying this equality, using the completeness of the system $\{H_k^{(i)}(\xi), k \in \mathbb{N}_0\}$ of harmonic polynomials, namely the equality (3.7.9), and orthogonality on ∂S of harmonic polynomials of various degrees, we transform the integral $I_1^\varepsilon(x)$

$$\begin{aligned} I_1^\varepsilon(x) &= \sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)\int_0^{|x|-\varepsilon} r^{n-1}\,dr \\ &\times\frac{1}{\omega_n}\int_{|\eta|=1} r^{2m+s+k}H_s(\eta)H_k^{(i)}(\eta)\,ds_\eta = \frac{|x|^{-(2s+n-2)}(|x|-\varepsilon)^{2m+2s+n}}{(2s+n-2)(2m+2s+n)} \\ &\times\sum_{i=1}^{h_s}H_s^{(i)}(x)\frac{1}{\omega_n}\int_{|\eta|=1}H_s(\eta)H_s^{(i)}(\eta)\,ds_\eta \\ &= |x|^{-(2s+n-2)}\frac{(|x|-\varepsilon)^{2m+2s+n}}{(2s+n-2)(2m+2s+n)}H_s(x). \end{aligned}$$

Now let us calculate the second integral $I_2^\varepsilon(x)$. Again we use (3.7.10) to represent the function $E(x,\xi) = E(\xi,x)$. Due to the uniform convergence of the series under the integral over ξ: $|x|+\varepsilon < |\xi| < 1$ for $|x| < 1$ (the equality (3.7.10) is applicable for $a = |x|+\varepsilon$ since $|x| < a < |\xi|$) we can change the order of integration and summation. That is why

$$
\begin{aligned}
I_2^\varepsilon(x) &= \frac{1}{\omega_n}\int_{|x|+\varepsilon<|\xi|<1} E(\xi,x)|\xi|^{2m}H_s(\xi)\,d\xi \\
&= \frac{1}{\omega_n}\int_{|x|+\varepsilon<|\xi|<1}\sum_{k=0}^{\infty}\frac{|\xi|^{-(2k+n-2)}}{2k+n-2}\sum_{i=1}^{h_k}|\xi|^{2m}H_s(\xi)H_k^{(i)}(\xi)\,d\xi H_k^{(i)}(x) \\
&= \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)\frac{1}{\omega_n}\int_{|x|+\varepsilon<|\xi|<1}|\xi|^{2m-(2k+n-2)}H_s(\xi)H_k^{(i)}(\xi)\,d\xi.
\end{aligned}
$$

Applying the equality (3.7.11), again using (3.7.9) and the orthogonality on ∂S of harmonic polynomials of various degrees, we obtain

$$
\begin{aligned}
I_2^\varepsilon(x) &= \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)\int_{|x|+\varepsilon}^{1} r^{2m-(2k+n-2)}r^{n-1}\,dr \\
&\quad\times\frac{1}{\omega_n}\int_{|\eta|=1} r^{k+s}H_s(\eta)H_k^{(i)}(\eta)\,ds_\eta = \frac{1-(|x|+\varepsilon)^{2m+2}}{(2m+2)(2s+n-2)} \\
&\quad\times\sum_{i=1}^{h_s}H_s^{(i)}(x)\frac{1}{\omega_n}\int_{|\eta|=1}H_s(\eta)H_s^{(i)}(\eta)\,ds_\eta = \frac{1-(|x|+\varepsilon)^{2m+2}}{(2m+2)(2s+n-2)}H_s(x).
\end{aligned}
$$

Thus, taking into account the calculated values of the integrals $I_1^\varepsilon(\tau)$ and $I_2^\varepsilon(\tau)$ we find

$$
\begin{aligned}
\frac{1}{\omega_n}\int_S E(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi &= \lim_{\varepsilon\to+0}(I_1^\varepsilon(x)+I_2^\varepsilon(x)) \\
&= \lim_{\varepsilon\to+0}\Big(|x|^{-(2s+n-2)}\frac{(|x|-\varepsilon)^{2m+2s+n}}{(2s+n-2)(2m+2s+n)}+\frac{1-(|x|+\varepsilon)^{2m+2}}{(2m+2)(2s+n-2)}\Big) \\
&\quad\times H_s(x) = \Big(\frac{|x|^{2m+2}}{(2m+2s+n)}+\frac{1-|x|^{2m+2}}{(2m+2)}\Big)\frac{H_s(x)}{2s+n-2} \\
&= -\frac{|x|^{2m+2}H_s(x)}{(2m+2)(2m+2s+n)}+\frac{H_s(x)}{(2m+2)(2s+n-2)}.
\end{aligned}
$$

Hence

$$
I(x) = \frac{|x|^{2m+2}H_s(x)}{(2m+2)(2m+2s+n)}-\frac{H_s(x)}{(2m+2)(2s+n-2)}. \tag{3.7.12}
$$

The resulting integral generalizes the equality (3.6.32) obtained for $m = 0$. Now let us calculate the second integral

$$
J(x) = \frac{1}{\omega_n}\int_S E_0(x,\xi)|\xi|^{2m}H_s(\xi)\,d\xi.
$$

It is easy to see that according to (3.7.10) for $|\xi| < |x|$ we have

$$\hat{E}(x,\xi) = \Lambda_x \sum_{k=0}^{\infty} \frac{|x|^{-(2k+n-2)}}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi)$$
$$= -\sum_{k=0}^{\infty} \frac{k+n-2}{2k+n-2} |x|^{-(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi).$$

Therefore, since for $x \in S$ and $\xi \in \overline{S}$ we have $\big||x| \cdot \xi\big| = |x||\xi| < 1 = \big|x/|x|\big|$, then the function $\varphi(t) = E\big(\frac{x}{|x|}, t|x|\xi\big)$ for $x \in S$, $\xi \in \overline{S}$ and $t \in [0,1]$ has no singularities and is continuous in $t \in [0,1]$. Then

$$E_0(x,\xi) = \int_0^1 \Big(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big) + 1\Big) \frac{dt}{t}$$
$$= -\int_0^1 \sum_{k=1}^{\infty} \frac{k+n-2}{2k+n-2} t^{k-1} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi)\, dt$$
$$= -\sum_{k=1}^{\infty} \frac{k+n-2}{k(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}(x) H_k^{(i)}(\xi). \quad (3.7.13)$$

Changing the order of integration and summation is legal, since the series converges uniformly in $t \in [0,1]$ for $x \in S$ and $\xi \in \overline{S}$. Moreover, the series in the resulting equality converges uniformly in $\xi \in \overline{S}$ as $x \in S$. This follows easily from Lemma 3.34, similar to the equality (3.7.10). Therefore, in the following equalities it is legal to change the order of integration over $\xi \in S$ and summation over k

$$J(x)$$
$$= \frac{1}{\omega_n} \int_S E_0(x,\xi) |\xi|^{2m} H_s(\xi)\, d\xi = \frac{1}{\omega_n} \int_0^1 \rho^{n-1}\, d\rho \int_{\partial S} E_0(x,\rho\xi) \rho^{2m} H_s(\rho\xi)\, ds_\xi$$
$$= -\int_0^1 d\rho \int_{\partial S} \sum_{k=1}^{\infty} \frac{k+n-2}{k(2k+n-2)} \rho^{2m+s+k+n-1} \frac{1}{\omega_n} \sum_{i=1}^{h_k} H_k^{(i)}(\xi) H_s(\xi) H_k^{(i)}(x)\, ds_\xi$$
$$= -\sum_{k=1}^{\infty} \frac{k+n-2}{k(2k+n-2)} \int_0^1 \rho^{2m+s+k+n-1} d\rho \sum_{i=1}^{h_k} \frac{1}{\omega_n} \int_{\partial S} H_k^{(i)}(\xi) H_s(\xi)\, ds_\xi H_k^{(i)}(x).$$

Using (3.7.9) and the orthogonality on ∂S of harmonic polynomials of various degrees we find

$$J(x) = -\frac{s+n-2}{s(2s+n-2)(2m+2s+n)} \sum_{i=1}^{h_s} H_s^{(i)}(x) \frac{1}{\omega_n} \int_{\partial S} H_s^{(i)}(\xi) H_s(\xi)\, ds_\xi$$
$$= -\frac{1}{2s+n-2} \frac{s+n-2}{s(2m+2s+n)} H_s(x).$$

Thus, the function $u(x)$ from (3.7.6) is a polynomial and has the form

$$u(x) = -\frac{1}{\omega_n} \int_S (E(x,\xi) - E_0(x,\xi))|\xi|^{2m} H_s(\xi)\, d\xi$$
$$= \frac{|x|^{2m+2} H_s(x)}{(2m+2)(2m+2s+n)} - \frac{H_s(x)}{2s+n-2}\left(\frac{1}{2m+2} - \frac{s+n-2}{s(2m+2s+n)}\right).$$

Since

$$\frac{1}{2m+2} - \frac{s+n-2}{s(2m+2s+n)} = \frac{s(2m+2s+n) + (2m+2)(s+n-2)}{s(2m+2)(2m+2s+n)}$$
$$= \frac{s(2m+2) + s(2s+n-2) + (2m+2)(s+n-2)}{s(2m+2)(2m+2s+n)}$$
$$= \frac{(2s+n-2)(2m+s+n)}{s(2m+2)(2m+2s+n)},$$

then

$$u(x) = \frac{|x|^{2m+2} - (2m+s+2)/s}{(2m+2)(2m+2s+n)} H_s(x),$$

which coincides with the equality (3.7.2) for $f(x) = |x|^{2m} H_s(x)$ and $m = k$. The theorem is proved. □

Example 3.12. Solution of the Neumann problem (3.7.1) for $f(x) = x_i|x|^2$, where $1 \le i \le n$, according to the equality (3.7.2) for $k = 1$ and $s = 1$, looks like

$$u(x) = -\frac{1}{\omega_n} \int_S \mathcal{N}(x,\xi)\xi_i|\xi|^2\, d\xi = \frac{|x|^4 - 1 - 4}{4(2+2+n)} x_i = \frac{|x|^4 - 5}{4(n+4)} x_i.$$

3.7.2 Green's function of the Neumann problem

Let us get rid of the polynomial condition for the function $f(x)$ in the representation of the solution (3.7.6) of the Neumann problem (3.7.1).

Theorem 3.30. *Let the function* $f \in C^1(\overline{S})$ *be such that* $\int_S f(\xi)\,d\xi = 0$*, then the solution to the Neumann problem* (3.7.1) *can be represented as* (3.7.6)*, where the function* $\mathcal{N}(x,\xi)$ *is found from* (3.7.7)*, and the function* $E_0(x,\xi)$ *found from* (3.7.8) *is harmonic in* S *by* x *for* $\xi \in \overline{S}$*.*

Proof. Due to the property of the operator Λ, which preserves the harmonicity of the function, the function $\hat{E}(x,\xi) = \Lambda_x E(x,\xi)$ is harmonic in $x,\xi \in S$ for $x \neq \xi$. Further, it is known [28] that since the function $\hat{E}(x,\xi)$ is harmonic in x and ξ for $x \neq \xi$, then the function

$$v(x,\xi) = \frac{1}{|x|^{n-2}}\hat{E}(x/|x|^2,\xi)$$

is also harmonic under the condition $x/|x|^2 \neq \xi$. Since in our case $\xi \in \overline{S}$, then for $x \in S$ we have $x/|x|^2 \in C\overline{S}$ and this means this condition is satisfied. Let us transform the function $v(x,\xi)$. Since

$$\begin{aligned}\hat{E}(x,\xi) = \frac{\Lambda_x\big(|x-\xi|^2\big)^{(2-n)/2}}{n-2} &= \frac{1}{n-2}\sum_{i=1}^n x_i \frac{2-n}{2} 2(x_i-\xi_i)|x-\xi|^{-n} \\ &= -\sum_{i=1}^n (x_i^2 - x_i\xi_i)|x-\xi|^{-n} = -\frac{|x|^2 - x\cdot\xi}{|x-\xi|^n},\end{aligned}$$

then we get

$$\begin{aligned}v(x,\xi) = \frac{1}{|x|^{n-2}}\hat{E}\big(x/|x|^2,\xi\big) &= -\frac{1-x\cdot\xi}{|x|^{n-2}|x|^2}\frac{1}{\big|x/|x|^2-\xi\big|^n} \\ &= -\frac{1-x\cdot\xi}{\big|x/|x|-|x|\xi\big|^n} = \hat{E}\big(x/|x|,|x|\xi\big).\end{aligned}$$

In addition, it is easy to see that $\hat{E}\big(x/|x|,t|x|\xi\big)\big|_{t=0} = -1$ and therefore since the function $\hat{E}\big(x/|x|,t|x|\xi\big)$ is differentiable with respect to $t \in [0,1]$, then the integrand function in the integral

$$E_0(x,\xi) = \int_0^1 \Big(\hat{E}\Big(\frac{x}{|x|},t|x|\xi\Big)+1\Big)\frac{dt}{t}$$

has no singularity at $t=0$. Therefore, the function $E_0(x,\xi)$ is defined for $x \in S$ and $\xi \in \overline{S}$. Moreover, since the function $\hat{E}\big(x/|x|,t|x|\xi\big)$ is harmonic in $x,\xi \in S$, then the function $E_0(x,\xi)$ has the same property. So the function

$$J(x) = \frac{1}{\omega_n}\int_S E_0(x,\xi)f(\xi)\,d\xi,$$

is harmonic in S. Therefore, for $f \in C^1(\overline{S})$, based on the properties of the volume potential, the function $u(x)$ satisfies the equation (3.7.1)

$$\Delta u(x) = \Delta\Big(-\frac{1}{\omega_n}\int_S E(x,\xi) f(\xi)\,d\xi\Big) + \Delta J(x) = f(x).$$

Let us check the boundary conditions of the Neumann problem. It is not hard to see that

$$\Lambda_x \hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big) = \Lambda_x \hat{E}\Big(\frac{tx}{|tx|}, |tx|\xi\Big) = t\Big(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big)\Big)'_t.$$

Therefore, taking into account the calculations performed above, we have

$$\begin{aligned}\Lambda_x E_0(x,\xi) = \int_0^1 \Lambda_x\Big(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big) + 1\Big)\frac{dt}{t} = \int_0^1 \Big(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big)\Big)'_t\,dt \\ = \hat{E}\Big(\frac{x}{|x|}, |x|\xi\Big) - \hat{E}\Big(\frac{x}{|x|}, 0\Big) = \hat{E}\Big(\frac{x}{|x|}, |x|\xi\Big) + 1.\end{aligned}$$

Using these calculations we find

$$\Lambda_x \mathcal{N}(x,\xi)\Big|_{x\in\partial S} = \Big(\Lambda_x E(x,\xi) - \hat{E}\Big(\frac{x}{|x|}, |x|\xi\Big) - 1\Big)\Big|_{|x|=1} = -1.$$

Thus we obtain

$$\begin{aligned}\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \Lambda u(x)\Big|_{\partial S} = \Lambda_x\Big(-\frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi) f(\xi)\,d\xi\Big)\Big|_{\partial S} \\ = -\frac{1}{\omega_n}\int_S \Lambda_x \mathcal{N}(x,\xi)\Big|_{\partial S} f(\xi)\,d\xi = \frac{1}{\omega_n}\int_S f(\xi)\,d\xi = 0.\end{aligned}$$

Entering the operator Λ_x under the integral sign is legal due to the properties of potential type integrals [183, p. 25] since the singularity of the function $\Lambda_x E(x,\xi)$ for $x = \xi$ is of order $n-1$ and therefore it is integrable with respect to $\xi \in \overline{S}$. The passage to the limit with respect to $x \to \partial S$ can be entered under the integral sign since an integral of potential type with a singularity of type $\Lambda_x E(x,\xi)$ is a continuous function with respect to $x \in \overline{S}$. Neumann's condition is met. The theorem is proved. □

Remark 3.25. In [157] another representation of the Green's function of the Neumann problem in the unit ball is found in the form

$$N(x,y) = \frac{1}{\omega_n}\big(E(x,y) + E(x|y|, y/|y|) + E_1(x,y)\big),$$

where

$$E_1(x,y) = \int_0^1 \left(\left|sx|y| - y/|y|\right|^{2-n} - 1\right)\frac{ds}{s},$$

which differs from that obtained in the equality (3.7.7).

Remark 3.26. Let $P_m(x)$ be a homogeneous polynomial of degree m, then by virtue of Theorem 3.29 the equality holds

$$-\frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi)P_m(\xi)\,d\xi = Q_{m+2}(\xi),$$

where $Q_{m+2}(x)$ is some polynomial of degree $m+2$. If, in accordance with the Almansi formula (2.1.5), we write the polynomial $P_m(x)$ in the form

$$P_m(x) = \sum_{k=0}^{m-2k\ge 0} |x|^{2k} H_{m-2k}(x),$$

where $H_{m-2k}(x)$ are homogeneous harmonic polynomials of degree $m-2k \ge 0$ (the formula for calculating them depending on $P_m(x)$ can be found in Lemma 1.11 on page 52), then by virtue of the equality (3.7.2) we get

$$Q_{m+2}(x) = \sum_{k=0}^{m-2k\ge 0} \frac{|x|^{2k+2} - (m+2)/(m-2k)}{(2k+2)(2m-2k+n)} H_{m-2k}(x).$$

3.7.3 Green's function of the Robin problem

Now consider the Robin problem for the Laplace equation

$$\Delta u(x) = f(x),\ x\in S;\quad \left(\frac{\partial u}{\partial \nu} + \lambda u\right)\Big|_{\partial S} = 0, \tag{3.7.14}$$

where $\lambda \in \mathbb{R}$ and $n > 2$.

First, let the function $f(x)$ be a polynomial in problem (3.7.14). We construct a solution to such a problem.

Lemma 3.37. *Let $f(x) = |x|^{2k}H_s(x)$, then a solution to problem* (3.7.14) *exists if and only if $\lambda + s \ne 0$ and this solution has the form*

$$u(x) = u_d(x) - \frac{H_s(x)}{(\lambda+s)(2k+2s+n)}, \tag{3.7.15}$$

where $u_d(x)$ *is the solution to the Dirichlet problem with zero boundary conditions from* (3.7.4). *If* $-\lambda \notin \mathbb{N}_0$*, then the solution* (3.7.15) *is unique, and if* $-\lambda \in \mathbb{N}_0$*, then the solution* (3.7.15) *is unique up to arbitrary homogeneous harmonic polynomials of degree* $-\lambda$*. If* $\lambda + s = 0$*, then a solution to problem* (3.7.14) *does not exist.*

Proof. By virtue of Theorem 1.1 on page 15 a solution to the problem (3.7.14) exists if and only if the condition is satisfied

$$-\lambda \in \mathbb{N}_0 \Rightarrow \forall \hat{H}_{-\lambda}(x) \qquad \int_S \hat{H}_{-\lambda}(\xi)|\xi|^{2k} H_s(\xi)\, d\xi = 0,$$

where $\hat{H}_{-\lambda}(x)$ is a homogeneous harmonic polynomial of degree $-\lambda$ and the solution is unique up to arbitrary homogeneous harmonic polynomials of degree $-\lambda \in \mathbb{N}_0$. If $-\lambda \notin \mathbb{N}_0$, then there are no additional conditions for the existence of solutions to the problem (3.7.14).

It is easy to see that for $-\lambda \in \mathbb{N}_0$ according to (3.7.11)

$$\int_S \hat{H}_{-\lambda}(\xi)|\xi|^{2k} H_s(\xi)\, d\xi = \frac{1}{2k+s+n-\lambda} \int_{|\eta|=1} \hat{H}_{-\lambda}(\eta) H_s(\eta)\, ds_\eta.$$

This expression for $-\lambda \neq s$, due to the orthogonality of homogeneous harmonic polynomials of various degrees, is equal to zero, i.e. the condition for the solvability of the problem is satisfied. The solution to the problem is unique up to arbitrary homogeneous harmonic polynomials of degree $-\lambda \in \mathbb{N}_0$. If $-\lambda = s$, then the integral on the right, when choosing $\hat{H}_{-\lambda}(x) = H_s(x)$, does not vanish and therefore a solution to the problem (3.7.14) does not exist.

In Lemma 3.36 a method is proposed for finding a solution to the Neumann problem for polynomial $f(x)$, but we skip a similar step for the problem (3.7.14) and only check that the equality (3.7.15) specifies the solution to the problem (3.7.14). It is not hard to see that

$$\Delta u(x) = \Delta u_d(x) = |x|^{2k} H_s(x)$$

and keeping in mind (3.7.4) we find

$$\begin{aligned}(\Lambda + \lambda) u_d(x)\big|_{\partial S} &= \frac{(2k+s+\lambda+2)|x|^{2k+2} - (s+\lambda)}{(2k+2)(2k+2s+n)} H_s(x)\big|_{\partial S} \\ &= \frac{1}{2k+2s+n} H_s(x)\big|_{\partial S},\end{aligned}$$

and therefore on ∂S

$$(\Lambda+\lambda)u(x)=\frac{H_s(x)}{2k+2s+n}-\frac{(\lambda+s)H_s(x)}{(\lambda+s)(2k+2s+n)}=0.$$

The solution (3.7.15) is defined for $\lambda+s\neq 0$. The lemma is proved. □

Let us get rid of the polynomial condition for the function $f(x)$ in problem (3.7.14). Let us present some arguments that clarify the origin of the function $E_\lambda(x,\xi)$ from (3.7.18). Note that the equality (3.7.15) can be rewritten in a form similar to (3.7.2) from Lemma 3.36

$$u(x)=\frac{|x|^{2k+2}-(2k+2+s+\lambda)/(s+\lambda)}{(2k+2)(2k+2s+n)}H_s(x).$$

From the equality (3.7.5) it is clear that in the numerator of the fraction from (3.7.2) the denominator s arises due to the integral $\int_0^1\cdot\, t^{-1}dt$. If now instead of this integral we take the integral $\int_0^1\cdot\, t^{\lambda-1}dt$, then the denominator $s+\lambda$ will probably appear in the numerator from (3.7.5). The following statement is true.

Theorem 3.31. *Let $f\in C^1(\overline{S})$ and $\lambda>0$. Then the solution to problem* (3.7.14) *can be represented in the form*

$$u(x)=-\frac{1}{\omega_n}\int_S \mathcal{N}_\lambda(x,\xi)f(\xi)\,d\xi, \tag{3.7.16}$$

where the Green's function $\mathcal{N}_\lambda(x,\xi)$ has the form

$$\mathcal{N}_\lambda(x,\xi)=E(x,\xi)-E_\lambda(x,\xi), \tag{3.7.17}$$

the function $E_\lambda(x,\xi)$ is written as

$$E_\lambda(x,\xi)=\int_0^1\hat{E}_\lambda\Big(\frac{x}{|x|},t|x|\xi\Big)t^{\lambda-1}\,dt \tag{3.7.18}$$

and $\hat{E}_\lambda(x,\xi)=(\Lambda_x+\lambda)E(x,\xi)$.

Proof. It is easy to see that, by analogy with the function $\hat{E}_0(x,\xi)$, the function $\hat{E}_\lambda(x,\xi)$ is harmonic in $x,\xi\in S$ for $x\, ne\xi$, which means the function $\hat{E}_\lambda\big(\frac{x}{|x|},t|x|\xi\big)$ is defined for $x\in S$, $\xi\in\overline{S}$ and the function $\hat{E}_\lambda\big(\frac{x}{|x|},t|x|\xi\big)t^{\lambda-1}$ is integrable with respect to $t\in[0,1]$ for $\lambda>0$. Moreover, similar to the proof of Theorem 3.30, it is easy to verify that the function $\hat{E}_\lambda\big(\frac{x}{|x|}, t|x|\xi\big)$, and therefore $E_\lambda(x,\xi)$ are harmonic for $x \in S$ and $\xi \in \overline{S}$. Therefore the function

$$J_\lambda(x) = \frac{1}{\omega_n}\int_S E_\lambda(x,\xi) f(\xi)\, d\xi,$$

is harmonic in S. Therefore, for $f \in C^1(\overline{S})$, based on the properties of the volume potential with density $f(x)$, the function $u(x)$ from (3.7.16) satisfies the equation (3.7.14)

$$\Delta u(x) = \Delta\Big(-\frac{1}{\omega_n}\int_S E(x,\xi) f(\xi)\, d\xi\Big) + \Delta J_\lambda(x) = f(x).$$

Let us check the boundary conditions of the problem (3.7.14). It is not hard to see that

$$\Lambda_x \hat{E}_\lambda\Big(\frac{x}{|x|}, t|x|\xi\Big) = \Lambda_x \hat{E}_\lambda\Big(\frac{tx}{|tx|}, |tx|\xi\Big) = t\Big(\hat{E}_\lambda\Big(\frac{x}{|x|}, t|x|\xi\Big)\Big)'_t.$$

Therefore, for $\lambda > 0$ we find

$$\begin{aligned}\Lambda_x E_\lambda(x,\xi) &= \int_0^1 \Lambda_x \hat{E}_\lambda\Big(\frac{x}{|x|}, t|x|\xi\Big) t^{\lambda-1}\, dt = \int_0^1 \Big(\hat{E}_\lambda\Big(\frac{x}{|x|}, t|x|\xi\Big)\Big)'_t t^\lambda\, dt \\ &= t^\lambda \hat{E}_\lambda\Big(\frac{x}{|x|}, t|x|\xi\Big)\Big|_0^1 - \lambda E_\lambda(x,\xi) = \hat{E}_\lambda\Big(\frac{x}{|x|}, |x|\xi\Big) - \lambda E_\lambda(x,\xi),\end{aligned}$$

which means

$$(\Lambda_x + \lambda) E_\lambda(x,\xi) = \hat{E}_\lambda\Big(\frac{x}{|x|}, |x|\xi\Big).$$

It should be noted that above, for example in (3.5.18) and Lemma 3.23, it is shown that the integral operator used above is the inverse of the operator $\Lambda + \lambda$ in the class of harmonic functions. Using the resulting formula, we write

$$(\Lambda_x + \lambda)\mathcal{N}_\lambda(x,\xi)\big|_{x\in\partial S} = \Big((\Lambda_x + \lambda)E(x,\xi) - \hat{E}(x,\xi)\Big)\Big|_{|x|=1} = 0.$$

Thus we have

$$\frac{\partial u}{\partial \nu} + \lambda u\Big|_{\partial S} = (\Lambda + \lambda)u(x)\Big|_{\partial S} = \frac{1}{\omega_n}\int_S (\Lambda_x + \lambda)\mathcal{N}_\lambda(x,\xi) f(\xi)\, d\xi\Big|_{\partial S} = 0.$$

Entering the operator Λ_x under the integral sign is legal due to the properties of potential type integrals [183] since the singularity of the function $\Lambda_x E(x,\xi)$ for $x = \xi$ is of order $n-1$ and therefore it is integrable over $\xi \in \overline{S}$. The passage to the limit with respect to $x \to \partial S$ can be entered under the integral sign, since a potential-type integral with the singularity $\Lambda_x E(x, \xi)$ is a continuous function with respect to $x \in \overline{S}$. The conditions of the problem (3.7.14) are met. The theorem is proved. $\square$

Remark 3.27. The case $\lambda = 0$ is not covered by Theorem 3.31 and therefore Theorem 3.30 complements Theorem 3.31. The equality (3.7.7) is a natural continuation of the equality (3.7.17) for the case of $\lambda = 0$.

Chapter 4

Navier and Riquier-Neumann boundary value problems

This chapter examines the issues of finding solutions to the Navier [146, 166] and Riquier-Neumann [101] boundary value problems for the polyharmonic equation in the unit ball. In Theorem 1.4 from Section 1.2, based on the system of normalized functions for the Laplace operator, the solvability of the Navier problem for the polyharmonic equation is already investigated and a solution method is even proposed, but it is a recurrent method, which is not always convenient for specific tasks. As a motivation for this topic, we note that in Theorem 2.8 from Section 2.3 the Green's function of the Dirichlet problem is constructed, and in Theorem 3.7 from section 3.2 an integral representation of the solution to the Neumann problem is given. In addition, in Section 3.7 the Green's function $\mathcal{N}(x,\xi)$ of the Neumann problem for the Poisson equation is found. Therefore, it is natural to want, using the results obtained above, to construct Green's functions also for other problems for the polyharmonic equation in the unit ball, namely the Navier and Riquier-Neumann boundary value problems.

Chapter 4 is organized as follows. In Section 4.1 the Navier and Riquier-Neumann problems for the biharmonic equation in a ball are studied. Theorem 4.1 gives an integral representation of functions from the class $u \in C^4(D) \cap C^3(\bar{D})$, then the Green's function of the Navier problem is defined in Theorem 4.2 and in the Theorem 4.3 an integral representation of the solution to this problem is given. Theorem 4.6 defines the Green's function of the Riquier-Neumann problem, and Theorem 4.7 finds the integral representation of the solution to this problem through the Green's function. Section 4.1.4 provides two examples of finding solutions to the considered homogeneous problems with a polynomial right-hand side of the biharmonic equation. Theorem 4.5 provides a formula for finding a solution to the Neumann problem for the Poisson equation with polynomial data.

Section 4.2 gives an integral representation of functions from $C^{2m}(D)\cap C^{2m-1}(\bar{D})$. First, an elementary solution $E_{2m}(x,\xi)$ of the polyharmonic equation is determined, which generalizes the elementary solutions $E_4(x,\xi)$ from (4.1.2) given in the previous Section 4.1 and $E(x,\xi)$ for the m-harmonic case. Lemmas 4.1 and 4.2 prove some properties of the function $E_{2m}(x,\xi)$. Theorem 4.8 using $E_{2m}(x,\xi)$ establishes the integral representation of functions from the class $u \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$ in a bounded domain $D \subset \mathbb{R}^n$ with a smooth boundary, which generalizes the equality (4.1.3). Example 4.1 is given.

In Section 4.3 necessary and sufficient conditions for the solvability of the Riquier-Neumann problem for an inhomogeneous polyharmonic equation in the unit ball are obtained. First, in Theorem 4.9, using Lemmas 4.3 and 4.4, the solvability of the Riquier-Neumann problem for a homogeneous polyharmonic equation is investigated. Operator formulas are given that allow one to find a solution to the Riquier-Neumann problem. Further, in Theorem 4.10, based on Theorem 4.9, necessary and sufficient conditions for the solvability of the Riquier-Neumann problem for an inhomogeneous polyharmonic equation in the unit ball are obtained.

Section 4.4 considers the Navier problem (in [146] it is called the Riquier problem) for the polyharmonic equation in the unit ball. Based on the results of Section 4.1, the definition of the Green's function $G^r_{2m}(x,\xi)$ of the Navier problem is given. Then, in Lemma 4.5, the property of the auxiliary function $E^r_{2k}(x,\xi)$ is proved. After this, in Theorem 4.11, a recurrent equality for the Green's function $G^r_{2m}(x,\xi)$ is established. Finally, in Theorem 4.12, an explicit formula for representing the solution to the Navier problem is derived. In Example 4.2 the integral with the kernel of the Green's functions $G^r_{2m}(x,\xi)$ for $m = 1, 2, 3$ of a homogeneous harmonic polynomial is calculated and the simple Navier problem for the 3-harmonic equation is solved.

Section 4.5 presents a solution to the Riquier-Neumann problem for the polyharmonic equation in the unit ball. Section 4.5.2 gives Definition 4.5 of the Green's function $\mathcal{N}_{2m}(x,\xi)$ of the Riquier-Neumann problem. Theorem 4.13 proves a recurrent formula for a certain Green's function. Theorem 4.14 from Section 4.5.3 contains an integral representation of the solution to the Riquier-Neumann problem. Theorem 4.15 proves that the function found in Theorem 4.14 is indeed a solution to the Riquier-Neumann problem in the unit ball. Finally, Theorem 4.16 from Section 4.5.4 considers a special case of the Riquier-Neumann problem for the homogeneous polyharmonic equation and gives Example 4.4 of solving the Riquier-Neumann problem with simple boundary data.

4.1 Green's functions of the Navier and Riquier-Neumann problems for the biharmonic equation

In this section, prefacing the results obtained for the polyharmonic equation, we study the Navier and Riquier-Neumann problems for the biharmonic equation in the unit ball. First, Theorem 4.1 gives an integral representation of functions from the class $u \in C^4(D) \cap C^3(\bar{D})$. Then Theorem 4.3 defines the Green's function of the Navier problem and Theorem 4.3 gives an integral representation of the solution to this problem. Next, we study the Riquier-Neumann problem. Theorem 4.6 defines the Green's function of the Riquier-Neumann problem, and Theorem 4.7 finds the integral representation of the solution to this problem through the Green's function. Two examples of solving the considered homogeneous problems with a polynomial right-hand side of the equation are given in Section 4.1.4. Theorem 4.5 considers the Neumann problem for the Poisson equation with polynomial data. The results of Section 4.1 are published in papers [117, 119].

We present several formulas from Section 2.2 so as not to send the reader too far back in the text. The Green's function of the Dirichlet problem for the Poisson equation in the ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$ is be denoted as

$$G_2(x,\xi) = E(x,\xi) - E\Big(\frac{x}{|x|}, |x|\xi\Big), \tag{4.1.1}$$

where $E(x,\xi)$ is an elementary solution to Laplace equation. We write the elementary solution of the biharmonic equation (2.2.7) in the form

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n > 4,\ n = 3 \\ -\dfrac{1}{4}\ln|x-\xi|, & n = 4 \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi| - 1\big), & n = 2 \end{cases}. \tag{4.1.2}$$

4.1.1 Integral representation of functions

Let us present the integral representation of a function from $u \in C^4(D) \cap C^3(\bar{D})$, where $D \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary ∂D.

Theorem 4.1. *The function $u \in C^4(D) \cap C^3(\bar{D})$ has the following integral representation*

$$u(x) = \frac{1}{\omega_n}\Big(- \int_{\partial D} E_4(x,\xi)\frac{\partial \Delta u}{\partial \nu}\, ds_\xi + \int_{\partial D} \frac{\partial E_4(x,\xi)}{\partial \nu}\Delta u\, ds_\xi \\ - \int_{\partial D} \Delta_\xi E_4(x,\xi)\frac{\partial u}{\partial \nu}\, ds_\xi + \int_{\partial D} \frac{\partial \Delta_\xi E_4(x,\xi)}{\partial \nu} u(\xi)\, ds_\xi \\ + \int_D E_4(x,\xi)\Delta^2 u(\xi)\, d\xi\Big), \quad (4.1.3)$$

where $\omega_n = |\partial S|$ is the area of the unit sphere in $\mathbb{R}^n$ and ν is the outer unit normal to ∂D.

Proof. Consider the functions $u, v \in C^2(D) \cap C^1(\bar{D})$. It is easy to see that for them in the domain D the identity holds

$$v\Delta u - u\Delta v = \operatorname{div}(v\nabla u) - \operatorname{div}(u\nabla v).$$

Let us integrate this identity over the domain D and use the divergence theorem [28]

$$\int_D (v\Delta u - u\Delta v)\, d\xi = \int_{\partial D} \Big(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big)\, ds_\xi. \quad (4.1.4)$$

Substituting $u = \Delta u$, as well as $v = \Delta v$, we get two equalities

$$\int_D (v\Delta^2 u - \Delta u\Delta v)\, d\xi = \int_{\partial D} \Big(v\frac{\partial \Delta u}{\partial \nu} - \Delta u\frac{\partial v}{\partial \nu}\Big)\, ds_\xi,$$
$$\int_D (\Delta v\Delta u - u\Delta^2 v)\, d\xi = \int_{\partial D} \Big(\Delta v\frac{\partial u}{\partial \nu} - u\frac{\partial \Delta v}{\partial \nu}\Big)\, ds_\xi.$$

Adding these equalities we obtain

$$\int_D (v\Delta^2 u - u\Delta^2 v)\, d\xi = \int_{\partial D} \Big(v\frac{\partial \Delta u}{\partial \nu} - \Delta u\frac{\partial v}{\partial \nu} + \Delta v\frac{\partial u}{\partial \nu} - u\frac{\partial \Delta v}{\partial \nu}\Big)\, ds_\xi. \quad (4.1.5)$$

Obviously, the resulting formula is correct for $u, v \in C^4(D) \cap C^3(\bar{D})$. Let us select a point $x \in D$ from the domain D together with a closed ball $|\xi - x| \le \delta$ of such radius $\delta > 0$ that this ball belongs to the domain D. We denote the remaining part of the domain D by D_δ. It is obvious that $\partial D_\delta = \partial D \cup \{\xi \in \mathbb{R}^n : |\xi - x| = \delta\}$. Let us apply the equality (4.1.5) for the domain $D = D_\delta$ in the case when $v(\xi) = E_4(x,\xi)$ and take into account that $\Delta_\xi^2 E_4(x,\xi) = 0$ and $\Delta_\xi E_4(x,\xi) = -E(x,\xi)$ for $x \ne \xi \in D$

Obviously, the resulting formula is correct for $u, v \in C^4(D) \cap C^3(\bar{D})$. Let us select a point $x \in D$ from the domain D together with a closed ball $|\xi - x| \le \delta$ of such radius

$\delta > 0$ that this ball belongs to the domain D. We denote the remaining part of the domain D by D_δ. It is obvious that $\partial D_\delta = \partial D \cup \{\xi \in \mathbb{R}^n : |\xi - x| = \delta\}$. Let us apply the equality (4.1.5) for the domain $D = D_\delta$ in the case when $v(\xi) = E_4(x,\xi)$ and take into account that $\Delta_\xi^2 E_4(x,\xi) = 0$ and $\Delta_\xi E_4(x,\xi) = -E(x,\xi)$ for $x \neq \xi \in D$

$$\int_{D_\delta} E_4(x,\xi)\Delta^2 u(\xi)\,d\xi$$
$$= \int_{\partial D} \Big(E_4(x,\xi)\frac{\partial \Delta u}{\partial \nu} - \Delta u \frac{\partial E_4(x,\xi)}{\partial \nu} - E(x,\xi)\frac{\partial u}{\partial \nu} + u\frac{\partial E(x,\xi)}{\partial \nu}\Big)\,ds_\xi$$
$$- \int_{|\xi - x| = \delta} \Big(E_4(x,\xi)\frac{\partial \Delta u}{\partial \nu} - \Delta u \frac{\partial E_4(x,\xi)}{\partial \nu} - E(x,\xi)\frac{\partial u}{\partial \nu} + u\frac{\partial E(x,\xi)}{\partial \nu}\Big)\,ds_\xi. \quad (4.1.6)$$

The minus sign arose due to the fact that the internal normal to part of the boundary of the domain D_δ – the sphere $|\xi - x| = \delta$ in the divergence theorem is replaced by an external normal. Let us calculate the second integral on the right in the resulting formula, which we divide into four terms I_1^δ, I_2^δ, I_4^δ, I_4^δ in order.

I. Let $n > 4$, or $n = 3$. By virtue of the equality (4.1.2) $E_4(x,\xi)|_{|x-\xi|=\delta} = \delta^{4-n}/(2(n-2)(n-4))$ and that means

$$|I_1^\delta| \le \int_{|\xi - x| = \delta} |E_4(x,\xi)|\Big|\frac{\partial \Delta u}{\partial \nu}\Big|\,ds_\xi \le \frac{\delta^{4-n}\max_{\xi\in\bar D}\|\nabla \Delta u\|}{2(n-2)|n-4|}\omega_n \delta^{n-1} \le \mathrm{const}\,\delta^3.$$

If $n = 4$, then $E_4(x,\xi)|_{|x-\xi|=\delta} = -\frac{1}{4}\ln\delta$ and then similarly to the previous

$$|I_1^\delta| \le \frac{|\ln\delta|\max_{\xi\in\bar D}\|\nabla\Delta u\|}{4}\omega_4\delta^3 \le \mathrm{const}\,|\ln\delta|\delta^3.$$

If $n = 2$, then $E_4(x,\xi)|_{|x-\xi|=\delta} = \delta^2|\ln\delta - 1|/4$ and then

$$|I_1^\delta| \le \frac{\delta^2|\ln\delta - 1|\max_{\xi\in\bar D}\|\nabla\Delta u\|}{4}\omega_2\delta \le \mathrm{const}\,|\ln\delta - 1|\delta^3.$$

II. Consider the next integral I_2^δ for $n > 4$, or $n = 3$. Taking into account

$$\frac{\partial E_4(x,\xi)}{\partial \nu}\Big|_{|x-\xi|=\delta} = -\frac{1}{2(n-2)}\sum_{i=1}^n \frac{\xi_i - x_i}{|x-\xi|}\frac{\xi_i - x_i}{|x-\xi|^{n-2}} = -\frac{\delta^{3-n}}{2(n-2)}$$

we have

$$|I_2^\delta| \le \int_{|\xi - x| = \delta}\Big|\frac{\partial E_4(x,\xi)}{\partial\nu}\Big||\Delta u|\,ds_\xi \le \frac{\delta^{3-n}\max_{\xi\in\bar D}|\Delta u|}{2(n-2)}\omega_n\delta^{n-1} \le \mathrm{const}\,\delta^2.$$

If $n = 4$, then

$$\frac{\partial E_4(x,\xi)}{\partial \nu}\Big|_{|x-\xi|=\delta} = -\frac{1}{4}\sum_{i=1}^{n}\frac{\xi_i - x_i}{|x-\xi|}\frac{\xi_i - x_i}{|x-\xi|^2} = -\frac{1}{4\delta},$$

which coincides with the previous case for $n = 4$ and therefore we obtain a similar estimate $|I_2^\delta| \le \operatorname{const}\delta^2$. If $n = 2$, then

$$\begin{aligned}\frac{\partial E_4(x,\xi)}{\partial \nu}\Big|_{|x-\xi|=\delta} &= \frac{1}{4}\sum_{i=1}^{n} 2\frac{\xi_i - x_i}{|x-\xi|}(\xi_i - x_i)(\ln|x-\xi| - 1) + \frac{1}{4}|x-\xi| \\ &= \frac{|x-\xi|}{4}(2\ln|x-\xi| - 1) = \frac{\delta}{4}(2\ln\delta - 1)\end{aligned}$$

and then

$$|I_2^\delta| \le \frac{\delta|2\ln\delta - 1|\max_{\xi\in\bar{D}}|\Delta u|}{4}\omega_2\delta \le \operatorname{const}\delta^2|2\ln\delta - 1|.$$

III. Consider the next integral I_3^δ for $n > 2$. Taking into account the following equality $E(x,\xi)|_{|x-\xi|=\delta} = \delta^{2-n}/(n-2)$ we have

$$|I_3^\delta| \le \int_{|\xi-x|=\delta}|E(x,\xi)|\left|\frac{\partial u}{\partial \nu}\right| ds_\xi \le \frac{\delta^{2-n}\max_{\xi\in\bar{D}}\|\nabla u\|}{n-2}\omega_n\delta^{n-1} \le \operatorname{const}\delta.$$

If $n = 2$, then since $E(x,\xi)|_{|x-\xi|=\delta} = -\ln\delta$, we get

$$|I_3^\delta| \le |\ln\delta|\max_{\xi\in\bar{D}}\|\nabla u\|\omega_2\delta \le \operatorname{const}\delta|\ln\delta|.$$

In all cases considered, $I_k^\delta \underset{x\in D}{\rightrightarrows} 0$ for $\delta \to 0$, $k = 1, 2, 3$.

IV. Let us examine the last integral I_4^δ. Since for $n \ge 2$

$$\frac{\partial E(x,\xi)}{\partial \nu}\Big|_{|x-\xi|=\delta} = -\sum_{i=1}^{n}\frac{\xi_i - x_i}{|\xi - x|}\frac{\xi_i - x_i}{|\xi - x|^n} = -\delta^{1-n},$$

then we have

$$\begin{aligned}\left|I_4^\delta - u(x)\int_{|\xi-x|=\delta}\frac{\partial E(x,\xi)}{\partial \nu}ds_\xi\right| &\le \int_{|\xi-x|=\delta}|u(\xi) - u(x)|\left|\frac{\partial E(x,\xi)}{\partial \nu}\right| ds_\xi \\ &\le \delta^{1-n}\max_{|\xi-x|\le\delta}|u(\xi) - u(x)|\omega_n\delta^{n-1} = \omega_f(\delta)\omega_n,\end{aligned}$$

where $\omega_f(\delta)$ is the modulus of continuity of the function f. By Cantor's theorem $\omega_f(\delta) \to 0$ for $\delta \to 0$, which means

$$I_4^\delta = \left(I_4^\delta - u(x)\int_{|\xi-x|=\delta} \frac{\partial E(x,\xi)}{\partial \nu}\, ds_\xi\right) + u(x)\frac{\omega_n \delta^{n-1}}{\delta^{n-1}} \to \omega_n u(x),\ \delta \to 0.$$

Let us passing to the limit at $\delta \to 0$ in the equality (4.1.6). Due to the integrable singularity in the volume integral on the left, at which

$$\lim_{\delta\to 0}\int_{D_\delta} E_4(x,\xi) f(\xi)\, d\xi = \int_D E_4(x,\xi) f(\xi)\, d\xi$$

and taking into account the found limits of the integrals I_k^δ for $k = 1,2,3,4$ we obtain

$$\int_D E_4(x,\xi)\Delta^2 u(\xi)\, d\xi = \int_{\partial D}\Big(E_4(x,\xi)\frac{\partial \Delta u}{\partial \nu} - \frac{\partial E_4(x,\xi)}{\partial \nu}\Delta u \\ - E(x,\xi)\frac{\partial u}{\partial \nu} + \frac{\partial E(x,\xi)}{\partial \nu} u(\xi)\Big)\, ds_\xi + \omega_n u(x).$$

Transferring all the integrals from the right side of the formula to the left, dividing by ω_n and taking into account the equality $\Delta_\xi E_4(x,\xi) = -E(x,\xi)$ we obtain the representation (4.1.3) to be proved. □

Remark 4.1. The formula (4.1.3) for $n \geq 3$ remains valid if we replace the elementary solution $E_4(x,\xi)$ with the fundamental solution $\mathcal{E}_4(x,\xi)$ from (2.3.7) because, as noted in Remark 2.6, these functions are the same.

4.1.2 Green's function of the Navier boundary value problem

The problems below are considered in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$. The Navier problem (in [20] it is also called the Dirichlet-2 problem) consists of finding the function $u \in C^4(S) \cap C^3(\bar{S})$, which is a solution to the following boundary value problem for inhomogeneous biharmonic equation

$$\begin{gathered} \Delta^2 u(x) = f(x), \quad x \in S, \\ u|_{\partial S} = \varphi_0(\xi), \quad \Delta u|_{\partial S} = \varphi_1(\xi), \quad \xi \in \partial S. \end{gathered} \tag{4.1.7}$$

Definition 4.1. Function of the form

$$G_4^r(x,\xi) = E_4(x,\xi) + g_4^r(x,\xi), \tag{4.1.8}$$

where $g_4^r(x,\xi) \in C_\xi^3(\bar{S})$ is a biharmonic function in the variables $x,\xi \in S$, for which for $x \in S$ the following conditions hold

$$G_4^r(x,\xi)|_{\xi\in\partial S} = \Delta_\xi G_4^r(x,\xi)|_{\xi\in\partial S} = 0$$

we call the Green's function of the Navier problem (4.1.7).

In what follows we also consider another function

$$E_4^r(x,y) = \frac{1}{\omega_n}\int_S E(x,y)E(y,\xi)\,dy.$$

Theorem 4.2. *The Green's function $G_4^r(x,\xi)$ of the Navier problem* (4.1.7) *satisfies the equality*

$$G_4^r(x,\xi) = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy, \tag{4.1.9}$$

where the function $G_2(x,y)$ is defined in (4.1.1).

Proof. First, let us prove that the function (4.1.9) has the form (4.1.8). To do this, we establish that the function $g_0(x,\xi) = E_4^r(x,\xi) - E_4(x,\xi)$ is harmonic in $\xi \neq x \in S$, where $E_4(x,y)$ is defined in (4.1.2). Suppose $x \neq \xi \in S$. Let us denote $S_\delta = S \setminus \{y \in \mathbb{R}^n : |x-y| \le \delta\}$ and take $\delta > 0$ so small that $\xi \in S_\delta$, i.e. $|x-\xi| > \delta$ and $\{y \in \mathbb{R}^n : |x-y| \le \delta\} \subset S$. Then

$$E_4^r(x,\xi) = \frac{1}{\omega_n}\Big(\int_{S_\delta} + \int_{|x-y|\le\delta}\Big)E(x,y)E(y,\xi)\,dy.$$

Since $\xi \in S_\delta$ and $x \notin S_\delta$, then by the property of the volume potential

$$\Delta_\xi \frac{1}{\omega_n}\int_{S_\delta} E(x,y)E(y,\xi)\,dy = -E(x,\xi). \tag{4.1.10}$$

Let $n > 4$ or $n = 3$. Other cases are treated similarly. Let us denote

$$F(x,\xi) = \int_{|x-y|\le\delta} E(x,y)E(y,\xi)\,dy = \frac{1}{(n-2)^2}\int_{|\eta|\le\delta} |\eta|^{2-n}|x-\xi-\eta|^{2-n}\,d\eta.$$

The integrand in the last integral has a singularity only at the origin ($|x-\xi-\eta| \ge |x-\xi|-\delta > 0$) and therefore differentiation of the integral with respect to ξ is possible under the integral sign, which means $F(x,\xi)$ is a harmonic function with respect to ξ. Therefore, by the property of the function $E_4(x,\xi)$

$$\Delta_\xi g_0(x,\xi) = \Delta_\xi(E_4^r(x,\xi) - E_4(x,\xi)) = -E(x,\xi) - (-E(x,\xi)) = 0$$

for $x \neq \xi$. The function $E_4(x,\xi)$ for $\xi = x$ has the singularity $\mathrm{O}(|x-\xi|^{4-n})$ if $n > 4$. Further

$$\begin{aligned} E_4^r(x,\xi) &= C\int_{|\xi+\eta|<1} |x-\xi-\eta|^{n-2}|\eta|^{n-2}\,d\eta \le C\int_{|\eta|<2} |z-\eta|^{2-n}|\eta|^{2-n}\,d\eta \\ &= C\Big(\int_{|z|<|\eta|<2} + \int_{|\eta|<|z|}\Big)|z-\eta|^{2-n}|\eta|^{2-n}\,d\eta \equiv I_1 + I_2, \end{aligned}$$

where $z = x - \xi$. For the first integral I_1 we have

$$I_1 \le C|z|^{3-n}\int_{|\eta|<2} |z-\eta|^{2-n}|\eta|^{-1}\,d\eta = \mathrm{O}(|z|^{3-n}),$$

since according to [183, p.27] the integral in this equality is bounded. Besides

$$\begin{aligned} I_2 &= C|z|^{2-n}|z|^n\int_{|\theta|z||<|z|} |z-\theta|z||^{2-n}|\theta|^{2-n}\,d\theta \\ &= C|z|^{4-n}\int_{|\theta|<1} |\hat z-\theta|^{2-n}|\theta|^{2-n}\,d\theta, \end{aligned}$$

where $\hat z = z/|z|$. Since the last integral is continuous in $\hat z \in \partial S$, then $I_2 = \mathrm{O}(|x - \xi|^{4-n})$. This means $g_0(x,\xi) = \mathrm{O}(|z|^{3-n})$ and by the theorem on erasing singularities [183] the function $g_0(x,\xi)$ is harmonic in ξS. Since the functions $E_4^r(x,\xi)$ and $E_4(x,\xi)$ are symmetric, the function $g_0(x,\xi)$ is also harmonic in $x \in S$.

Now let us transform the function $G_4^r(x,\xi)$. Using (4.1.1) we write

$$\begin{aligned} G_4^r(x,\xi) = E_4(x,\xi) + g_0(x,\xi) + \frac{1}{\omega_n}\Big(&\int_S E(x,y)g^d(y,\xi)\,dy \\ &+ \int_S g^d(x,y)E(y,\xi)\,dy + \int_S g^d(x,y)g^d(y,\xi)\,dy\Big) \\ &\equiv E_4(x,\xi) + g_4^r(x,\xi), \quad (4.1.11) \end{aligned}$$

where $g^d(x,\xi) = -E\big(\frac{x}{|x|}, |x|\xi\big)$. The first and third integrals in the resulting equality are harmonic functions in ξ, since the singularity in the first integral is integrable, and differentiation does not increase this singularity.

The second integral, by the property of the volume potential, is a biharmonic function in ξ, since the function $g^d(x,y) = -E\big(\frac{x}{|x|}, |x|y\big)$ has bounded derivatives with respect to y in $\bar S$ for $x \in S$. Similar statements are true for the variable x.

Let us check the boundary conditions for the function $G_4^r(x,\xi)$. By the property of the Green's function $G_2(x,\xi)$ and the continuity of the volume potential, we have

$$G_4^r(x,\xi)|_{\xi\in\partial S} = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)|_{\xi\in\partial S}\,dy = 0,$$

where $x \in S$. Using (4.1.11) and the properties of the volume potential at $x \in S$ we find

$$\Delta_\xi G_4^r(x,\xi)|_{\xi\in\partial S} = \Big(-E(x,\xi) - g^d(x,\xi)\Big)|_{\xi\in\partial S} = -G_2(x,\xi)|_{\xi\in\partial S} = 0.$$

Since the functions $E_4(x,\xi)$ and $g^d(x,y)$ are symmetric, the function $G_4^r(x,\xi)$ from (4.1.9) is also symmetric. From (4.1.11) it is clear that $g_4^r(x,\xi) \in C_\xi^3(\bar S)$ for $x \in S$ since the same are the functions $g^d(x,\xi)$ and $g_0(x,\xi)$. The theorem is proved. □

Theorem 4.3. *The function $u \in C^4(S) \cap C^3(\bar S)$, which is a solution to the Navier problem* (4.1.7)*, can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\frac{\partial\Delta_\xi G_4^r(x,\xi)}{\partial\nu}\varphi_0(\xi)\,ds_\xi + \frac{1}{\omega_n}\int_{\partial S}\frac{\partial G_4^r(x,\xi)}{\partial\nu}\varphi_1(\xi)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4^r(x,\xi)f(\xi)\,d\xi. \quad (4.1.12)$$

If $\varphi_0 \in C(\partial S)$, $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ and $f \in C^1(\bar S)$, then the function $u(x)$ from (4.1.12) *is a solution to the Navier problem* (4.1.7).

Proof. Let $u \in C^4(S) \cap C^3(\bar S)$ be a solution to the Navier problem (4.1.7). If we use the equality (4.1.3) and similarly derived equality

$$0 = \int_{\partial S}\Big(-g_4^r(x,\xi)\frac{\partial\Delta u}{\partial\nu} + \frac{\partial g_4^r(x,\xi)}{\partial\nu}\Delta u - \Delta_\xi g_4^r(x,\xi)\frac{\partial u}{\partial\nu} + \frac{\partial\Delta_\xi g_4^r(x,\xi)}{\partial\nu}u(\xi)\Big)\,ds_\xi + \int_S g_4^r(x,\xi)f(\xi)\,d\xi, \quad (4.1.13)$$

then we get

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\Big(-G_4^r(x,\xi)\frac{\partial \Delta u}{\partial \nu} + \frac{\partial G_4^r(x,\xi)}{\partial \nu}\Delta u$$
$$- \Delta_\xi G_4^r(x,\xi)\frac{\partial u}{\partial \nu} + \frac{\partial \Delta_\xi G_4^r(x,\xi)}{\partial \nu}u(\xi)\Big)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4^r(x,\xi)f(\xi)\,d\xi.$$

Using here the equalities $G_4^r(x,\xi)|_{\xi\in\partial S} = \Delta_\xi G_4^r(x,\xi)|_{\xi\in\partial S} = 0$ for $x \in S$ we get (4.1.12).

Let us check that the function $u(x)$ determined from (4.1.12) is a solution to the Navier problem (4.1.7) for $\varphi_0 \in C(\partial S)$, $\varphi_1 \in C^1(\partial S)$ and $f \in C^1(\bar{S})$. Let us denote the first, second and third integrals from (4.1.12) by $u_1(x)$, $u_2(x)$ and $u_3(x)$, respectively. Since $\Delta_\xi G_4^r(x,\xi) = -G_2(x,\xi)$ for $x \neq \xi$ (proven similarly to (4.1.10)), then, as it is known, the function

$$u_1(x) = -\frac{1}{\omega_n}\int_{\partial S}\frac{\partial G_2(x,\xi)}{\partial \nu}\varphi_0(\xi)\,ds_\xi \equiv w_{\varphi_0}(x)$$

is a solution to the Dirichlet problem for the Laplace equation in S with value φ_0 on the boundary. Further, by virtue of Fubini's theorem, we obtain

$$u_2(x) = \frac{1}{\omega_n}\int_S G_2(x,y)\Big(\frac{1}{\omega_n}\int_{\partial S}\frac{\partial G_2(y,\xi)}{\partial \nu}\varphi_1(\xi)\,ds_\xi\Big)\,dy$$
$$= -\frac{1}{\omega_n}\int_S G_2(x,y)w_{\varphi_1}(y)\,dy,$$

which means $\Delta_x u_2(x) = w_{\varphi_1}(x)$ and $u_2(x)|_{\partial S} = 0$. In this case, the function $w_{\varphi_1}(x)$ must be subject to the condition $w_{\varphi_1} \in C^1(\bar{S})$, and this condition is satisfied if $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ [4, Lemma 2.7]. Thus, the biharmonic function $u_1(x) + u_2(x)$ is a solution to the Navier problem (4.1.7) for the homogeneous biharmonic equation. Finally, by the property of the function $G_2(x,\xi)$,

$$\Delta^2 u_3(x) = -\frac{1}{\omega_n}\Delta\int_S G_2(x,\xi)f(\xi)\,d\xi = f(x)$$

and $u_3(x)|_{\partial S} = \Delta u_3(x)|_{\partial S} = 0$. The theorem is proved. □

The solvability of the Navier problem also follows from Theorem 3.24, but since a more complex problem is considered, the smoothness of φ_0 and φ_1 is higher.

4.1.3 Riquier-Neumann boundary value problem

The Riquier-Neumann problem [101] (in [20] it is also called the Neumann-2 problem) is to find the function $u \in C^4(S) \cap C^3(\bar{S})$, which is a solution to the following boundary value problem for the inhomogeneous biharmonic equation

$$\begin{gathered}\Delta^2 u(x) = f(x), \quad x \in S, \\ \frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_0(\xi), \quad \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = \varphi_1(\xi), \quad \xi \in \partial S.\end{gathered} \tag{4.1.14}$$

First, a few notes on the Neumann boundary value problem. In Section 3.7 the Green's function

$$\mathcal{N}(x,\xi) = E(x,\xi) - E_0(x,\xi)$$

for the Neumann problem for the Poisson equation in the ball S is constructed, where the harmonic function $E_0(x,\xi)$ is written in the form

$$E_0(x,\xi) = \int_0^1 \left(\hat{E}\Big(\frac{x}{|x|}, t|x|\xi\Big) + 1\right)\frac{dt}{t}$$

and $\hat{E}(x,\xi) = \Lambda_x E(x,\xi)$. It is easy to see that the function $\mathcal{N}(x,\xi)$ is symmetric. The following result is an addition to Theorem 3.30 on page 316.

Theorem 4.4. *Let $f \in C^1(\bar{S})$ and $\psi \in C(\partial S)$, then the solution to the Neumann problem*

$$\Delta u = f, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = \psi,$$

provided that

$$\int_{\partial S} \psi(\xi)\, ds_\xi = \int_S f(\xi)\, d\xi, \tag{4.1.15}$$

can be written, up to a constant, in the form

$$v_\psi(x) = \frac{1}{\omega_n}\int_{\partial S} \mathcal{N}(x,\xi)\psi(\xi)\, ds_\xi - \frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi) f(\xi)\, d\xi. \tag{4.1.16}$$

Proof. First note that $\Lambda u = \frac{\partial u}{\partial \nu}$ on ∂S. Theorem 3.30 established that the second integral in (4.1.16) is a solution to the inhomogeneous Poisson equation with $f \in C^1(\bar{S})$ on the right-hand side and that

$$\Lambda_x \int_S \mathcal{N}(x,\xi) f(\xi)\, d\xi\Big|_{x\in\partial S} = -\int_S f(\xi)\, d\xi. \tag{4.1.17}$$

Let us consider the first integral in the equality (4.1.15). It is clear that by virtue of the definition of the function $\mathcal{N}(x,\xi)$ it is harmonic in S. Using the symmetry of $G_2(x,\xi)$ it is easy to calculate that for $\xi\in\partial S$

$$\Lambda_x\mathcal{N}(x,\xi)=\Lambda_x E(x,\xi)-\hat{E}\Big(\frac{x}{|x|},t|x|\xi\Big)-1$$
$$=-\Lambda_\xi G_2(x,\xi)-1=-\frac{\partial G_2(x,\xi)}{\partial\nu}-1,$$

and therefore, by the property of the Green's function $G_2(x,\xi)$ of the Dirichlet problem in S

$$\frac{1}{\omega_n}\int_{\partial S}\Lambda_x\mathcal{N}(x,\xi)\psi(\xi)\,ds_\xi\Big|_{x\in\partial S}=-\frac{1}{\omega_n}\int_{\partial S}\frac{\partial G_2(x,\xi)}{\partial\nu}\psi(\xi)\,ds_\xi\Big|_{x\in\partial S}$$
$$-\frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi=\psi(x)-\frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi.$$

From here we get

$$\frac{\partial v_\psi(x)}{\partial\nu}\Big|_{x\in\partial S}=\psi(x)-\frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi+\frac{1}{\omega_n}\int_S f(\xi)\,d\xi.$$

If we use the condition (4.1.15) here, then the harmonic in S function $v_\psi(x)$ also satisfies the boundary condition of the Neumann problem. The theorem is proved. □

Remark 4.2. During the proof of Theorem 4.4, we get an equality that is useful later

$$\frac{1}{\omega_n}\int_{\partial S}\frac{\partial\mathcal{N}(x,\xi)}{\partial\nu_x}\varphi(\xi)\,ds_\xi|_{x\in\partial S}=\varphi(x)-\frac{1}{\omega_n}\int_{\partial S}\varphi(\xi)\,d\xi,\tag{4.1.18}$$

where $\varphi\in C(\partial S)$.

The proved theorem is not very convenient to apply in the case of polynomial data. Similar to Theorem 2.15 on polynomial solutions to the Dirichlet problem, the following statement is true for the Neumann problem.

Theorem 4.5. *Let $P(x)$ and $Q(x)$ be polynomials, then the solution to the Neumann problem*

$$\Delta u(x)=Q(x),\ x\in S;\quad \frac{\partial u}{\partial\nu}\Big|_{\partial S}=P(x)|_{\partial S},$$

exists only in the case $v(0)=0$, where

$$v(x) = P(x) + \frac{|x|^2-1}{2}\int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^s}{(2s+2)!!(2s)!!}$$
$$\times \Delta^s((\Lambda+2)Q - \Delta P)(\alpha x)\,\alpha^{n/2-1}\,d\alpha \quad (4.1.19)$$

and can be written as

$$u(x) = \int_0^1 v(tx)\frac{dt}{t} + C. \qquad (4.1.20)$$

Proof. According to Theorem 3.12 on page 206 the solution to the Neumann problem is represented in the form (4.1.20), where the function $v(x)$ is the solution to the Dirichlet problem

$$\Delta v(x) = (\Lambda+2)Q(x),\ x \in S; \quad v|_{\partial S} = P(x)|_{\partial S}$$

According to Remark 3.9, a solution to the Neumann problem exists only in the case $v(0) = 0$. If we use Theorem 2.15 for the Dirichlet problem written above, we obtain the equality (4.1.19). The theorem is proved. □

Let us return to the Riquier-Neumann problem for the biharmonic equation (4.1.14).

Definition 4.2. Function of the form

$$G_4^{rn}(x,\xi) = E_4(x,\xi) + g_4^{rn}(x,\xi), \qquad (4.1.21)$$

where $g_4^{rn}(x,\xi) \in C^4(S)\cap C^3(\bar S)$ is a biharmonic function in the variables $x,\xi \in S$, for which the equalities hold for $x \in S$

$$\frac{\partial G_4^{rn}(x,\xi)}{\partial \nu}\Big|_{\xi\in\partial S} = 0, \quad \frac{\partial \Delta_\xi G_4^{rn}(x,\xi)}{\partial \nu}\Big|_{\xi\in\partial S} = 1. \qquad (4.1.22)$$

we call the Green's function of the Riquier-Neumann problem (4.1.14).

Theorem 4.6. *The Green's function $G_4^{rn}(x,\xi)$ of the Riquier-Neumann problem* (4.1.14) *can be represented as*

$$G_4^{rn}(x,\xi)$$
$$= \frac{1}{\omega_n}\Big(\int_S \mathcal{N}(x,y)\mathcal{N}(y,\xi)\,dy - \frac{1}{\tau_n}\int_S \mathcal{N}(x,y)\,dy \cdot \int_S \mathcal{N}(y,\xi)\,dy\Big), \quad (4.1.23)$$

where $\tau_n = |S|$ is the volume of S. The Green's function $G_4^{rn}(x,\xi)$ is symmetric.

Proof. First, let us prove that the function (4.1.23) has the form (4.1.21). It is not hard to see that

$$G_4^{rn}(x,\xi) = E_4^r(x,\xi) - \frac{1}{\omega_n}\Big(\int_S E(x,y)E_0(y,\xi)\,dy + \int_S E_0(x,y)E(y,\xi)\,dy$$
$$- \int_S E_0(x,y)E_0(y,\xi)\,dy + \frac{1}{\tau_n}\int_S \mathcal{N}(x,y)\,dy \cdot \int_S \mathcal{N}(y,\xi)\,dy\Big)$$
$$\equiv E_4(x,\xi) + g_4^{rn}(x,\xi).$$

Theorem 4.2 states that $E_4^r(x,\xi) = E_4(x,\xi) + g_0(x,\xi)$. The first and third integrals in the resulting equality are harmonic functions in ξ, since the singularity in the first integral is integrable, and differentiation does not increase this singularity. The second and fourth integrals, by the property of the volume potential, are biharmonic functions with respect to ξ, since the harmonic function $E_0(x,\xi)$ has bounded derivatives with respect to ξ in $\bar{S}$ for $x \in S$.

Let us check the boundary conditions for $G_4^{rn}(x,\xi)$. By the property of the Green's function $\mathcal{N}(x,\xi)$ (4.1.17) and the continuous differentiability of the volume potential for $x \in S$ we have

$$\frac{\partial G_4^{rn}(x,\xi)}{\partial \nu}\Big|_{\xi\in\partial S} = \frac{1}{\omega_n}\Big(-\int_S \mathcal{N}(x,y)\,dy + \frac{1}{\tau_n}\int_S \mathcal{N}(x,y)\,dy \cdot \tau_n\Big) = 0.$$

In Theorem 3.30 it is also established that $\Lambda_\xi \mathcal{N}(x,\xi)|_{\xi\in\partial S} = -1$ and since

$$\Delta_\xi G_4^{rn}(x,\xi) = -\mathcal{N}(x,\xi) + \frac{1}{|S|}\int_S \mathcal{N}(x,y)\,dy$$

for $x \neq \xi$ (proved similarly to (4.1.10)), then for $x \in S$

$$\frac{\partial \Delta_\xi G_4^{rn}(x,\xi)}{\partial \nu}\Big|_{\xi\in\partial S} = 1.$$

The symmetry of $G_4^{rn}(x,\xi)$ follows from the symmetry of $\mathcal{N}(x,\xi)$. The theorem is proved. □

Theorem 4.7. *Let the function $u \in C^4(S) \cap C^3(\bar{S})$ be a solution to the Riquier-Neumann problem* (4.1.14)*, then it can be represented as*

$$u(x) = -\frac{1}{\omega_n}\int_{\partial S} \Delta_\xi G_4^{rn}(x,\xi)\varphi_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial S} G_4^{rn}(x,\xi)\varphi_1(\xi)\,ds_\xi$$

$$+\frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi + C. \quad (4.1.24)$$

Moreover, if $\varphi_0 \in C(\partial S)$, $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ *and* $f \in C^1(\bar{S})$ *, then the function* $u(x)$ *from* (4.1.24) *is a solution to the Riquier-Neumann problem* (4.1.14) *provided that*

$$\int_{\partial S}\varphi_1(\xi)\,ds_\xi = \int_S f(\xi)\,d\xi. \quad (4.1.25)$$

Proof. Let $u(x)$ be the solution to the Riquier-Neumann problem (4.1.14). Similar to the proof of the theorem 4.3, if we use the equality (4.1.3) and then (4.1.13) with $g_4^r(x,\xi) = g_4^{rn}(x,\ xi)$, then we get

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\Big(-G_4^{rn}(x,\xi)\frac{\partial\Delta u}{\partial\nu} + \frac{\partial G_4^{rn}(x,\xi)}{\partial\nu}\Delta u \\ - \Delta_\xi G_4^{rn}(x,\xi)\frac{\partial u}{\partial\nu} + \frac{\partial\Delta_\xi G_4^{rn}(x,\xi)}{\partial\nu}u(\xi)\Big)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi.$$

If we now recall the equalities (4.1.22), we obtain the equality (4.1.24) for $C = \frac{1}{\omega_n}\int_{\partial S}u(\xi)\,ds_\xi$.

Let us check that the function $u(x)$ from (4.1.24) is a solution to the Navier problem for $\varphi_0 \in C(\partial S)$, $\varphi_1 \in C^1(\partial S)$ and $f \in C^1(\bar{S})$. Let us denote the first, second and third integrals from (4.1.24) by $v_1(x)$, $v_2(x)$ and $v_3(x)$, respectively. Since for $x \neq \xi$

$$\Delta_\xi G_4^{rn}(x,\xi) = -\mathcal{N}(x,\xi) + \frac{1}{\tau_n}\int_S \mathcal{N}(x,y)\,dy,$$

then, by virtue of (4.1.16), the function

$$v_1(x) = \frac{1}{\omega_n}\int_{\partial S}\mathcal{N}(x,\xi)\varphi_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_S \mathcal{N}(x,y)\Big(\frac{1}{\tau_n}\int_{\partial S}\varphi_0(\xi)\,ds_\xi\Big)\,dy$$

is a solution to the Neumann problem for the Laplace equation in S with the value φ_0 on the boundary and with the value $\frac{1}{\tau_n}\int_{\partial S}\varphi_0(\xi)\,ds_\xi$ on the right parts of the equation (the required condition for the equality of the integrals is satisfied). This means $v_1(x)$ is a biharmonic function satisfying the conditions $\frac{\partial v_1}{\partial\nu}|_{\partial S} = \varphi_0(x)$, $\frac{\partial\ Deltav_1}{\partial\nu}|_{\partial S} = 0$. Further, by virtue of Fubini's theorem, we will have

$$v_2(x) = -\frac{1}{\omega_n}\int_S \mathcal{N}(x,y)\Big(\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}(y,\xi)\varphi_1(\xi)\,ds_\xi$$

$$-\int_{\partial S}\frac{1}{\tau_n}\int_S \mathcal{N}(z,\xi)\,dz\,\varphi_1(\xi)\,ds_\xi\Big)\,dy,$$

whence, due to (4.1.16) (the integral over y of the coefficient in parentheses for the function $\mathcal{N}(x,y)$ is equal to 0)

$$\Delta_x v_2(x)=\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}(x,\xi)\varphi_1(\xi)\,ds_\xi-\int_{\partial S}\frac{1}{\tau_n}\int_S \mathcal{N}(z,\xi)\,dz\,\varphi_1(\xi)\,ds_\xi$$
$$\equiv v_{\varphi_1;0}(x)-C$$

and $\frac{\partial v_2}{\partial\nu}|_{\partial S}=0$. In addition, $\Delta_x^2 v_2(x)=0$ and by (4.1.18)

$$\frac{\partial\Delta v_2}{\partial\nu}\Big|_{\partial S}=\varphi_1(x)-\frac{1}{\omega_n}\int_{\partial S}\varphi_1(\xi)\,d\xi.$$

When calculating $\Delta_x v_2(x)$ on the function $v_{\varphi_1;0}(x)$, the condition $v_{\varphi_1;0}\in C^1(\bar S)$ must be imposed under the integral, and this holds if $\varphi_1\in C^{1+\varepsilon}(\partial S)$ [4]. Finally, by the property of the functions $G_4^{rn}(x,\xi)$ and $\mathcal{N}(x,\xi)$

$$\Delta^2 v_3(x)=-\frac{1}{\omega_n}\Delta\int_S \mathcal{N}(x,\xi)f(\xi)\,d\xi=f(x)$$

at $f\in C^1(\bar S)$ and since it is possible to enter a derivative under the sign of the volume potential and because of the equality $\Lambda_x\mathcal{N}(x,\xi)|_{x\in\partial S}=-1$ we get

$$\frac{\partial v_3}{\partial\nu}\Big|_{x\in\partial S}=0,\qquad \frac{\partial\Delta v_3}{\partial\nu}\Big|_{x\in\partial S}=\frac{1}{\omega_n}\int_S f(\xi)\,d\xi.$$

Thus, the function $u(x)=v_1(x)+v_2(x)+v_3(x)$ is a solution to the following Riquier-Neumann problem

$$\Delta^2 u(x)=0+0+f(x)=f(x),$$
$$\frac{\partial u}{\partial\nu}\Big|_{x\in\partial S}=\varphi_0(x)+0+0=\varphi_0(x),$$
$$\frac{\partial\Delta u}{\partial\nu}\Big|_{x\in\partial S}=0+\varphi_1(x)-\frac{1}{\omega_n}\int_{\partial S}\varphi_1(\xi)\,d\xi+\frac{1}{\omega_n}\int_S f(\xi)\,d\xi.$$

Therefore, if the conditions (4.1.25) are met, then the function $u(x)$ from (4.1.24) is a solution to the problem (4.1.14). The theorem is proved. □

Remark 4.3. A necessary and sufficient condition for the solvability of the Riquier-Neumann problem for a polyharmonic equation is also obtained in [101] and this condition has the form (4.1.25).

4.1.4 Examples

Let $\{H_k^{(i)}(x) : i = 1, \dots, h_k,\ k \in \mathbb{N}_0\}$ be a complete system of homogeneous degree $k \in \mathbb{N}_0$ harmonic polynomials orthogonal on ∂S normalized so that $\int_{\partial S}(H_k^{(i)}(\xi))^2\, ds_\xi = \omega_n$, and $h_k = \frac{2k+n-2}{n-2}\binom{k+n-3}{n-3}$ for $n > 2$, $h_k = 2$ for $n = 2$ (see Definition 1.5).

Let us find solutions to the homogeneous Navier and Riquier-Neumann boundary value problems considered above for the inhomogeneous biharmonic equation in S with $f(x) = |x|^{2l} H_m^{(j)}(x)$, $l \in \mathbb{N}_0$.

1. Navier boundary value problem. By Theorem 4.2 the Green's function of the Navier problem $G_4^r(x, \xi)$ has the form

$$G_4^r(x,\xi) = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy.$$

It is known that the function $G_2(x,\xi)$ is symmetric and from Lemma 2.8 it follows that for $n > 2$ and $|x| > |\xi|$

$$G_2(x,\xi) = \sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}-1}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi).$$

Let us calculate the following integral

$$\begin{aligned} I(x) &= \frac{1}{\omega_n}\int_S G_2(x,\xi)f(\xi)\,d\xi = \int_0^1 \rho^{n-1}d\rho\frac{1}{\omega_n}\int_{|\xi|=1} G_2(x,\rho\xi)f(\rho\xi)\,ds_\xi \\ &= \lim_{\varepsilon\to 0}\Big(\int_0^{|x|-\varepsilon} + \int_{|x|+\varepsilon}^1\Big)\rho^{n-1}d\rho\frac{1}{\omega_n}\int_{|\xi|=1} G_2(x,\rho\xi)f(\rho\xi)\,ds_\xi \\ &\equiv \lim_{\varepsilon\to+0}(I_1^\varepsilon(x) + I_2^\varepsilon(x)), \end{aligned}$$

If $x = 0$, then the integral $I_1^\varepsilon(x)$ must be omitted. Let us calculate the first integral $I_1^\varepsilon(x)$. Since $|\xi| < |x| - \varepsilon \equiv a < |x|$, then, in accordance with Lemmas 2.7 and 3.34, the series below converges uniformly in ξ and so summation and integration can be interchanged

$$I_1^\varepsilon(x) = \int_0^{|x|-\varepsilon}\rho^{2l+m+n-1+k}\sum_{k=0}^{\infty}\frac{|x|^{-(2k+n-2)}-1}{2k+n-2}$$

$$\times \sum_{i=1}^{h_k} H_k^{(i)}(x) \frac{1}{\omega_n} \int_{|\xi|=1} H_k^{(i)}(\xi) H_m^{(j)}(\xi)\, ds_\xi.$$

Using the orthogonality of the system $\{H_k^{(i)}(x) : i = 1, \ldots, h_k, k \in \mathbb{N}_0\}$ we find

$$I_1^\varepsilon(x) = \frac{|x|^{-(2m+n-2)} - 1}{2m+n-2} \frac{(|x| - \varepsilon)^{2l+2m+n}}{2l+2m+n} H_m^{(j)}(x).$$

Passing to the limit at $\varepsilon \to +0$ we obtain

$$\lim_{\varepsilon \to +0} I_1^\varepsilon(x) = \frac{|x|^{2l+2} - |x|^{2l+2m+n}}{(2m+n-2)(2l+2m+n)} H_m^{(j)}(x).$$

Similarly, using the symmetry $G_2(x, \xi)$ we find

$$I_2^\varepsilon(x) = \int_{|x|+\varepsilon}^1 \rho^{2l+2m+n-1} \frac{\rho^{-(2m+n-2)} - 1}{2m+n-2}\, d\rho\, H_m^{(j)}(x) = \frac{1}{2m+n-2}$$
$$\times \Big(\frac{1}{2l+2} - \frac{1}{2l+2m+n} - \frac{(|x|+\varepsilon)^{2l+2}}{2l+2} + \frac{(|x|+\varepsilon)^{2l+2m+n}}{2l+2m+n} \Big) H_m^{(j)}(x).$$

Passing to the limit at $\varepsilon \to +0$ we obtain

$$\lim_{\varepsilon \to +0} I_2^\varepsilon(x) = \frac{1}{2m+n-2} \Big(\frac{1 - |x|^{2l+2}}{2l+2} - \frac{1 - |x|^{2l+2m+n}}{2l+2m+n} \Big) H_m^{(j)}(x).$$

Thus we have

$$\frac{1}{\omega_n} \int_S G_2(x, \xi) |\xi|^{2l} H_m^{(j)}(\xi)\, d\xi = I_1^0(x) + I_2^0(x) = \frac{1}{C_{l,m}} (1 - |x|^{2l+2}) H_m^{(j)}(x).$$

where $C_{l,m} = (2l+2)(2l+2m+n)$. Therefore, by Fubini's theorem,

$$\frac{1}{\omega_n} \int_S G_4^r(x, \xi) |\xi|^{2l} H_m^{(j)}(\xi)\, d\xi = \frac{1}{\omega_n} \int_S G_2(x, y) \frac{1}{\omega_n} \int_S G_2(y, \xi) |\xi|^{2l}$$
$$\times H_m^{(j)}(\xi)\, d\xi\, dy = \frac{1}{C_{l,m}} \frac{1}{\omega_n} \int_S G_2(x, y)(1 - |y|^{2l+2}) H_m^{(j)}(y)\, dy$$
$$= \Big(\frac{|x|^{2l+4} - 1}{C_{l,m} C_{l+1,m}} + \frac{1 - |x|^2}{C_{l,m} C_{0,m}} \Big) H_m^{(j)}(x). \quad (4.1.26)$$

2. Riquier-Neumann boundary value problem. By Theorem 4.6 the Green's function of this problem $G_4^{rn}(x,\xi)$ is written in the form

$$G_4^{rn}(x,\xi) = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}(x,y)\mathcal{N}(y,\xi)\,dy - \frac{n}{\omega_n}\int_S \mathcal{N}(x,y)\,dy \cdot \int_S \mathcal{N}(y,\xi)\,dy\Big).$$

The solvability condition for the homogeneous Riquier-Neumann problem for $f(x) = |x|^{2l}H_m^{(j)}(x)$ has the form $\int_S |\xi|^{2l}H_m^{(j)}(\xi)\,d\xi = 0$, which is possible only for $m > 0$, since for such m the equality $\int_{\partial S} H_m^{(j)}(\xi)\,ds_\xi = 0$ holds. Theorem 3.29 proved the equality

$$\frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi = -\frac{|x|^{2l+2} - (2l+2+m)/m}{C_{l,m}}H_m^{(j)}(x), \tag{4.1.27}$$

which we use. By virtue of Fubini's theorem, we write

$$\begin{aligned}\frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi &= \frac{1}{\omega_n}\int_S \mathcal{N}(x,y)\frac{1}{\omega_n}\int_S \mathcal{N}(y,\xi)f(\xi)\,d\xi\,dy \\ &- \frac{n}{\omega_n}\int_S \mathcal{N}(x,y)\,dy\cdot\frac{1}{\omega_n}\int_S\Big(\frac{1}{\omega_n}\int_S \mathcal{N}(y,\xi)\,dy\Big)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi \\ &\equiv J_1(x) + J_2(x).\end{aligned}$$

Let us calculate the integral $J_1(x)$. Due to (4.1.27) we have

$$\begin{aligned}J_1(x) &= -\frac{1}{\omega_n}\int_S \mathcal{N}(x,y)\frac{|y|^{2l+2} - (2l+2+m)/m}{C_{l,m}}H_m^{(j)}(y)\,dy \\ &= \Big(\frac{|x|^{2l+4} - (2l+4+m)/m}{C_{l,m}C_{l+1,m}} - \frac{2l+2+m}{mC_{l,m}}\frac{|x|^2 - (m+2)/m}{C_{0,m}}\Big)H_m^{(j)}(x).\end{aligned}$$

From the equality (3.7.13) of Theorem 3.29 it follows that

$$E_0(x,\xi) = -\sum_{k=1}^{\infty}\frac{k+n-2}{k(2k+n-2)}\sum_{i=1}^{h_k}H_k^{(i)}(x)H_k^{(i)}(\xi),$$

where $x \in S$ and $\xi \in \overline{S}$. Therefore, taking into account Lemma 3.34, we have the following representation of the function $\mathcal{N}(x,\xi)$ for $|\xi| > |x|$

$$\mathcal{N}(x,\xi) = \frac{|\xi|^{-(n-2)}}{n-2}$$

$$+\sum_{k=1}^{\infty}\frac{|\xi|^{-(2k+n-2)}+(k+n-2)/k}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}(x)H_k^{(i)}(\xi). \quad (4.1.28)$$

From here we find

$$\frac{1}{\omega_n}\int_S \mathcal{N}(x,\xi)\,d\xi=\frac{1}{\omega_n}\Big(\int_0^{|x|}+\int_{|x|}^1\Big)\rho^{n-1}\int_{\partial S}\mathcal{N}(x,\rho\xi)\,ds_\xi\,d\rho$$
$$=\frac{1}{n-2}\Big(\frac{|x|^2}{n}+\frac{1}{2}-\frac{|x|^2}{2}\Big)=\frac{1}{2(n-2)}-\frac{|x|^2}{2n}. \quad (4.1.29)$$

Therefore, using the symmetry of $\mathcal{N}(x,\xi)$, we obtain

$$\frac{1}{\omega_n}\int_S\Big(\frac{1}{\omega_n}\int_S \mathcal{N}(y,\xi)\,dy\Big)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi$$
$$=\frac{1}{\omega_n}\int_S\Big(\frac{|\xi|^{2l}}{2(n-2)}-\frac{|\xi|^{2l+2}}{2n}\Big)H_m^{(j)}(\xi)\,d\xi=0,$$

whence it follows $J_2(x)=0$ and therefore for $m>0$

$$\frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)|\xi|^{2l}H_m^{(j)}(\xi)\,d\xi=\Big(\frac{|x|^{2l+4}-(2l+4+m)/m}{C_{l,m}C_{l+1,m}}$$
$$-\frac{2l+2+m}{mC_{l,m}}\cdot\frac{|x|^2-(m+2)/m}{C_{0,m}}\Big)H_m^{(j)}(x). \quad (4.1.30)$$

Since the coefficients of $H_m^{(j)}(x)$ on the right-hand sides of (4.1.26) and (4.1.30) do not depend on j, then instead of $H_m^{(j)}(x)$ in them we can take an arbitrary homogeneous harmonic polynomial of degree m.

4.2 Integral representation of functions from $C^{2m}(D)\cap C^{2m-1}(\bar{D})$

In this section, an elementary solution $E_{2m}(x,\xi)$ of the polyharmonic equation is defined. It generalizes the elementary solutions $E_6(x,\xi)$ from [110], $E_4(x,\xi)$ from (4.1.2) and $E(x,\xi)$ given in the previous Section 4.1 to the m-harmonic case. Lemmas 4.1 and 4.2 prove some properties of the function $E_{2m}(x,\xi)$. Theorem 4.8 using $E_{2m}(x,\xi)$ establishes the integral representation of functions from the class $C^{2m}(D)\cap C^{2m-1}(\bar{D})$ in a bounded domain $D\subset\mathbb{R}^n$ with a smooth boundary, which generalizes the equality (4.1.3). Example 4.1 is given. The main results of Section 4.2 are published in [125].

4.2.1 Elementary solution $E_{2m}(x,\xi)$

By analogy with the elementary solutions $E_4(x,\xi)$ and $E(x,\xi)$, we introduce an elementary solution to the m-harmonic equation. Let $m\in\mathbb{N}$. Then the set $\mathbb{N}\setminus\{1\}$ can be divided into two disjoint sets $\mathbb{N}_m=\{n\in\mathbb{N}: n>2m>1\}\cup(2\mathbb{N}+1)$ and its complement $\mathbb{N}_m^c=\{2,4,\ldots,2m\}$. Since the set $\mathbb{N}_m^c$ is finite, then $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}_{m-1}^c\subset\mathbb{N}_m^c$, and therefore $\mathbb{N}_m\subset\mathbb{N}_{m-1}$. It should be kept in mind that $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Definition 4.3. We call an elementary solution of the m-harmonic equation $\Delta^m u=0$ a function of the form

$$E_{2m}(x,\xi)=\begin{cases}\dfrac{(-1)^m|x-\xi|^{2m-n}}{(2-n,2)_m(2,2)_{m-1}}, & n\in\mathbb{N}_m,\\[2ex] \dfrac{(-1)^m|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}}\Big(\ln|x-\xi|-\sum\limits_{k=1}^{m-n/2}\frac{1}{2k}-\sum\limits_{k=n/2}^{m-1}\frac{1}{2k}\Big), & n\in\mathbb{N}_m^c.\end{cases}\tag{4.2.1}$$

where $(a,b)_k=a(a+b)\ldots(a+kb-b)$ and $(a,b)_0=1$. The symbol $(a,b)_k^*$ means that if there is 0 among the factors $a,(a+b),\ldots,(a+kb-b)$ included in $(a,b)_k$, then it should be replaced by 1. For example, $(-2,2)_3^*=(-2)\cdot1\cdot2=-4$. In addition, if in the sums included in (4.2.1) the upper index becomes less than the lower index, then the sum is considered equal to zero. Note that $(2-n,2)_m=(2-n)(4-n)\ldots(2m-n)\neq0$ for $n\in\mathbb{N}_m$ and therefore the first part of the equality (4.2.1) is defined correctly.

It should be noted that in [110] an elementary solution for the 3-harmonic equation is defined in the form

$$E_6(x,\xi) = \begin{cases} \dfrac{1}{2\cdot 4(n-2)(n-4)(n-6)}|x-\xi|^{6-n}, & n \ge 3, n \ne 4, 6, \\ -\dfrac{1}{2\cdot 4(n-2)(n-4)}\ln|x-\xi|, & n = 6 \\ \dfrac{|x-\xi|^2}{2n(n-2)^2}\Big(\ln|x-\xi| - \dfrac{n+2}{2n}\Big), & n = 4 \\ -\dfrac{|x-\xi|^4}{8n(n+2)}\Big(\ln|x-\xi| - \dfrac{n+2}{2n} - \dfrac{n+6}{4(n+2)}\Big), & n = 2 \end{cases} \tag{4.2.2}$$

and on its basis the Green's function of the Dirichlet problem for the 3-harmonic equation is constructed. The connection between all the elementary solutions mentioned above is established in the following lemma.

Lemma 4.1. *The function $E_{2m}(x,\xi)$ coincides with the elementary functions $E(x,\xi)$, $E_4(x,\xi)$ from* (4.1.2) *and $E_6(x,\xi)$ from* (4.2.2) *for $m = 1$, $m = 2$ and $m = 3$, respectively.*

Proof. Let $m = 1$. Then $\mathbb{N}_1^c = \{2\}$, $(2-n,2)_1 = 2-n$, $(2,2)_0 = 1$, and therefore $E_2(x,\xi) = E(x,\xi)$ for $n > 2$. If $n = 2$, then $(0,2)_1^* = 1$, $m - n/2 = 0$, $m - 1 = 0$ and therefore in the second equality from (4.2.1) the numerical sums are absent and therefore $E_2(x,\xi) = E(x,\xi)$ for $n = 2$.

Let $m = 3$. Then $\mathbb{N}_3^c = \{2,4,6\}$, $(2-n,2)_3 = (2-n)(4-n)(6-n)$, $(2,2)_2 = 2\cdot 4$, and that means

$$E_6(x,\xi) = \frac{|x-\xi|^{6-n}}{8(n-2)(n-4)(n-6)}, \quad n \in \mathbb{N}_3,$$

which coincides with (4.2.2) for $n = 3, 5$ or $n > 6$. If $n = 2$, then $(2-n,2)_3^* = 2\cdot 4$, $(2,2)_2 = 8$, $m - n/2 = 2$, $m - 1 = 2$ and therefore in the second equality from (4.2.1) the sum of 2 numeric sums without a sign has the form $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$ and therefore

$$E_6(x,\xi) = -\frac{|x-\xi|^4}{8\cdot 8}\Big(\ln|x-\xi| - \frac{3}{2}\Big),$$

which is the same as (4.2.2). If $n = 4$, then we have $(2-n,2)_3^* = -2\cdot 2$, $(2,2)_2 = 8$, $m - n/2 = 1$, $m - 1 = 2$ and therefore in the second equality from (4.2.1) both numerical sums without a sign have only one term $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Therefore in this case

$$E_6(x,\xi) = -\frac{|x-\xi|^2}{-4\cdot 8}\Big(\ln|x-\xi| - \frac{3}{4}\Big),$$

which is the same as (4.2.2). Finally, for $n = 6$ we find $(2-n,2)_3^* = -4\cdot(-2)$, $(2,2)_2 = 8$, $m-n/2 = 0$, $m-1 = 2$ and therefore in the second equality from (4.2.1) there is no numerical sums and therefore we get (4.2.2)

$$E_6(x,\xi) = -\frac{1}{64}\ln|x-\xi|.$$

Using similar reasoning, it is also easy to verify that the function $E_{2m}(x,\xi)$ for $m = 2$ coincides with $E_4(x,\xi)$ from (4.1.2). □

Remark 4.4. S.L. Sobolev [162, p. 521] considered the fundamental solutions $G_{m,n}(x)$ of the polyharmonic equation, which differ slightly from the elementary solutions $E_{2m}(x,\xi)$. For example, for odd n

$$G_{m,n}(x) = \chi_{m,n}|x|^{2m-n}, \quad \chi_{m,n} = \frac{(-1)^{(n-1)/2}}{\Gamma(m)\Gamma(m-n/2+1)2^{2m}\pi^{n/2-1}}.$$

Considering that $2^{m-1}\Gamma(m) = (2,2)_{m-1}$,

$$2^m\Gamma(m-n/2+1) = (2m-n)\dots(2-n)\Gamma(1-n/2) = (2-n,2)_m\Gamma(1-n/2)$$

and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ we can write

$$\chi_{m,n} = \frac{1}{\omega_n}\frac{(-1)^{(n-1)/2}\pi}{(2,2)_{m-1}(2-n,2)_m\Gamma(n/2)\Gamma(1-n/2)}.$$

If we now notice that $\Gamma(n/2)\Gamma(1-n/2) = \frac{\pi}{\sin(n\pi/2)} = \pi(-1)^{(n-1)/2}$, then we get

$$\chi_{m,n} = \frac{1}{\omega_n}\frac{1}{(2,2)_{m-1}(2-n,2)_m},$$

however, the function $E_{2m}(x,\xi)$ also contains the factor $(-1)^m$. This factor is additionally taken into account in Theorem 4.8. For $n\in\mathbb{N}_m^c$ the difference between these functions is somewhat greater and it is associated with the following property of the function $E_{2m}(x,\xi)$, which $G_{m,n}(x)$ does not possesses.

Lemma 4.2. *The symmetric function $E_{2m}(x,\xi)$ defined for $x\neq\xi$ satisfies the equalities*

$$\Delta_\xi E_{2m}(x,\xi) = -E_{2(m-1)}(x,\xi), \quad \Delta_\xi E_2(x,\xi) = 0.$$

Proof. From the equality (4.2.1) it immediately follows that the function $E_{2m}(x, \xi)$ is symmetric.

1^0. Let $n \in \mathbb{N}_m \subset \mathbb{N}_{m-1}$ and $m > 1$. Consider $\alpha \in \mathbb{R}$. It is easy to check that

$$\frac{\partial}{\partial \xi_i} |x-\xi|^\alpha = \alpha\,(\xi_i - x_i)\,|x-\xi|^{\alpha-2}$$

and that means

$$\frac{\partial^2}{\partial \xi_i^2} |x-\xi|^\alpha = \alpha \left(|x-\xi|^{\alpha-2} + (\alpha-2)\,(\xi_i - x_i)^2\,|x-\xi|^{\alpha-4} \right).$$

Therefore we have

$$\Delta_\xi |x-\xi|^\alpha = \alpha(\alpha+n-2)|x-\xi|^{\alpha-2}. \tag{4.2.3}$$

For $\alpha = 2m-n$ it follows that

$$\Delta_\xi |x-\xi|^{2m-n} = (2m-n)(2m-2)|x-\xi|^{2m-2-n},$$

which means the equality holds

$$\begin{aligned}\Delta_\xi E_{2m}(x,\xi) = \frac{(-1)^m (2m-n)(2m-2)}{(2-n,2)_m (2,2)_{m-1}} |x-\xi|^{2m-n} \\ = -\frac{(-1)^{m-1}|x-\xi|^{2m-2-n}}{(2-n,2)_{m-1}(2,2)_{m-2}} = -E_{2(m-1)}(x,\xi).\end{aligned}$$

2^0. Let now $n \in \mathbb{N}_m^c = \{2, 4, \ldots, 2m\}$ and $m > 1$. For $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\Big(|x-\xi|^\alpha \ln|x-\xi| \Big) \\ = \alpha(\xi_i - x_i)|x-\xi|^{\alpha-2}\ln|x-\xi| + |x-\xi|^\alpha \frac{\xi_i - x_i}{|x-\xi|^2},\end{aligned} \tag{4.2.4}$$

whence it follows that

$$\begin{aligned}\frac{\partial^2}{\partial \xi_i^2}\Big(|x-\xi|^\alpha \ln|x-\xi| \Big) = \alpha|x-\xi|^{\alpha-2}\ln|x-\xi| \\ + \alpha(\alpha-2)(\xi_i - x_i)^2|x-\xi|^{\alpha-4}\ln|x-\xi| + \alpha|x-\xi|^{\alpha-2}\frac{(\xi_i - x_i)^2}{|x-\xi|^2}\end{aligned}$$

$$+ |x-\xi|^{\alpha-2} + (\alpha-2)|x-\xi|^{\alpha-4}(\xi_i - x_i)^2$$

and therefore if $\alpha \neq 0$ and $\alpha \neq 2-n$, then

$$\begin{aligned}\Delta_\xi\Big(|x-\xi|^\alpha \ln|x-\xi|\Big) &= n\alpha|x-\xi|^{\alpha-2}\ln|x-\xi| \\ &\quad + \alpha(\alpha-2)|x-\xi|^{\alpha-2}\ln|x-\xi| + (\alpha+n+\alpha-2)|x-\xi|^{\alpha-2} \\ &= \alpha(\alpha+n-2)|x-\xi|^{\alpha-2}\Big(\ln|x-\xi| + \frac{1}{\alpha+n-2} + \frac{1}{\alpha}\Big).\end{aligned}$$

In the case $\alpha = 0$ we get

$$\Delta_\xi\big(|x-\xi|^0 \ln|x-\xi|\big) = (n-2)|x-\xi|^{-2},$$

and for $\alpha = 2-n$ we find

$$\Delta_\xi\big(|x-\xi|^{2-n} \ln|x-\xi|\big) = (2-n)|x-\xi|^{-n}.$$

Therefore, taking into account also (4.2.3), for $\alpha \neq 0$ and $\alpha \neq 2-n$ we have

$$\begin{aligned}&\Delta_\xi\Big(|x-\xi|^\alpha\big(\ln|x-\xi| + C\big)\Big) \\ &\qquad = \alpha(\alpha+n-2)|x-\xi|^{\alpha-2}\Big(\ln|x-\xi| + \frac{1}{\alpha+n-2} + \frac{1}{\alpha} + C\Big),\end{aligned}$$

where C is some constant. Therefore, if $n \in \mathbb{N}_m^c = \{2,4,\dots,2m\}$ and $n \neq 2m$, and $m > 1$, i.e. $n \in \mathbb{N}_{m-1}^c = \{2,4,\dots,2m-2\}$, then we get

$$\begin{aligned}\Delta_\xi E_{2m}(x,\xi) &= \Delta_\xi \frac{(-1)^m|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}}\Big(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} - \sum_{k=n/2}^{m-1}\frac{1}{2k}\Big) \\ &= (-1)^m \frac{(2m-n)(2m-2)|x-\xi|^{2m-2-n}}{(2-n,2)_m^*(2,2)_{m-1}}\Big(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} \\ &\quad - \sum_{k=n/2}^{m-1}\frac{1}{2k} + \frac{1}{2m-2} + \frac{1}{2m-n}\Big) = -\frac{(-1)^{m-1}|x-\xi|^{2m-2-n}}{(2-n,2)_{m-1}^*(2,2)_{m-2}} \\ &\quad \times \Big(\ln|x-\xi| - \sum_{k=1}^{m-1-n/2}\frac{1}{2k} - \sum_{k=n/2}^{m-2}\frac{1}{2k}\Big) = -E_{2m-2}(x,\xi).\end{aligned}$$

In the case $n = 2m$ and $m > 1$ we have $n \in \mathbb{N}_m^c \setminus \mathbb{N}_{m-1}^c$, i.e. $n \in \mathbb{N}_{m-1}$ and then (see case $\alpha = 0$ above)

$$\Delta_\xi E_{2m}(x,\xi) = \Delta_\xi\Big(\frac{(-1)^m|x-\xi|^0}{(2-n,2)^*_m(2,2)_{m-1}}\ln|x-\xi|\Big)$$
$$= \frac{(-1)^m(n-2)|x-\xi|^{-2}}{(2-n,2)_{m-1}(2,2)_{m-1}} = -\frac{(-1)^{m-1}|x-\xi|^{-2}}{(2-n,2)_{m-1}(2,2)_{m-2}} = -E_{2m-2}(x,\xi),$$

since $(2,2)_{m-1} = 2\cdot 4\ldots(2m-2) = (2,2)_{m-2}(n-2)$.

3^0. The remaining case is $m = 1$. From Lemma 4.1 it follows that $E_2(x,\xi) = E(x,\xi)$, and therefore we have $\Delta_\xi E_2(x,\xi) = 0$ for $x \neq \xi$. The lemma is proved. □

4.2.2 Integral representation

The formula for representing the harmonic function $u(x)$ in a bounded domain $D \subset \mathbb{R}^n$ with a piecewise smooth boundary ∂D is well known [183, c.361]. In the notation made, it has the form

$$u(x) = \frac{1}{\omega_n}\int_{\partial D}\Big(E_2(x,\xi)\frac{\partial u}{\partial\nu} - \frac{\partial E_2(x,\xi)}{\partial\nu}u\Big)\,ds_\xi.$$

where $\omega_n = |\partial S|$ is the area of the unit sphere in $\mathbb{R}^n$, ν is the outer unit normal to ∂D. In addition, we can recall the integral representation of a function from the class $C^4(D)\cap C^3(\bar D)$ from Theorem 4.1. Let us present a similar integral representation of a function from the class $u \in C^{2m}(D)\cap C^{2m-1}(\bar D)$, where $D\subset\mathbb{R}^n$ is a bounded domain with piecewise smooth boundary ∂D.

Theorem 4.8. *The function $u \in C^{2m}(D)\cap C^{2m-1}(\bar D)$ has the following integral representation*

$$u(x) = \frac{1}{\omega_n}\int_{\partial D}\sum_{k=0}^{m-1}(-1)^k\Big(E_{2k+2}(x,\xi)\frac{\partial\Delta^k u}{\partial\nu} - \frac{\partial E_{2k+2}(x,\xi)}{\partial\nu}\Delta^k u\Big)\,ds_\xi$$
$$+\frac{(-1)^m}{\omega_n}\int_D E_{2m}(x,\xi)\Delta^{2m}u(\xi)\,d\xi, \quad (4.2.5)$$

where $\omega_n = |\partial S|$ is the area of the unit sphere in $\mathbb{R}^n$, ν is the outer unit normal to ∂D.

Proof. Consider the functions $u, v \in C^{2m}(D)\cap C^{2m-1}(\bar D)$ and use the equality (4.1.4)

$$\int_D(v\Delta u - u\Delta v)\,d\xi = \int_{\partial D}\Big(v\frac{\partial u}{\partial\nu} - \frac{\partial v}{\partial\nu}u\Big)\,ds_\xi,$$

where ν is the outer unit normal to ∂D. Substituting here $u = \Delta^{m-k-1}u$, as well as $v = \Delta^k v$, where $k = 0, \dots, m-1$ we get

$$\int_D \left(\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u\right) d\xi$$
$$= \int_{\partial D} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u\right) ds_\xi.$$

Let us sum the resulting equality over $k = 0, \dots, m-1$

$$\int_D \sum_{k=0}^{m-1} \left(\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u\right) d\xi$$
$$= \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u\right) ds_\xi.$$

Since

$$\sum_{k=0}^{m-1} \left(\Delta^k v \Delta^{m-k} u - \Delta^{k+1} v \Delta^{m-k-1} u\right) = \sum_{k=0}^{m-1} \Delta^k v \Delta^{m-k} u$$
$$- \sum_{k=0}^{m-1} \Delta^{k+1} v \Delta^{m-k-1} u = \sum_{k=0}^{m-1} \Delta^k v \Delta^{m-k} u - \sum_{k=1}^{m} \Delta^k v \Delta^{m-k} u = v\Delta^m u - u\Delta^m v,$$

then we get

$$\int_D \left(v\Delta^m u - u\Delta^m v\right) d\xi$$
$$= \int_{\partial D} \sum_{k=0}^{m-1} \left(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u\right) ds_\xi. \quad (4.2.6)$$

Obviously, the resulting equality is correct for $u, v \in C^{2m}(D) \cap C^{2m-1}(\bar{D})$. Let us select from the domain D a point $x \in D$ together with a closed ball $|\xi - x| \leq \delta$ of such radius $0 < \delta < 1$ that this ball is located in the domain D. We denote the remaining part of the domain D by D_δ. It is obvious that by construction $\partial D_\delta = \partial D \cup \{\xi \in \mathbb{R}^n : |\xi - x| = \delta\}$. Let us apply the equality (4.2.6) to the domain $D = D_\delta$ in the case where $v(\xi) = E_{2m}(x, \xi)$. At the same time, we take into account that by virtue of Lemma 4.2 $\Delta_\xi^2 E_{2m}(x, \xi) = -E_{2m-2}(x, \xi)$ and $\Delta_\xi E_2(x, \xi) = 0$ when $\xi \in D_\delta$

$$\begin{aligned}
&\int_{D_\delta} E_{2m}(x,\xi)\Delta^{2m}u(\xi)\,d\xi \\
&\quad= \int_{\partial D}\sum_{k=0}^{m-1}(-1)^k\Big(E_{2m-2k}(x,\xi)\frac{\partial\Delta^{m-k-1}u}{\partial\nu}-\frac{\partial E_{2m-2k}(x,\xi)}{\partial\nu}\Delta^{m-k-1}u\Big)\,ds_\xi \\
&\qquad\qquad -\sum_{k=0}^{m-1}(-1)^k\Big(\int_{|\xi-x|=\delta}E_{2m-2k}(x,\xi)\frac{\partial\Delta^{m-k-1}u}{\partial\nu}\,ds_\xi \\
&\qquad\qquad\qquad -\int_{|\xi-x|=\delta}\frac{\partial E_{2m-2k}(x,\xi)}{\partial\nu}\Delta^{m-k-1}u\,ds_\xi\Big). \qquad (4.2.7)
\end{aligned}$$

The minus sign in front of the last sum arose due to the fact that the internal normal ν to a part of the boundary of the domain D_δ – the sphere $\{\xi : |\xi - x| = \delta\}$ is replaced by outer normal to this sphere. Let us denote

$$I_k^\delta = (-1)^{k+1}\int_{|\xi-x|=\delta}E_{2m-2k}(x,\xi)\frac{\partial\Delta^{m-k-1}u}{\partial\nu}\,ds_\xi,$$
$$J_k^\delta = (-1)^{k}\int_{|\xi-x|=\delta}\frac{\partial E_{2m-2k}(x,\xi)}{\partial\nu}\Delta^{m-k-1}u\,ds_\xi,$$

where $k = 0,\dots,m-1$. Let us estimate the dependence of the quantities I_k^δ and J_k^δ on δ.

1^0. Let $n \in \mathbb{N}_m$. According to the equality (4.2.1) we have

$$E_{2m}(x,\xi)|_{|x-\xi|=\delta} = \frac{(-1)^m\delta^{2m-n}}{(2-n,2)_m(2,2)_{m-1}}$$

and since $n \in \mathbb{N}_m \Rightarrow n \in \mathbb{N}_{2m-2k}$ for $k = 0,\dots,m-1$ we find

$$\begin{aligned}
|I_k^\delta| &\le \int_{|\xi-x|=\delta}|E_{2m-2k}(x,\xi)|\Big|\frac{\partial\Delta^{m-k-1}u}{\partial\nu}\Big|\,ds_\xi \le \\
&\le \delta^{2m-2k-n}\frac{\max_{\xi\in\bar D}\|\nabla\Delta^{m-k-1}u\|}{|(2-n,2)_m|(2,2)_{m-1}}\omega_n\delta^{n-1} \le C_k^{(1)}\delta^{2m-2k-1} \le C_k^{(1)}\delta, \quad (4.2.8)
\end{aligned}$$

where $C_k^{(1)}$ is a constant depending on u, m, and n. Therefore $I_k^\delta \to 0$ at $\delta \to 0$.

Next, consider the value of J_k^δ at $k = 0,\dots,m-2$. It is not hard to see that

$$\frac{\partial|x-\xi|^\alpha}{\partial\nu}\Big|_{|x-\xi|=\delta} = \alpha\sum_{i=1}^n\frac{\xi_i-x_i}{|x-\xi|}(\xi_i-x_i)|x-\xi|^{\alpha-2}\Big|_{|x-\xi|=\delta} = \alpha\delta^{\alpha-1}$$

and therefore

$$|J_k^\delta| \le \int_{|\xi-x|=\delta} \left|\frac{\partial E_{2m-2k}(x,\xi)}{\partial \nu}\right| |\Delta^{m-k-1}u|\, ds_\xi \le \delta^{2m-2k-n-1}\omega_n\delta^{n-1}$$
$$\times \frac{|2m-2k-n| \max_{\xi\in\bar D}|\Delta^{m-k-1}u|}{|(2-n,2)_m|(2,2)_{m-1}} \le C_k^{(2)}\delta^{2m-2k-2} \le C_k^{(2)}\delta^2, \quad (4.2.9)$$

whence it follows that $J_k^\delta \to 0$ at $\delta \to 0$.

Now consider the value J_{m-1}^δ. In this case $E_{2m-2(m-1)}(x,\xi) = E_2(x,\xi)$ ($n \ne 2$ since $n \in \mathbb{N}_m$) and that means

$$\frac{\partial E_2(x,\xi)}{\partial \nu}\Big|_{|x-\xi|=\delta} = -\sum_{i=1}^n \frac{\xi_i - x_i}{|\xi - x|}\frac{\xi_i - x_i}{|\xi - x|^n} = -\delta^{1-n}. \quad (4.2.10)$$

Therefore we have

$$\left|J_{m-1}^\delta - (-1)^{m-1}u(x)\int_{|\xi-x|=\delta}\frac{\partial E_2(x,\xi)}{\partial\nu}\, ds_\xi\right|$$
$$\le \int_{|\xi-x|=\delta}|u(\xi)-u(x)|\left|\frac{\partial E_2(x,\xi)}{\partial\nu}\right| ds_\xi \le \delta^{1-n}\max_{|\xi-x|\le\delta}|u(\xi)-u(x)|\omega_n\delta^{n-1}$$
$$= \omega_f(\delta)\omega_n,$$

where $\omega_f(\delta)$ is the modulus of continuity of the function f. By Cantor's theorem $\omega_f(\delta) \to 0$ for $\delta \to 0$, which means

$$J_{m-1}^\delta = \left(J_{m-1}^\delta - (-1)^{m-1}u(x)\int_{|\xi-x|=\delta}\frac{\partial E_2(x,\xi)}{\partial\nu}\, ds_\xi\right)$$
$$+ (-1)^m u(x)\frac{\omega_n\delta^{n-1}}{\delta^{n-1}} \to (-1)^m\omega_n u(x), \quad (4.2.11)$$

at $\delta \to 0$.

2^0. Let $n \in \mathbb{N}_m^c$. According to the equality (4.2.1) we have

$$E_{2m}(x,\xi) = \frac{(-1)^m|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}}\left(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} - \sum_{k=n/2}^{m-1}\frac{1}{2k}\right).$$

Let us estimate the value of I_k^δ. If it turns out that for a given k the inclusion $n \in \mathbb{N}_{2m-2k}$ holds, then we obtain the estimate (4.2.8), otherwise

$$|I_k^\delta| \leq \int_{|\xi-x|=\delta} |E_{2m-2k}(x,\xi)| \Big|\frac{\partial \Delta^{m-k-1}u}{\partial \nu}\Big| \, ds_\xi \leq \delta^{2m-2k-n}$$
$$\times \frac{\max_{\xi\in\bar{D}} \|\nabla\Delta^{m-k-1}u\|}{|(2-n,2)_m^*|(2,2)_{m-1}} \omega_n \delta^{n-1} \leq C_k^{(3)} \delta^{2m-2k-1} \leq C_k^{(3)}\delta\Big|\ln\delta - \sum_{k=1}^{m-1}\frac{1}{k}\Big|.$$

In both cases $I_k^\delta \to 0$ at $\delta \to 0$, where $k = 0,\ldots,m-1$.

Next, consider the value of J_k^δ at $k = 0,\ldots,m-1$. In accordance with (4.2.4) we write

$$\frac{\partial}{\partial\nu}\Big(|x-\xi|^\alpha\big(\ln|x-\xi|+C\big)\Big)\Big|_{|x-\xi|=\delta}$$
$$= \sum_{i=1}^n \frac{(\xi_i-x_i)^2}{|x-\xi|}\big(\alpha\ln|x-\xi|+\alpha C+1\big)|x-\xi|^{\alpha-2}\Big|_{|x-\xi|=\delta} = \delta^{\alpha-1}(\alpha\ln\delta+\alpha C+1).$$

If $k \leq m-2$ is such that $n \in \mathbb{N}_{2m-2k}$, then we have the estimate (4.2.9), otherwise $n \in \mathbb{N}^c_{2m-2k}$ and then

$$|J_k^\delta| \leq \int_{|\xi-x|=\delta} \Big|\frac{\partial E_{2m-2k}(x,\xi)}{\partial\nu}\Big| |\Delta^{m-k-1}u| \, ds_\xi$$
$$\leq \delta^{2m-2k-n-1} C_k^{(4)} |\ln\delta| \max_{\xi\in\bar{D}} |\Delta^{m-k-1}u| \omega_n \delta^{n-1}$$
$$\leq C_k^{(5)}\delta^{2m-2k-2}|\ln\delta| \leq C_k^{(5)}\delta^2|\ln\delta|$$

for small δ. It follows that $J_k^\delta \to 0$ at $\delta \to 0$.

Finally, consider the value of J_{m-1}^δ when $n \in \mathbb{N}^*_{2m-2m+2} = \mathbb{N}^*_2 = \{2\}$. In this case $E_{2m-2(m-1)}(x,\xi) = E_2(x,\xi) = -\ln|x-\xi|$ and therefore, in accordance with (4.2.4) for $\alpha = 0$,

$$\frac{\partial E_2(x,\xi)}{\partial\nu}\Big|_{|x-\xi|=\delta} = -\sum_{i=1}^2 \frac{\xi_i-x_i}{|\xi-x|}\frac{\xi_i-x_i}{|\xi-x|^2} = -\delta^{-1} = -\delta^{1-n},$$

which is the same as (4.2.10). Therefore, the derivation of the equality (4.2.11), which is based on the written equality, is similar, and therefore $J_{m-1}^\delta \to (-1)^m\omega_n u(x)$ at $\delta \to 0$ in this case too.

Thus, in all considered cases $I_k^\delta \to 0$, $k = 0,1,\ldots,m-1$, $J_k^\delta \to 0$ $k = 0,1,\ldots,m-2$, and $J_{m-1}^\delta \to (-1)^m\omega_n u(x)$ at $\delta \to 0$. Taking these remarks into account, let us pass in the equality (4.2.7) to the limit at $\delta \to 0$. Since $2m-n \geq 2-n$, the singularity in the volume integral on the left is integrable. Therefore the equality holds true

$$\lim_{\delta\to 0}\int_{D_\delta} E_{2m}(x,\xi)f(\xi)\,d\xi = \int_D E_{2m}(x,\xi)f(\xi)\,d\xi$$

and therefore

$$\int_D E_{2m}(x,\xi)\Delta^{2m}u(\xi)\,d\xi = \int_{\partial D}\sum_{k=0}^{m-1}(-1)^k\Big(E_{2m-2k}(x,\xi)\frac{\partial\Delta^{m-k-1}u}{\partial\nu} - \frac{\partial E_{2m-2k}(x,\xi)}{\partial\nu}\Delta^{m-k-1}u\Big)\,ds_\xi + (-1)^m\omega_n u(x). \quad (4.2.12)$$

Transferring all the integrals from the right side of the resulting equality to the left side, dividing both sides by $(-1)^m\omega_n$ and changing the summation index $k \to m-k-1$, we obtain the equality (4.2.5). The theorem is proved. □

Example 4.1. For $m = 2$, from (4.2.5) we obtain a representation of the solution to the inhomogeneous biharmonic equation

$$u(x) = \frac{1}{\omega_n}\int_{\partial D}\Big(E_2(x,\xi)\frac{\partial u}{\partial\nu}(\xi) - \frac{\partial E_2(x,\xi)}{\partial\nu}u(\xi)\Big)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial D}\Big(E_4(x,\xi)\frac{\partial\Delta u}{\partial\nu} - \frac{\partial E_4(x,\xi)}{\partial\nu}\Delta u\Big)\,ds_\xi + \frac{1}{\omega_n}\int_D E_4(x,\xi)\Delta^2 u(\xi)\,d\xi,$$

which coincides, up to simple transformations, with that obtained in Theorem 4.1.

4.3 Riquier-Neumann boundary value problem

In this section, necessary and sufficient conditions for the solvability of the Riquier-Neumann boundary value problem for the inhomogeneous polyharmonic equation in the unit ball are obtained. First, in Theorem 4.9, using Lemmas 4.3 and 4.4, the solvability of the Riquier-Neumann problem for the homogeneous polyharmonic equation is investigated. Quite cumbersome formulas are given (see, for example, the equality (4.3.20)) that allow one to find a solution to the Riquier-Neumann problem. Further, in Theorem 4.10, based on Theorem 4.9, necessary and sufficient conditions for the solvability of the Riquier-Neumann problem for the inhomogeneous polyharmonic equation in the unit ball are obtained. The main results of Section 4.3 are published in [101].

Let S be the unit ball in $\mathbb{R}^n$. Consider in S the following Riquier-Neumann boundary value problem for the inhomogeneous polyharmonic equation

$$\Delta^m u = f(x), \quad x \in S; \tag{4.3.1}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_0, \quad \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = \varphi_1, \ldots, \frac{\partial \Delta^{m-1} u}{\partial \nu}\Big|_{\partial S} = \varphi_{m-1}, \tag{4.3.2}$$

where $\dfrac{\partial}{\partial \nu}$ is the outer normal derivative to the unit sphere, $f(x)$ and $\varphi_i(s)$ for $i = 0, \ldots, m-1$ are given functions on S and ∂S, respectively. This problem generalizes the well-known Navier problem [47, p. 33], which is also called the Requier problem [68]. For the biharmonic equation, such a problem is a special case of a more general problem studied in Section 3.5. Riquier-Neumann boundary conditions arise, for example, in a boundary value problem when modeling the displacements of a flat plate.

By a solution to the problem (4.3.1)-(4.3.2) we mean a solution $u(x)$ of the equation $\Delta^m u = f(x)$ in S for which $\nu \cdot \nabla u(x) \to \varphi_0(s)$, $\nu \cdot \nabla \Delta u(x) \to \varphi_1(s)$, $\ldots, \nu \cdot \nabla \Delta^{m-1} u(x) \to \varphi_{m-1}(s)$ for $x \to \partial S$, where $(-\nu)$ is the internal normal at point $s \in \partial S$ passing through the point $x \in S$.

4.3.1 Homogeneous equation

Let us consider the problem (4.3.1)-(4.3.2) for the homogeneous polyharmonic equation.

Theorem 4.9. *Let $f(x) = 0$ and $\varphi_i \in C(\partial S)$ for $i = 0, 1, \ldots, m-1$. Then for the existence of a solution to the Riquier-Neumann problem* (4.3.1)-(4.3.2) *for the homogeneous polyharmonic equation* (4.3.1) *it is necessary and sufficient that the condition be satisfied*

$$\int_{\partial S} \varphi_{m-1}(x)\, ds_x = 0. \tag{4.3.3}$$

The solution to the problem under consideration is unique up to an arbitrary constant.

Proof. Necessity. Let the problem have a solution. It is known that any polyharmonic function in S can be represented using the Almansi formula in the form $u(x) = u_0(x) + |x|^2 u_1(x) + \ldots + |x|^{2m-2} u_{m-1}(x)$, where $u_j(x)$ for $j = 0, 1, \ldots, m-1$ are functions harmonic in S (see, for example, [162, p.531] or [71]). Let $v(x)$ be some function harmonic in S. It is easy to verify that the equalities hold true

$$\Delta(|x|^{2k}v(x)) = \Delta(|x|^{2k})v(x) + 2\sum_{i=1}^{n} 2k|x|^{2k-2}x_i D_{x_i}v(x) + |x|^{2k}\Delta v(x)$$
$$= 2k(2k+n-2)|x|^{2k-2} + 4k|x|^{2k-2}\Lambda v(x) = 2k|x|^{2k-2}(2k+n-2+2\Lambda)v(x)$$

since

$$\Delta|x|^{2k} = \sum_{i=1}^{n} D_{x_i}^2(x_1^2+\ldots+x_n^2)^k = \sum_{i=1}^{n} D_{x_i}2kx_i(x_1^2+\ldots+x_n^2)^{k-1}$$
$$= 2k\sum_{i=1}^{n}[(x_1^2+\ldots+x_n^2)^{k-1}+(2k-2)x_i^2(x_1^2+\ldots+x_n^2)^{k-2}] = 2k(2k+n-2)|x|^{2k-2}.$$

Therefore we have

$$\Delta u(x) = \sum_{j=0}^{m-1}\Delta(|x|^{2j}u_j(x)) = \sum_{j=1}^{m-1}2j|x|^{2j-2}(2j-2+n+2\Lambda)u_j(x),$$

whence, taking into account that all functions $\Lambda u_j(x)$ for $j = 0,1,\ldots,m-1$ are harmonic in S, we find

$$\Delta^2 u(x) = \sum_{j=2}^{m-1}2j(2j-2)|x|^{2j-4}(2j-2+n+2\Lambda)(2j-4+n+2\Lambda)u_j(x).$$

Similarly calculate

$$\Delta^{2i}u(x) = \sum_{j=i}^{m-1}|x|^{2j-2i}\frac{(2j)!!}{(2j-2i)!!}\prod_{k=1}^{i}(2j-2k+n+2\Lambda)u_j(x), \tag{4.3.4}$$

where, in the case when the superscript in a product becomes less than the lower index, then the product should be considered equal to 1.

Let us consider the boundary conditions (4.3.2). Let ν be the outer normal to ∂S. Since the internal normal to ∂S passing through the point $x \in S$ has the form $-\nu = -x/|x|$, then

$$\nu\cdot\nabla u(x)\big|_{\partial S} = \sum_{i=1}^{n}\frac{x_i}{|x|}D_{x_i}u\big|_{\partial S} = \Lambda u\big|_{\partial S}$$

and therefore the boundary conditions (4.3.2) can be rewritten in the form

$$\Lambda u\big|_{\partial S} = \varphi_0, \quad \Lambda\Delta u\big|_{\partial S} = \varphi_1, \dots, \Lambda\Delta^{m-1}u\big|_{\partial S} = \varphi_{m-1}. \qquad (4.3.5)$$

Let $v_i(x)$ be functions harmonic in S such that $v_i\big|_{\partial S} = \varphi_i$ for $i = 0, \dots, m-1$, i.e. $v_i(x)$ is a solution to the corresponding Dirichlet problem for the Laplace equation in S. Then

$$(\Lambda\Delta^{2i}u - v_i)\big|_{\partial S} = 0, \quad i = 0, \dots, m-1. \qquad (4.3.6)$$

Let $v(x)$ again be some harmonic function in S. Since the operator Λ is a linear homogeneous differential operator of the first order, then

$$\Lambda(|x|^{2i}v(x)) = |x|^{2i}(\Lambda + 2i)v(x).$$

Therefore, recalling the formula (4.3.4), let us transform (4.3.6) to the form

$$\sum_{j=i}^{m-1}\Big(|x|^{2j-2i}\frac{(2j)!!}{(2j-2i)!!}(\Lambda + 2j - 2i)\prod_{k=1}^{i}(2j - 2k + n + 2\Lambda)u_j(x)\Big) - v_i\Big|_{\partial S} = 0,$$

where $i = 0, \dots, m-1$. From here it immediately follows

$$\Big(\sum_{j=i}^{m-1}\frac{(2j)!!}{(2j-2i)!!}(\Lambda + 2j - 2i)\prod_{k=1}^{i}(2j - 2k + n + 2\Lambda)u_j(x) - v_i\Big)\Big|_{\partial S} = 0. \qquad (4.3.7)$$

Since inside the outer parentheses in (4.3.7) there are functions harmonic in S that are continuous up to the boundary, then by virtue of the uniqueness theorem for the solution of the Dirichlet problem in S (Theorem 2.16) we obtain a system of m differential equations for functions $u_j(x)$, $j = 0, 1, \dots, m-1$, harmonic in S with an upper triangular operator matrix

$$\sum_{j=i}^{m-1}\frac{(2j)!!}{(2j-2i)!!}(\Lambda + 2j - 2i)\prod_{k=1}^{i}(2j - 2k + n + 2\Lambda)u_j(x) = v_i(x), \qquad (4.3.8)$$

where $i = 0, 1, \dots, m-1$, $x \in \overline{S}$, with a right-hand side that is harmonic in S and continuous in $\overline{S}$. In particular, for $i = 0$, $i = 1$ and $i = m-1$ the equations of the system (4.3.8) have the form

$$\Lambda u_0(x) + (\Lambda + 2)u_1(x) + (\Lambda + 4)u_2(x) + \dots + (\Lambda + 2m - 2)u_{m-1}(x) = v_0(x),$$
$$2\Lambda(n + 2\Lambda)u_1(x) + 4(\Lambda + 2)(n + 2 + 2\Lambda)u_2(x) + \dots$$

$$+(2m-2)(\Lambda+2m-4)(2m-4+n+2\Lambda)u_{m-1}(x)=v_1(x),$$

$$\dots$$

$$(2m-2)!!\Lambda\prod_{k=1}^{m-1}(2m-2k-2+n+2\Lambda)u_{m-1}(x)=v_{m-1}(x). \qquad (4.3.9)$$

The system (4.3.8) can be rewritten in matrix form

$$A(\Lambda)U(x)=V(x), \quad x\in\overline{S}, \qquad (4.3.10)$$

where is denoted

$$A(\Lambda)=\begin{pmatrix}\Lambda & \Lambda+2 & \Lambda+4 & \dots\\ 0 & 2\Lambda(n+2\Lambda) & 4(\Lambda+2)(n+2+2\Lambda) & \dots\\ 0 & 0 & 8\Lambda(n+2\Lambda)(n+2+2\Lambda) & \dots\\ & & \dots & \end{pmatrix}$$

and $U=(u_0,u_1,u_2,\dots,u_{m-1})^T$, $V=(v_0,v_1,v_2,\dots,v_{m-1})^T$. The coefficients of the matrix operator $A(\Lambda)=(\lambda_{ij}(\Lambda))_{i,j=0}^{m-1}$ have the form

$$\lambda_{ij}(\Lambda)=\frac{4^i j!}{(j-i)!}(\Lambda+2j-2i)\prod_{k=1}^{i}(\Lambda+j-k+n/2), \qquad (4.3.11)$$

for $j\ge i$ and $\lambda_{ij}(\Lambda)=0$ for $j<i$. So, every solution $u(x)=u_0(x)+|x|^2u_1(x)+\dots+|x|^{2m-2}u_{m-1}(x)$ of the problem (4.3.1)-(4.3.2) for $f=0$ generates a solution to the system of equations (4.3.10). The opposite statement is also true, i.e. if $U(x)=(u_0(x),u_1(x),\dots,u_{m-1}(x))^T$ is a solution to the system of equations (4.3.10), then the polyharmonic function $u(x)=u_0(x)+|x|^2u_1(x)+\dots+|x|^{2m-2}u_{m-1}(x)$ satisfy the conditions (4.3.7), and therefore the conditions (4.3.6) and (4.3.5) and consequently the boundary conditions (4.3.2) are satisfied.

Let us solve the system of equations (4.3.8). We consider the last equation (4.3.9) of this system. Let us denote in it

$$w(x)=(2m-2)!!\prod_{k=1}^{m-1}(2m-2k-2+n+2\Lambda)u_{m-1}(x).$$

Then we have the equation $\Lambda w(x)=v_{m-1}(x)$ in $\overline{S}$, in which $w(x)$ and $v_{m-1}(x)$ are harmonic functions in S. This equation has a solution if and only if $v_{m-1}(0)=0$ and it is unique up to a constant. Indeed, if $v_{m-1}(x)$ and $w(x)$ are harmonic functions in S, then in the neighborhood of the origin they have the form

$$w(x) = w(0) + \sum_{i=1}^{n} w_i x_i + O(|x|^2), \quad v_{m-1}(x) = v_{m-1}(0) + \sum_{i=1}^{n} v_{m-1}^i x_i + O(|x|^2).$$

Therefore, in the vicinity of the origin the equality must be satisfied

$$\sum_{i=1}^{n} w_i \cdot 1 \cdot x_i + O(|x|^2) = v_{m-1}(0) + \sum_{i=1}^{n} v_{m-1}^i x_i + O(|x|^2).$$

Assuming $x = 0$ in it, we obtain $v_{m-1}(0) = 0$. A necessary condition for the existence of a solution to the equation $\Lambda w(x) = v_{m-1}(x)$ is found. The sufficiency of the obtained condition for the equation $\Lambda w(x) = v_{m-1}(x)$ follows from the representation of the solution to this equation in the form (see the proof of Theorem 3.12 on page 206 or (3.5.18))

$$w(x) = M_0 v_{m-1}(x) + C \equiv \int_0^1 v_{m-1}(tx)\,\frac{dt}{t} + C, \tag{4.3.12}$$

which is true if $v_{m-1}(0) = 0$, i.e. if the integral converges. It is easy to verify that under the condition $v_{m-1}(0) = 0$ the equality holds true in S

$$\Lambda M_0 v_{m-1}(x) = \Lambda \int_0^1 v_{m-1}(tx)\,\frac{dt}{t} = \int_0^1 (v_{m-1}(tx))'_t\, dt = v_{m-1}(x).$$

Note that $M_0 v_{m-1}(x)|_{x=0} = 0$ and therefore the operator M_0 can be applied to the function $v_{m-1}(x)$ several times. So, the necessary condition for the existence of a solution to the system (4.3.8), and therefore to the problem (4.3.1)-(4.3.2) for $f = 0$ has the form $v_{m-1}(0) = 0$.

Let us prove the sufficiency of the condition $v_{m-1}(0) = 0$. We construct a solution to the system (4.3.10). Let, first, the right side of the system (4.3.10) be a constant vector.

Lemma 4.3. *A solution to the system* (4.3.10) *with a constant right-hand side* $V(x) = V_0$ *exists if and only if the last component of the vector* V_0 *is equal to zero. This solution is unique up to the vector* $(C, 0, \dots, 0)^T$.

Proof. Let $V_0 = (v_0, \dots, v_{m-1})^T$. Then, according to what was said before the lemma, the condition $v_{m-1} = 0$ must be satisfied. Suppose that a solution $U(x) =$

$(u_0(x), u_1(x), \dots, u_{m-1}(x))^T$ of the system (4.3.10) exists. According to Theorem 1.8 on page 50 in some neighborhood of the origin this solution can be represented in the form

$$U(x) = \sum_\nu U_\nu G_\nu(x),$$

where $G_\nu(x)$ (see Theorem 1.7) is a complete system of homogeneous degree ν_1 harmonic polynomials orthogonal on ∂S, $\nu = (\nu_1, \dots, \nu_n)$, $\nu_k \geq \nu_{k+1} \geq 0$, and U_ν is a vector of coefficients and the series can be differentiated under the sum sign. Since the operator Λ is homogeneous, the equality $A(\Lambda)U_\alpha x^\alpha = A(\alpha)U_\alpha x^\alpha$ holds true. Therefore, the system (4.3.10) takes the form

$$\sum_\nu A(\nu_1)U_\nu G_\nu(x) = V_0$$

and therefore the coefficients U_ν of its solution must satisfy the equalities $A(\nu_1)U_\nu = 0$ for all $\nu \neq 0$ and $A(0)U_0 = V_0$, where $U_0 = U_{(0,\dots,0)} \equiv (u_0, u_1, \dots, u_{m-1})^T$. From (4.3.11) we find

$$\lambda_{ii}(\nu_1) = \nu_1 4^i i! \prod_{k=1}^{i} (i - k + n/2 + \nu_1)$$

and therefore $\lambda_{ii}(\nu_1) > 0$ for $\nu_1 > 0$. It follows that for $\nu_1 > 0$

$$\det A(\nu_1) = \prod_{i=0}^{m-1} \lambda_{ii}(\nu_1) > 0$$

and therefore, for $\nu_1 > 0$, the system of algebraic equations $A(\nu_1)U_\nu = 0$ has only the zero solution $U_\nu = 0$. Let us study the case $\nu = 0$. We have the equation

$$A(0)U_0 = V_0. \tag{4.3.13}$$

The coefficients of the matrix $A(0)$ have the form

$$\lambda_{ij}(0) = (2j - 2i)\frac{4^i j!}{(j-i)!} \prod_{k=1}^{i} (j - k + n/2)$$

for $j \geq i$ and $\lambda_{ij}(0) = 0$ for $j < i$. It can be seen that all coefficients $\lambda_{ij}(0)$ located above the main diagonal ($j > i$) are positive, and all other coefficients are zero. Therefore, the last equation of the system of algebraic equations (4.3.13) is trivial $0 = 0$ and this system is equivalent to the quadratic $(m-1) \times (m-1)$ system of equations $A^* U_0^* = V_0^*$ with truncated unknowns $U_0^* = (u_1, u_2, \dots, u_{m-1})^T$ and the truncated right-hand side $V_0^* = (v_0, v_1, \ldots, v_{m-2})^T$. If we find this solution, then the vector U_0 can be obtained from U_0^* by adding to it the 1th arbitrary component $u_0 = C$. The coefficients of the new matrix A^* have the form

$$\lambda_{ij}^* = (2j - 2i)\frac{4^i j!}{(j-i)!}\prod_{k=1}^{i}(j - k + n/2),$$

for $i = 0, \ldots, m-2$, $j = i+1, \ldots, m-1$, and all other coefficients are zero. It follows that

$$\det A^* = \prod_{i=0}^{m-2}\lambda_{ii+1}^* = 2^{m-1}\prod_{i=0}^{m-2}4^i(i+1)!\prod_{k=1}^{i}(i + 1 - k + n/2) > 0.$$

This means $U_0^* = (A^*)^{-1}V_0^*$ and therefore $U(x) = U_0 = (C, U_0^*)$ is the desired solution to the system (4.3.10). The lemma is proved. □

Let us represent the right-hand side of the system (4.3.10) in the form $V(x) = V(0) + (V(x) - V(0)) \equiv V(0) + V^*(x)$ and search for its solution in the form $U(x) = U_0 + U^*(x)$. Then instead of (4.3.10) we get two systems

$$A(\Lambda)U_0 = V(0), \quad A(\Lambda)U^*(x) = V^*(x), \quad x \in \overline{S}, \tag{4.3.14}$$

where the vector $V^*(x)$ has the property $V^*(0) = 0$. We determine the solution to the first system U_0 using Lemma 4.3. The condition of this lemma is satisfied, since $v_{m-1}(0) = 0$. Let us solve the second system from (4.3.14).

Lemma 4.4. *A solution $U^*(x)$ to the second system from* (4.3.14) *exists for any harmonic in S and continuous in $\overline{S}$ function $V^*(x)$, having the property $V^*(0) = 0$.*

Proof. Consider the operator of the form (3.5.18)

$$M_\lambda v = \int_0^1 v(tx)t^{\lambda-1}\,dt,$$

which acts on functions $v(x)$ harmonic in S and continuous in $\overline{S}$ and defined for $\lambda > 0$. It is obvious that the operators M_λ and M_μ commute

$$\begin{aligned} M_\lambda M_\mu v(x) &= M_\lambda \int_0^1 v(tx)t^{\mu-1}\,dt = \int_0^1 \tau^{\lambda-1}\int_0^1 v(t\tau x)t^{\mu-1}\,dt\,d\tau \\ &= \int_0^1 t^{\mu-1}\int_0^1 v(\tau t x)\tau^{\lambda-1}\,d\tau\,dt = M_\mu M_\lambda v(x). \end{aligned} \tag{4.3.15}$$

Note that if $v(0) = 0$, then $M_\lambda v(x)|_{x=0} = 0$. For the operator M_λ for $\lambda > 0$ the following equalities hold true

$$\begin{aligned}(\Lambda+\lambda)M_\lambda v(x) &= (\Lambda+\lambda)\int_0^1 v(tx)t^{\lambda-1}\,dt \\ &= \int_0^1 \sum_{i=1}^n x_i v_{x_i}(tx)t^\lambda\,dt + \lambda M_\lambda v(x) = \int_0^1 (v(tx))'_t t^\lambda\,dt + \lambda M_\lambda v(x) \\ &= v(tx)t^\lambda\Big|_0^1 - \lambda\int_0^1 v(tx)t^{\lambda-1}\,dt + \lambda M_\lambda v(x) = v(x). \end{aligned} \tag{4.3.16}$$

For $\lambda > 0$ the inverse equality $M_\lambda(\Lambda+\lambda)v(x) = v(x)$ is also true since

$$\begin{aligned}M_\lambda \Lambda v(x) &= \int_0^1 \sum_{i=1}^n t x_i v_{x_i}(tx)t^{\lambda-1}\,dt = \int_0^1 (v(tx))'_t t^\lambda\,dt \\ &= v(tx)t^\lambda\Big|_0^1 - \lambda M_\lambda v = v - \lambda M_\lambda v.\end{aligned}$$

Moreover, the operators M_λ and $\Lambda+\mu$ also commute for $\lambda,\mu > 0$

$$\begin{aligned}M_\lambda(\Lambda+\mu)v(x) &= v(x) + (\mu-\lambda)M_\lambda v(x) \\ &= (\Lambda+\lambda)M_\lambda v(x) + (\mu-\lambda)M_\lambda v(x) = (\Lambda+\mu)M_\lambda v(x).\end{aligned} \tag{4.3.17}$$

For $\lambda = 0$ or $\mu = 0$ these equalities are also satisfied, but only if $v(0) = 0$.

Let $\mu > 0$, then $\det A(\mu) > 0$. We denote by $A^*(\mu)$ a matrix having the property $A(\mu)A^*(\mu) = \det A(\mu)$ and consider the matrix

$$B(\mu) = \frac{1}{\det A(\mu)}A^*(\mu).$$

It obviously has the property

$$A(\mu)B(\mu) = \frac{1}{\det A(\mu)}A(\mu)A^*(\mu) = E. \tag{4.3.18}$$

From (4.3.11) we find

$$\det A(\mu) = \prod_{i=0}^{m-1}\lambda_{ii}(\mu) = \mu^m \prod_{i=0}^{m-1} 4^i i! \prod_{k=1}^{i}(\mu + i - k + n/2)$$

$$= a_m \mu^m (\mu + n/2)^{m-1} (\mu + n/2 + 1)^{m-2} \cdot \ldots \cdot (\mu + n/2 + m - 2),$$

where $a_m = \prod_{i=0}^{m-1} 4^i i!$. Let us introduce the scalar operator

$$C(M) = \frac{1}{a_m} M_0^m M_{n/2}^{m-1} M_{n/2+1}^{m-2} \ldots M_{n/2+m-2},$$

for which, due to the equalities (4.3.15), (4.3.16) and (4.3.17) with the operators M_λ, the property holds true

$$\det A(\Lambda) C(M) v(x) = a_m \Lambda^m (\Lambda + n/2)^{m-1} (\Lambda + n/2 + 1)^{m-2} \ldots (\Lambda + n/2 + m - 2) \frac{1}{a_m} M_0^m M_{n/2}^{m-1} M_{n/2+1}^{m-2} \ldots M_{n/2+m-2} v(x) = v(x), \quad (4.3.19)$$

if $v(0) = 0$. Therefore the matrix operator

$$D(M) = A^*(\Lambda) C(M)$$

specifies the solution of the second system from (4.3.14) in the form

$$U^*(x) = D(M) V^*(x) = \sum_{j=0}^{m-1} D_j(M) v_j^*(x), \quad (4.3.20)$$

where $D_j(M)$ is the jth column of the matrix $D(M)$, and $v_j^*(x)$ are the elements of the vector $V^*(x)$. Indeed, due to the property (4.3.19) of the operator $C(M)$ the following equalities hold

$$A(\Lambda) U^*(x) = A(\Lambda) D(M) V^*(x) = A(\Lambda) A^*(\Lambda) C(M) V^*(x) = \det A(\Lambda) C(M) V^*(x) = V^*(x).$$

The operator $D(M)$ is applicable to the function $V^*(x)$ since, by construction, $V^*(0) = 0$. A similar method for representing polynomial solutions to a system of linear partial differential equations is proposed in [89, Theorem 2.4].

Let us determine what smoothness must be imposed on the function $V^*(x)$ to satisfy the boundary conditions (4.3.2). Let $a_{ij}^*(\Lambda)$ denote the elements of the matrix $A^*(\Lambda)$. They have the form $a_{ij}^*(\Lambda) = (-1)^{i+j} A_{ji}(\Lambda)$, where $A_{ij}(\Lambda)$ is an algebraic complement of element $a_{ij}(\Lambda)$ of matrix $A(\Lambda)$. It is known that $A_{ij}(\Lambda) = 0$ for $j < i$. Since $\deg a_{ij}(\Lambda) = i + 1$ $(i = 0, \ldots, m-1)$ and $\deg \det A(\Lambda) = m(m+1)/2 \equiv m_0$, and the minor $A_{ij}(\Lambda)$ consists of the sum of products of the elements of the matrix

$A(\Lambda)$, which do not include the elements of the ith row, then $\deg a^*_{ij}(\Lambda) \le m_0 - j - 1$. Therefore, in the equality (4.3.20) the differential part of the operator

$$D_j(M) = A^*_j(\Lambda)C(M) = \frac{1}{a_m}A^*_j(\Lambda)M_0^m M_{n/2}^{m-1} M_{n/2+1}^{m-2} \dots M_{n/2+m-2}$$

has a degree no greater than $m_0 - j - 1$. Since $(\Lambda+\lambda)M_\mu = (\Lambda+\mu)M_\mu + (\lambda-\mu)M_\mu = 1-(\lambda-\mu)M_\mu$, then the operator $D_j(M)$ is essentially an integral operator of order $j+1$ (it contains at least the product of the $(j+1)$th operator of the form M_λ after reduction of similar terms). This means that if $\varphi_i \in C(\partial S)$, $i = \overline{0, m-1}$, then $v^*_i(x) \in C(\overline{S})$ for every $i = \overline{0, m-1}$, i.e. the smoothness required in the theorem is sufficient for the vector $U^*(x)$ from (4.3.20) to be a solution to the second system from (4.3.14) in $\overline{S}$. The lemma is proved. □

Continuation of the proof of Theorem 4.9. Thus, the vector $U(x) = U_0 + U^*(x)$, found using Lemmas 4.3 and 4.4 is a solution system (4.3.10), which means the polyharmonic function of the form $u(x) = u_0(x) + |x|^2 u_1(x) + \dots + |x|^{2m-2}u_{m-1}(x)$ is a solution to the problem (4.3.1)-(4.3.2) for $f = 0$. The condition $v_{m-1}(0) = 0$ of the theorem, based on the Poisson formula, can be rewritten as (4.3.3). The sufficiency of the conditions of the theorem is established and the proof is completed. □

4.3.2 Inhomogeneous equation

Let us now consider the Riquier-Neumann boundary value problem for the inhomogeneous polyharmonic equation (4.3.1).

Theorem 4.10. *Let $f \in C^1(\overline{S})$ and $\varphi_i \in C(\partial S)$ for $i = 0, 1, \dots, m-1$. A solution to the Riquier-Neumann problem* (4.3.1)-(4.3.2) *exists if and only if the condition is satisfied*

$$\int_{\partial S} \varphi_{m-1}(x)\, ds_x = \int_S f(\xi)\, d\xi \tag{4.3.21}$$

and this solution is unique up to an arbitrary constant.

Proof. By the conditions of the theorem, $f \in C^1(\overline{S})$. Consider the volume potential $V[f](x)$ with density $f(x)$

$$V[f](x) = -\frac{1}{\omega_n}\int_S E(x,\xi)f(\xi)\, d\xi,$$

where $E(x,\xi) = (n-2)^{-1}|\xi - x|^{2-n}$ $(n > 2)$ is an elementary solution to the Laplace equation. Let us denote $V_m[f] = V[\underset{m}{\ldots} V[f]]$. It is known that $f \in C^1(\overline{S}) \Rightarrow V[f] \in C^1(\overline{S})$ and $\Delta V[f] = f$ in S [28, p. 58], and therefore $V_k[f] \in C^1(\overline{S})$ and $\Delta^k V_m[f] = V_{m-k}[f]$ for $k = 0, \ldots, m-1$, where $V_0[f] \equiv f$. Let us represent the solution $u(x)$ to the Riquier-Neumann problem (4.3.1)-(4.3.2) as a sum of functions $u(x) = V_m[f] + w(x)$. Then

$$\Delta^m u(x) - f = \Delta^m V_m[f] + \Delta^m w(x) - f = \Delta^m w(x)$$

and therefore with respect to $w(x)$ we obtain the following problem

$$\Delta^m w = 0, \quad x \in S, \tag{4.3.22}$$

$$\frac{\partial w}{\partial \nu}\Big|_{\partial S} = \tilde{\varphi}_0, \quad \frac{\partial \Delta w}{\partial \nu}\Big|_{\partial S} = \tilde{\varphi}_1, \ldots \frac{\partial \Delta^{m-1} w}{\partial \nu}\Big|_{\partial S} = \tilde{\varphi}_{m-1}, \tag{4.3.23}$$

where is denoted

$$\tilde{\varphi}_i(s) = \varphi_i(s) - \Lambda V_{m-i}[f].$$

Since $V_{m-i}[f] \in C^1(\overline{S})$ and $\varphi_i \in C(\partial S)$, then $\tilde{\varphi}_i \in C(\partial S)$ for $i = 0, \ldots, m-1$. According to Theorem 4.9, the necessary and sufficient condition for the solvability of the problem (4.3.22)-(4.3.23) has the form

$$0 = \int_{\partial S} \tilde{\varphi}_{m-1}(x)\, ds_x = \int_{\partial S} \varphi_{m-1}(x)\, ds_x - \int_{\partial S} \Lambda V[f]\, ds_x,$$

or

$$\int_{\partial S} \varphi_{m-1}(x)\, ds_x = -\frac{1}{\omega_n} \int_{\partial S} ds_x \int_S \Lambda_x E(x,\xi) f(\xi)\, d\xi = \int_S f(\xi)\, d\xi.$$

The solution $w(x)$ is unique up to a constant, which means the solution $u(x)$ also exists and is unique up to a constant. The equality of the integrals in the last formula is known, but we prove it in a different way. By assumption of the theorem, $f \in C^1(\overline{S})$. As shown in the proof of Theorem 3.11 on page 202

$$\int_S \Lambda_x E(x,\xi) f(\xi)\, d\xi = \int_S E(x,\xi)(\Lambda + 2) f(\xi)\, d\xi - \int_{\partial S} E(x,\xi) f(\xi)\, ds_\xi$$

and therefore, by Fubini's theorem (functions under integrals have an integrable singularity)

$$-\frac{1}{\omega_n} \int_{\partial S} ds_x \int_S \Lambda_x E(x,\xi) f(\xi)\, d\xi = -\int_S (\Lambda + 2) f(\xi)\, d\xi \frac{1}{\omega_n} \int_{\partial S} E(x,\xi)\, ds_x$$

$$+\int_{\partial S} f(\xi)\, ds_\xi \frac{1}{\omega_n}\int_{\partial S} E(x,\xi)\, ds_x = \frac{1}{n-2}\Big(\int_{\partial S} f(\xi)\, ds_\xi - \int_S (\Lambda+2) f(\xi)\, d\xi\Big).$$

The equality (3.6.30) states that if $H_m(x)$ is a homogeneous harmonic polynomial, then

$$\frac{1}{\omega_n}\int_{\partial S} E(x,\xi) H_m(x)\, ds_x = \frac{1}{2m+n-2} H_m(\xi)$$

and that means $\frac{1}{\omega_n}\int_{\partial S} E(x,\xi)\, ds_x = \frac{1}{n-2}$. Since according to the divergence theorem

$$-\int_S \Lambda f(\xi)\, d\xi = -\int_S \sum_{i=1}^n \xi_i f_{\xi_i}\, d\xi = -\int_S \sum_{i=1}^n (\xi_i f)_{\xi_i}\, d\xi + n\int_S f(\xi)\, d\xi$$
$$= -\int_{\partial S} \sum_{i=1}^n \xi_i^2 f\, d\xi + n\int_S f(\xi)\, d\xi = -\int_{\partial S} f(\xi)\, ds_\xi + n\int_S f(\xi)\, d\xi,$$

then

$$-\frac{1}{\omega_n}\int_{\partial S} ds_x \int_S \Lambda_x E(x,\xi) f(\xi)\, d\xi = \frac{1}{n-2}(n-2)\int_S f(\xi)\, d\xi.$$

The theorem is proved. □

4.4 Green's function of the Navier problem for the polyharmonic equation

This section considers the Navier boundary value problem for the polyharmonic equation in the unit ball. Theorem 1.4 from Section 1.2 proved the existence of a solution to the Navier problem for the polyharmonic equation. Now the question arises of finding this solution. One of the methods for representing solutions to boundary value problems for elliptic equations is a method based on constructing the Green's function of the problem. The presence of the Green's function of the boundary value problem allows one to obtain an integral representation of the solution to this problem.

In Section 4.1 the Green's function $G_4^r(x,\xi)$ of the Navier problem for the biharmonic equation is already obtained. Based on these results, Definition 4.4 gives the Green's function $G_{2m}^r(x,\xi)$ for the m-harmonic Navier problem. Then, in Lemma 4.5, the property of the auxiliary function $E_{2k}^r(x,\xi)$ is proved. After this, Theorem 4.11 establishes a recurrent formula for the Green's function $G_{2m}^r(x,\xi)$. Finally, in Theorem 4.12, using Theorem 4.8, an explicit equality is derived for representing the solution to the Navier problem in the unit ball through the boundary data of the problem and the right-hand side of the equation. In Example 4.2 the value of the integral with the Green's function kernel $G^r_{2m}(x,\xi)$ of a homogeneous harmonic polynomial is calculated and, on the basis of this, a simple Navier problem for the 3-harmonic equation is solved. The main results of Section 4.4 are published in [122].

4.4.1 Green's function of the Navier boundary value problem

The Navier problem [146, 166] is to find the function $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$, which is a solution to the following boundary value problem

$$\begin{gathered}\Delta^m u(x) = f(x), \quad x \in S,\\ u|_{\partial S} = \varphi_0(\xi),\ \Delta u|_{\partial S} = \varphi_1(\xi), \ldots, \Delta^{m-1} u|_{\partial S} = \varphi_{m-1}(\xi),\ \xi \in \partial S.\end{gathered} \tag{4.4.1}$$

In the case $m = 1$, the Navier problem turns into a Dirichlet problem.

Definition 4.4. Function of the form

$$G^r_{2m}(x,\xi) = E_{2m}(x,\xi) + g^r_{2m}(x,\xi), \tag{4.4.2}$$

where $g^r_{2m}(x,\xi)$ is an m-harmonic function in the variables $x,\xi \in S$ such that for $x \in S$ the equalities hold

$$G^r_{2m}(x,\xi)|_{\xi\in\partial S} = \Delta_\xi G^r_{2m}(x,\xi)|_{\xi\in\partial S} = \ldots = \Delta^{m-1}_\xi G^r_{2m}(x,\xi)|_{\xi\in\partial S} = 0, \tag{4.4.3}$$

we call the Green's function of the Navier problem (4.4.1).

Let $n \geq 3$. In the reasoning given below, the following recursively defined function is required

$$E^r_{2k}(x,\xi) = \frac{1}{\omega_n}\int_S E^r_{2k-2}(x,y)E_2(y,\xi)\,dy, \quad k \geq 2, \tag{4.4.4}$$

where $E^r_2(x,\xi) = E_2(x,\xi) = E(x,\xi)$. For example,

$$E^r_4(x,\xi) = \frac{1}{\omega_n}\int_S E_2(x,y)E_2(y,\xi)\,dy,$$

$$E^r_6(x,\xi) = \frac{1}{\omega_n^2}\int_S E_2(x,\eta)\int_S E_2(\eta,y)E_2(y,\xi)\,dy\,d\eta.$$

Lemma 4.5. *The function $E^r_{2m}(x,\xi)$ is defined for $\xi, x \in S$, $\xi \neq x$ and perhaps has a singularity for $\xi = x$ such that that $E^r_{2m}(x,\xi) \leq C|x-\xi|^{3-n}$, where C is some positive constant. For $\xi \neq x$ the equality $\Delta E^r_{2m}(x,\xi) = -E^r_{2m-2}(x,\xi)$ holds.*

Proof. Indeed, as shown in Theorem 4.2 the function $E_4^r(x,\xi)$ is defined for $\xi \neq x$ and its singularity for $\xi = x$ is not higher than $O(|x-\xi|^{3-n})$. Suppose by induction that $E_{2m-2}^r(x,\xi)$ is defined for $\xi \neq x$ and $E_{2m-2}^r(x,\xi) \leq C_1|x-\xi|^{3-n}$. Then for $\xi \neq x$ the integral from (4.4.4) defining the function $E_{2m}^r(x,\xi)$ has two integrable singularities for $y = x$ and $y = \xi$ of orders $n-3$ and $n-2$, respectively. This means $E_{2m}^r(x,\xi)$ is defined and continuous for $\xi \neq x \in S$ [183]. Besides

$$\begin{aligned}
E_{2m}^r(x,\xi) &\leq C_2\int_{|y|<1}|x-y|^{3-n}|y-\xi|^{2-n}\,dy \\
&= C_2\int_{|\xi+\eta|<1}|x-\xi-\eta|^{3-n}|\eta|^{n-2}\,d\eta \leq C_2\int_{|\eta|<2}|z-\eta|^{3-n}|\eta|^{2-n}\,d\eta \\
&= C_2\Big(\int_{|z|<|\eta|<2}+\int_{|\eta|<|z|}\Big)|z-\eta|^{3-n}|\eta|^{2-n}\,d\eta \equiv I_1+I_2,
\end{aligned}$$

where $z = x-\xi$ is denoted. For the first integral I_1, since $3-n \leq 0$, we have

$$I_1 \leq C_2|z|^{3-n}\int_{|\eta|<2}|z-\eta|^{3-n}|\eta|^{-1}\,d\eta = O(|z|^{3-n}),$$

since, according to [183, p.27], the integral is bounded. Besides

$$\begin{aligned}
I_2 &= C_2|z|^{2-n}|z|^n\int_{|\theta|z||<|z|}|z-\theta|z||^{3-n}|\theta|^{2-n}\,d\theta \\
&= C_2|z|^{5-n}\int_{|\theta|<1}|\hat{z}-\theta|^{3-n}|\theta|^{2-n}\,d\theta,
\end{aligned}$$

where $\hat{z} = z/|z|$. Since the integral is continuous for $\hat{z} \in \partial S$ [183], then $I_2 = O(|x-\xi|^{5-n})$. Thus $E_{2m}^r(x,\xi) = O(|x-\xi|^{3-n})$. The induction step is proved.

Suppose $\xi \neq x \in S$. Let us denote $S_\delta = S \setminus \{y \in \mathbb{R}^n : |x-y| \leq \delta\}$ and take $\delta > 0$ so small that $\xi \in S_\delta$, i.e. $|x-\xi| > \delta$ and $\{y \in \mathbb{R}^n : |x-y| \leq \delta\} \subset S$. Then

$$E_{2m}^r(x,\xi) = \frac{1}{\omega_n}\int_{S_\delta}E_{2m-2}^r(x,y)E_2(y,\xi)\,dy + \frac{1}{\omega_n}\int_{|y-x|\leq\delta}E_{2m-2}^r(x,y)E_2(y,\xi)\,dy.$$

Since $\xi \in S_\delta$ and $x \notin S_\delta$, then by the property of the volume potential

$$\frac{1}{\omega_n}\Delta_\xi\int_{S_\delta}E_{2m-2}^r(x,y)E_2(y,\xi)\,dy = -E_{2m-2}^r(x,\xi),$$

the second integral does not have a singularity under the integral over ξ and therefore is a harmonic function in ξ. Therefore, for $\xi \neq x$ we obtain the equality $\Delta_\xi E_{2m}^r(x,\xi) = -E_{2m-2}^r(x,\xi)$. The lemma is proved. □

Theorem 4.11. *Function $G^r_{2m}(x,\xi)$ for $\xi, x \in S$, defined recursively by the equality*

$$G^r_{2m}(x,\xi) = \frac{1}{\omega_n}\int_S G^r_{2m-2}(x,y)G_2(y,\xi)\,dy, \quad m \geq 2, \tag{4.4.5}$$

where $G^r_2(x,y) \equiv G_2(x,y)$ is the Green's function of the Dirichlet problem from (4.1.1)*, is the Green's function of the Navier problem* (4.4.1)*. The Green's function $G^r_{2m}(x,\xi)$ has continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$, $x \neq \xi$ and is symmetric.*

Proof. Let us prove the theorem using the method of mathematical induction on m. For $m = 2$ the statement of the theorem is proved in Theorem 4.2 and in this case

$$G^r_4(x,\xi) = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy,$$

and

$$g^r_4(x,\xi) = \frac{1}{\omega_n}\int_S E_2(x,y)g^r_2(y,\xi)\,dy + \frac{1}{\omega_n}\int_S g^r_2(x,y)E_2(y,\xi)\,dy + \frac{1}{\omega_n}\int_S g^r_2(x,y)g^r_2(y,\xi)\,dy,$$

where $g^r_2(x,\xi) = -E_2\left(\frac{x}{|x|}, |x|\xi\right)$. The singularities in the written integrals are such that, according to [183], the function $g^r_4(x,\xi)$, and therefore $G^r_{2m}(x,\xi)$, has continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$, $x \neq \xi$.

Let the theorem be true for some $m \geq 2$ and let also the function $g^r_{2m-2}(x,\xi)$ have continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$. Then, according to Definition 4.4, the Green's function $G^r_{2m-2}(x,\xi)$, found by the equality (4.4.5), has the representation

$$G^r_{2m-2}(x,\xi) = E_{2m-2}(x,\xi) + g^r_{2m-2}(x,\xi),$$

where $g^r_{2m-2}(x,\xi)$ is some $(m-1)$-harmonic function in S in the variable ξ for any $x \in S$.

Let us rewrite this equality in the form

$$G^r_{2m-2}(x,\xi) = E^r_{2m-2}(x,\xi) + (h_{2m-2}(x,\xi) + g^r_{2m-2}(x,\xi)), \tag{4.4.6}$$

where is denoted $h_{2k}(x,\xi) = E_{2k}(x,\xi) - E^r_{2k}(x,\xi)$.

Let us prove that the function $h_{2k}(x,\xi)$ is k-harmonic in S with respect to ξ. For $k=1$, by definition $E_2^r(x,\xi)=E_2(x,\xi)$, and therefore $h_2(x,\xi)=0$. Let the function $h_{2k-2}(x,\xi)$ be $(k-1)$-harmonic in S with respect to ξ. By the property of the function $E_{2k}(x,\xi)$ from Lemma 4.2 and by virtue of Lemma 4.5 for $\xi\neq x$ we write

$$\Delta_\xi h_{2k}(x,\xi)=\Delta_\xi(E_{2k}(x,\xi)-E_{2k}^r(x,\xi))$$
$$=-E_{2k-2}(x,\xi)+E_{2k-2}^r(x,\xi)\equiv -h_{2k-2}(x,\xi)$$

and therefore the function $h_{2k}(x,\xi)$ for $\xi\neq x$, as a solution to the Poisson equation, can be written in the form

$$h_{2k}(x,\xi)=-\frac{1}{\omega_n}\int_S E_2(\xi,y)h_{2k-2}(x,y)\,dy+H(x,\xi),$$

where $H(x,\xi)$ is some function harmonic in $\xi\neq x\in S$. Since the function $h_{2k-2}(x,\xi)$ is $(k-1)$-harmonic for all $\xi\in S$, then in this case, according to the property of the volume potential, it should be k-harmonic function for all $\xi\in S$. Consequently, the harmonic function $H(x,\xi)$ has the same singularity at $\xi=x$ as the function $h_{2k}(x,\xi)$. Since the function $E_{2k}(x,\xi)$, for $k\geq 2$ has the singularity at $\xi=x$ of order no more than $|x-\xi|^{3-n}$, and due to Lemma 4.5 it is the same as the function $E_{2k}^r(x,\xi)$, then $H(x,\xi)$ has a singularity at $\xi=x$ of order no more than $|x-\xi|^{3-n}$. By the singularity erasure theorem [183], the function $H(x,\xi)$ is harmonic in S for all $\xi\in S$. The induction step on k is proved.

So the function

$$\hat g_{2m-2}^r(x,\xi)=h_{2m-2}(x,\xi)+g_{2m-2}^r(x,\xi)$$

is $(m-1)$-harmonic in S with respect to ξ. Using the equality (4.4.6), taking into account this notation, according to the equality (4.4.5), we can write

$$G_{2m}^r(x,\xi)=\frac{1}{\omega_n}\int_S E_{2m-2}^r(x,y)G_2(y,\xi)\,dy+\frac{1}{\omega_n}\int_S \hat g_{2m-2}^r(x,y)G_2(y,\xi)\,dy,$$

from where, taking into account (4.1.1) and (4.4.4) we find

$$G_{2m}^r(x,\xi)=E_{2m}^r(x,\xi)+\frac{1}{\omega_n}\int_S E_{2m-2}^r(x,y)E_2\Big(\frac{y}{|y|},|y|\xi\Big)\,dy$$
$$+\frac{1}{\omega_n}\int_S \hat g_{2m-2}^r(x,y)G_2(y,\xi)\,dy. \quad (4.4.7)$$

The second term in the resulting equality is a harmonic function in $\xi\in S$, since such is the function $E_2\Big(\frac{y}{|y|},|y|\xi\Big)$. The third term of this equality, by the properties of the Green's function $G_2(y,\xi)$ and since $g^r_{2m-2}(x,\xi)$ and $h_{2m-2}(x,\xi)$ have continuous derivatives with respect to ξ in $\bar S$ for $x\in S$, satisfies the equality

$$\frac{1}{\omega_n}\Delta_\xi\int_S \hat g^r_{2m-2}(x,y)G_2(y,\xi)\,dy=-\hat g^r_{2m-2}(x,\xi),\quad \xi\in S.$$

Since the function $\hat g^r_{2m-2}(x,\xi)$ is $(m-1)$-harmonic in S, then the function on the left is m-harmonic in S and from (4.4.7) the equality (4.4.2) follows if we denote

$$g^r_{2m}(x,\xi)=\frac{1}{\omega_n}\int_S E^r_{2m-2}(x,y)E_2\Big(\frac{y}{|y|},|y|\xi\Big)\,dy \\ +\frac{1}{\omega_n}\int_S \hat g^r_{2m-2}(x,y)G_2(y,\xi)\,dy-h_{2m}(x,\xi),$$

besides by virtue of Lemmas 4.2 and 4.5

$$\Delta_\xi g^r_{2m}(x,\xi)=-h_{2m-2}(x,\xi)-g^r_{2m-2}(x,\xi)+h_{2m-2}(x,\xi)=-g^r_{2m-2}(x,\xi).$$

It is easy to see that, due to the singularity of the kernels under the integrals, the function $g^r_{2m}(x,\xi)$, and therefore $G^r_{2m}(x,\xi)$, has continuous derivatives with respect to ξ in $\bar S$ for $x\in S, x\neq\xi$ [183]. So, the function $G^r_{2m}(x,\xi)$ satisfies the first condition from Definition 4.4. The induction step on m is proved.

Symmetry. Since the function $G^r_2(x,\xi)=G_2(x,\xi)$ is symmetric, then from the equality (4.4.5) it is immediately clear that if the function $G^r_{2m-2}(x,\xi)$ is symmetric, then the function $G^r_{2m}(x,\xi)$ is also symmetric. In addition, the function $G^r_4(x,\xi)$ is symmetric. Therefore, from the representation (4.4.2) it follows that the m-harmonic function $g^r_{2m}(x,\xi)$ is also symmetric.

Let us check the boundary conditions for the function $G^r_{2m}(x,\xi)$ from Definition 4.4. Similar to Lemma 4.5 suppose $x\neq\xi\in S$. Let us denote $S_\delta=S\setminus\{y\in\mathbb{R}^n:|x-y|\le\delta\}$ and take $\delta>0$ so small that $\xi\in S_\delta$, i.e. $|x-\xi|>\delta$ and $\{y\in\mathbb{R}^n:|x-y|\le\delta\}\subset S$. Then

$$G^r_{2m}(x,\xi)=\frac{1}{\omega_n}\int_{S_\delta}G^r_{2m-2}(x,y)G_2(y,\xi)\,dy \\ +\frac{1}{\omega_n}\int_{|y-x|\le\delta}G^r_{2m-2}(x,y)G_2(y,\xi)\,dy. \quad (4.4.8)$$

Since $\xi\in S_\delta$ and $x\notin S_\delta$, then in the first integral $G^r_{2m-2}(x,y)$ does not have a singularity and its derivatives with respect to y are bounded in $\bar S_\delta$, and therefore by the property of the volume potential

$$\frac{1}{\omega_n}\Delta_\xi \int_{S_\delta} G^r_{2m-2}(x,y)G_2(y,\xi)\,dy = -G^r_{2m-2}(x,\xi).$$

The second integral does not have a singularity under the integral over ξ and the singularity at $x = y$ is integrable, which means it is a harmonic function in ξ (differentiation with respect to ξ does not increase the singularity). Therefore, for $\xi \neq x$ we obtain the equality $\Delta_\xi G^r_{2m}(x,\xi) = -G^r_{2m-2}(x,\xi)$. We can similarly find

$$\Delta^k_\xi G^r_{2m}(x,\xi) = (-1)^k G^r_{2m-2k}(x,\xi), \quad \xi \neq x. \tag{4.4.9}$$

Let us return to (4.4.8) and let $\xi \to \partial S$ for fixed $x \in S$. The first volume potential has a continuous kernel $G^r_{2m-2}(x,y)$ on S_δ and therefore, by the property of the Green's function $G_2(y,\xi)$ of the Dirichlet problem, tends to 0. The second volume potential has a singularity at $x = y$ no more than $|x-y|^{3-n}$, and therefore, due to the continuity of $G_2(y,\xi)$ on the integration domain, it is a continuous function over ξ [183]. Since $G_2(y,\xi) \to 0$ for $\xi \to \partial S$, the second integral also tends to 0. Thus $G^r_{2m}(x,\xi)|_{\xi\in\partial S} = 0$. Using the equality (4.4.9) we get

$$\Delta^k_\xi G^r_{2m}(x,\xi)|_{\xi\in\partial S} = (-1)^k G^r_{2m-2k}(x,\xi)|_{\xi\in\partial S} = 0.$$

Thus, the function $G^r_{2m}(x,\xi)$ also satisfies the second condition (4.4.3) from Definition 4.4. The theorem is proved. □

4.4.2 Solution of the Navier problem

Theorem 4.12. *The function $u \in C^{2m}(S) \cap C^{2m-1}(\bar S)$, which is a solution to the Navier problem* (4.4.1)*, for $x \in S$ can be represented in the form*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\sum_{k=1}^{m}(-1)^k \frac{\partial G^r_{2k}(x,\xi)}{\partial \nu}\varphi_{k-1}(\xi)\,ds_\xi + \frac{(-1)^m}{\omega_n}\int_S G^r_{2m}(x,\xi)f(\xi)\,d\xi, \tag{4.4.10}$$

where $G^r_{2m}(x,\xi)$ is the Green's function of the Navier problem.

If $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ for $k = 1,\dots,m-1$ and $f\ inC^1(\bar S)$, then the function $u(x)$ found from (4.4.10) *is a solution to the Navier problem* (4.4.1).

Proof. Let $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$ be the solution to the Navier problem (4.4.1). Let us use the equality (4.2.5) from Theorem 4.8 with $D = \{\xi : |\xi| < 1 - \varepsilon\}$, $\varepsilon \in (0, 1)$ and $E_{2m}(x, \xi) = G^r_{2m}(x, \xi)$. Due to the equality (4.4.2), the singularities of the functions $E_{2m}(x, \xi)$ and $G^r_{2m}(x, \xi)$ have the same character, and therefore, using reasoning similar to those made in Theorem 4.1, we get

$$u(x) = \frac{1}{\omega_n} \int_{|\xi|=1-\varepsilon} \sum_{k=0}^{m-1} (-1)^k \Big(G^r_{2k+2}(x, \xi) \frac{\partial \Delta^k u}{\partial \nu} - \frac{\partial G^r_{2k+2}(x, \xi)}{\partial \nu} \Delta^k u \Big)\, ds_\xi \\ + \frac{(-1)^m}{\omega_n} \int_{|\xi|<1-\varepsilon} G^r_{2m}(x, \xi) \Delta^m u(\xi)\, d\xi. \quad (4.4.11)$$

In this case, the equalities (4.4.9) are taken into account, and the fact that the functions $g^r_{2k}(x, \xi)$ from (4.4.2) are k-harmonic in S, and therefore sufficiently smooth in $\{\xi : |\xi| < 1 - \varepsilon\}$. Let us put $\Delta^m u(\xi) = f(\xi)$ in the resulting equality and go to the limit at $\varepsilon \to +0$. At the same time, we take into account that the functions $g^r_{2k}(x, \xi)$ (therefore the functions $G^r_{2k}(x, \xi)$) and their derivatives with respect to ξ are continuous with respect to ξ in $\{\xi : 1 - \varepsilon < |\xi| \le 1\}$ for fixed $x \in S$, and therefore surface integrals over $\{\xi : |\xi| = 1\}$ containing terms $G^r_{2k+2}(x, \xi)$ turn to 0, and due to the boundary conditions of the Navier problem we have $\Delta^k u(\xi) \to \varphi_k(\xi)$. Thus, from (4.4.11) in the limit at $\varepsilon \to +0$ and after replacing the summation index $k \to k - 1$ we obtain (4.4.10).

Let us check that the function $u(x)$ determined from (4.4.10) is a solution to the Navier problem (4.4.1) for $\varphi_k \in C^{1+\varepsilon}(\partial S)$, $k = 0, \ldots, m - 1$ and $f \in C^1(\bar{S})$. We transform the equality (4.4.10). Consider two operators

$$L[\varphi] = -\frac{1}{\omega_n} \int_{\partial S} \frac{\partial G_2}{\partial \nu_\xi}(x, \xi) \varphi(\xi)\, ds_\xi, \quad P[f] = -\frac{1}{\omega_n} \int_S G_2(x, \xi) f(\xi)\, d\xi. \quad (4.4.12)$$

Then, by virtue of (4.4.5), using Fubini's theorem, we write

$$\frac{(-1)^m}{\omega_n} \int_S G^r_{2m}(x, \xi) f(\xi)\, d\xi \\ = \frac{(-1)^{m-1}}{\omega_n} \int_S G^r_{2m-2}(x, y) \Big(-\frac{1}{\omega_n} \int_S G_2(y, \xi) f(\xi)\, d\xi \Big)\, dy \\ = \frac{(-1)^{m-1}}{\omega_n} \int_S G^r_{2m-2}(x, y) P[f](y)\, dy \\ \ldots = -\frac{1}{\omega_n} \int_S G^r_2(x, \xi) P^{m-1}[f](y)\, d\xi = P^m[f](x).$$

Let $f \in C^1(\bar{S})$, then $P[f] \in C^2(S)$ and $\Delta P[f] = f$ in S [28]. Since also $P[f] \in C^1(\bar{S})$ [183], we obtain $P^2[f] \in C^2(S)$, $\Delta P^2[f] = P[f]$ or $\Delta^2 P^2[f] = f$ in S. Continuing these arguments we have: if $f \in C^1(\bar{S})$, then $\Delta^m P^m[f] = f$ in S. In addition, by the property of the function $G_2(x,\xi)$, we have $P^k[f]|_{\partial S} = 0$ for $k = 0,\ldots,m-1$.

Let us transform the surface integrals into (4.4.10) for $k > 1$. By virtue of (4.4.5) and using Fubini's theorem, we write

$$\frac{(-1)^k}{\omega_n}\int_{\partial S}\frac{\partial G^r_{2k}(x,\xi)}{\partial \nu}\varphi_{k-1}(\xi)\,ds_\xi$$
$$= \frac{(-1)^{k-1}}{\omega_n}\int_S G^r_{2k-2}(x,y)\Big(-\frac{1}{\omega_n}\int_{\partial S}\frac{\partial G^r_2(y,\xi)}{\partial \nu}\varphi_{k-1}(\xi)\,ds_\xi\Big)\,dy$$
$$= \frac{(-1)^{k-1}}{\omega_n}\int_S G^r_{2k-2}(x,y)L[\varphi_{k-1}](y)\,dy = P^{k-1}[L[\varphi_{k-1}]](x).$$

It is known that if $\varphi_{k-1} \in C(\partial S)$, then $\Delta L[\varphi_{k-1}] = 0$ in S and $L[\varphi_{k-1}]|_{\partial S} = \varphi_{k-1}$. Therefore it is sufficient that $\varphi_0 \in C(\partial S)$. However, to be able to apply the Δ operator to the function $P[L[\varphi_{k-1}]]$ it is necessary that $L[\varphi_{k-1}] \in C^1(\bar{S})$. This condition is satisfied, for example, if $\varphi_{k-1} \in C^{1+\varepsilon}(\partial S)$ [4, Lemma 2.7]. Thus, similar to the volume potential, we obtain: if $\varphi_{k-1} \in C^{1+\varepsilon}(\partial S)$ for $k > 1$, then for $x \in S$

$$\Delta^{k-1}P^{k-1}[L[\varphi_{k-1}]]|_{\partial S} = L[\varphi_{k-1}]|_{\partial S} = \varphi_{k-1},\ \Delta^k P^{k-1}[L[\varphi_{k-1}]](x) = 0.$$

So, the smoothness of the functions φ_k for $k = 0,\ldots,m-1$ and f, required in the theorem, is sufficient for the function $u(x)$ from (4.4.10) be the solution to the Navier problem. The theorem proved. □

Corollary 4.1. Let the functions $u_k(x)$ and $w_f(x)$ be found from the equalities

$$u_k(x) = P^k[L[\varphi_k]](x), \quad w_f(x) = P^m[f](x),$$

where the operators $L[\cdot]$ and $P[\cdot]$ are defined in (4.4.12). Then the solution to the Navier problem (4.4.10) can be written in the form

$$u(x) = \sum_{k=0}^{m-1} u_k(x) + w_f(x).$$

Proof. Indeed, with the notation made, the function $u(x)$ from (4.4.10) is written in the indicated form. The function $u_k(x)$ is k-harmonic in S and

$$\Delta^q u_k(x)|_{\partial S} = \begin{cases} \varphi_k, & q = k \\ 0, & q \neq k \end{cases},$$

and the function $w_f(x)$ satisfies the equality $\Delta^m w_f(x) = f(x)$ in S and, in addition, $\Delta^k w_f(x)|_{\partial S} = 0$ for $k = 0, \dots, m-1$. The corollary is proved. □

Example 4.2. Let us give examples of calculating the simplest integrals with the kernel of the Green's function $G^r_{2m}(x,\xi)$ for $m = 2$ and $m = 3$. Let us choose as the function $f(x)$ a homogeneous harmonic polynomial $H_k(x)$ of degree $k \in \mathbb{N}_0$. From theorem 2.10 on page 127 for $m = 1$ it follows that

$$-\frac{1}{\omega_n}\int_{\partial S} G_2(x,\xi)|x|^{2l} H_k(\xi)\,d\xi = \frac{|x|^{2l+2}-1}{(2l+2)(2l+2k+n)} H_k(x).$$

Since $G^r_2(x,\xi) = G_2(x,\xi)$, then from here, for $l = 0$, we get

$$u_1[H_k] \equiv -\frac{1}{\omega_n}\int_{\partial S} G^r_2(x,\xi) H_k(\xi)\,d\xi = (|x|^2-1)\frac{H_k(x)}{2(2k+n)}.$$

Again using Theorem 2.10 for $m = 1$ we find

$$\begin{aligned} u_2[H_k] &\equiv \frac{1}{\omega_n}\int_{\partial S} G^r_4(x,\xi) H_k(\xi)\,d\xi = \left(\frac{|x|^4-1}{4(2k+2+n)} - \frac{|x|^2-1}{2(2k+n)}\right)\frac{H_k(x)}{2(2k+n)} \\ &= \left(\frac{|x|^4}{4(2k+2+n)} - \frac{|x|^2}{2(2k+n)} + \frac{2k+4+n}{4(2k+n)(2k+2+n)}\right)\frac{H_k(x)}{2(2k+n)} \end{aligned}$$

and similarly we get

$$\begin{aligned} u_3[H_k] \equiv -\frac{1}{\omega_n}\int_{\partial S} G^r_6(x,\xi) H_k(\xi)\,d\xi = \Big(&\frac{|x|^6-1}{24(2k+2+n)(2k+4+n)} \\ &- \frac{|x|^4-1}{8(2k+n)(2k+2+n)} + \frac{(2k+4+n)(|x|^2-1)}{8(2k+n)^2(2k+2+n)}\Big)\frac{H_k(x)}{2(2k+n)}. \end{aligned}$$

It is easy to check that the equalities hold true

$$\Delta u_1[H_k](x) = H_k(x),\ \Delta u_2[H_k](x) = u_1[H_k](x),\ \Delta u_3[H_k](x) = u_2[H_k](x),$$

and also $u_1[H_k]|_{\partial S} = u_2[H_k]|_{\partial S} = u_3[H_k]|_{\partial S} = 0$. Therefore, by Corollary 4.1, the following Navier problem for the 3-harmonic equation

$$\begin{gathered} \Delta^3 u(x) = H_{k_3}(x), \quad x \in S, \\ u|_{\partial S} = H_{k_0}(\xi),\ \Delta u|_{\partial S} = H_{k_1}(\xi),\ \Delta^2 u|_{\partial S} = H_{k_2}(\xi),\ \xi \in \partial S, \end{gathered}$$

where $H_{k_i}(x)$ are homogeneous harmonic polynomials, has the form

$$u(x) = H_{k_0}(x) + u_1[H_{k_1}](x) + u_2[H_{k_2}](x) + u_3[H_{k_3}](x).$$

4.5 Green's function of the Riquier-Neumann problem for the polyharmonic equation

This section considers the construction of a solution to the Riquier-Neumann boundary value problem for the polyharmonic equation in the unit ball. Theorems 4.9 and 4.10 from Section 4.3 proved the existence of a solution to the Riquier-Neumann problem for the polyharmonic equation. Now the question arises of finding this solution. As a solution method, a method based on constructing the Green's function of the problem is chosen. In Section 4.1 the Green's function $G_4^{rn}(x, \xi)$ of the Riquier-Neumann problem for the biharmonic equation is already obtained. Based on these results, the Green's function of the Riquier-Neumann problem for the polyharmonic equation is constructed. The results of Section 4.5 are published in [124, 126].

For ease of reading, first, in Section 4.5.1, for studying the Riquier-Neumann problem, the necessary formulas obtained earlier in other sections are given. Section 4.5.2 gives Definition 4.5 of the Green's function $\mathcal{N}_{2m}(x, \xi)$ of the Riquier-Neumann problem. Theorem 4.13 proves a recurrent equality for the defined Green's function. Example 4.3 is given, supplementing the equality from Theorem 3.29. Theorem 4.14 from Section 4.5.3 contains an integral representation of the solution to the Riquier-Neumann problem. Theorem 4.15 proves that the function found in Theorem 4.14 is indeed a solution to the Riquier-Neumann problem in the unit ball. Finally, Theorem 4.16 from Section 4.5.4 considers a special case of the Riquier-Neumann problem for a homogeneous polyharmonic equation and gives Example 4.4 of solving the Riquier-Neumann problem with simple boundary data.

The Riquier-Neumann problem [101] for the polyharmonic equation in S is to find the function $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$, which is a solution the following boundary value problem for an inhomogeneous polyharmonic equation

$$\begin{gathered} \Delta^m u(x) = f(x), \quad x \in S, \\ \frac{\partial u}{\partial \nu}\Big|_{\partial S} = \varphi_0(\xi), \ \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = \varphi_1(\xi), \ldots, \frac{\partial \Delta^{m-1} u}{\partial \nu}\Big|_{\partial S} = \varphi_{m-1}(\xi), \end{gathered} \tag{4.5.1}$$

where $\xi \in \partial S$. For $m = 1$, the Riquier-Neumann problem turns into the Neumann problem for the Poisson equation. The main results of Section 4.5 are published in [124].

4.5.1 Supporting information

In Theorem 3.30 from Section 3.7 the Green's function of the Neumann problem for the Poisson equation in S is defined in the form

$$\mathcal{N}_2(x,\xi) = E_2(x,\xi) - E_0(x,\xi), \tag{4.5.2}$$

where the function $E_0(x,\xi)$, harmonic in $x,\xi \in S$, is written in the form

$$E_0(x,\xi) = \int_0^1 \left(\hat{E}_2\Big(\frac{x}{|x|}, t|x|\xi\Big) + 1\right)\frac{dt}{t}$$

and $\hat{E}_2(x,\xi) = \Lambda_x E_2(x,\xi)$, where the subscript x indicates that the operator Λ is applied to the variables x. As noted above, on ∂S the equality $\Lambda u = \frac{\partial u}{\partial \nu}$ holds true. Since $\hat{E}_2(x,\xi) = -(|x|^2 - x\cdot\xi)/|x-\xi|^n$, then the function

$$\hat{E}_2\Big(\frac{x}{|x|}, t|x|\xi\Big) = -\frac{1-(x\cdot\xi)t}{(1-2t(x\cdot\xi)+|x|^2|\xi|^2t^2)^{n/2}}$$

is symmetric and therefore the function $E_0(x,\xi)$, and therefore the function $\mathcal{N}_2(x,\xi)$ are also symmetric. The function $\mathcal{N}_2(x,\xi)$ has the properties (see Theorems 3.30 and 4.4)

$$\Lambda_x\mathcal{N}_2(x,\xi) = \Lambda_x E_2(x,\xi) - (\Lambda_x E_2)\Big(\frac{x}{|x|}, |x|\xi\Big) - 1,\ x,\xi\in S,\ x\neq\xi, \tag{4.5.3}$$

$$\Lambda_x\mathcal{N}_2(x,\xi)\big|_{\xi\in\partial S} = -\frac{\partial G_2(x,\xi)}{\partial\nu_\xi} - 1, \quad x\in S,$$

and therefore the equalities hold true

$$\begin{gathered}\int_S \frac{\partial\mathcal{N}_2(x,\xi)}{\partial\nu_x} f(\xi)\,d\xi\big|_{x\in\partial S} = -\int_S f(\xi)\,d\xi,\\ \frac{1}{\omega_n}\int_{\partial S}\frac{\partial\mathcal{N}_2(x,\xi)}{\partial\nu_x}\psi(\xi)\,ds_\xi\big|_{x\in\partial S} = \psi(x)\big|_{\partial S} - \frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi.\end{gathered} \tag{4.5.4}$$

Theorem 4.4 shows that the solution to the Neumann problem for the Poisson equation in S

$$\Delta u(x) = f(x),\ x\in S; \quad \frac{\partial u(x)}{\partial\nu}\big|_{\partial S} = \psi(x),\ x\in\partial S$$

when the known condition is met, $\int_{\partial S}\psi(\xi)\,ds_\xi = \int_S f(\xi)\,d\xi$ is written in the form

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_2(x,\xi)\psi(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_S \mathcal{N}_2(x,\xi)f(\xi)\,d\xi + C.$$

In the arguments below, we also need the function $E_{2k}^r(x,\xi)$, defined in the equality (4.4.4) from the previous section

$$E_{2k}^r(x,\xi) = \frac{1}{\omega_n}\int_S E_{2k-2}^r(x,y)E_2(y,\xi)\,dy, \quad k \geq 2, \tag{4.5.5}$$

where $E_2^r(x,\xi) = E_2(x,\xi)$. One of its properties is described in Lemma 4.5, and in the proof of Theorem 4.11 it is established that, by the singularity erasure theorem [183], the function $h_{2k}(x,\xi) = E_{2k}(x,\xi) - E_{2k}^r(x,\xi)$ is k-harmonic in $\xi \in S$.

4.5.2 Green's function of the Riquier-Neumann boundary value problem

Now we define the Green's function of the Riquier-Neumann problem.

Definition 4.5. Function of the form

$$\mathcal{N}_{2m}(x,\xi) = E_{2m}(x,\xi) + g_{2m}^n(x,\xi), \quad m \geq 1, \tag{4.5.6}$$

where $g_{2m}^n(x,\xi)$ is an m-harmonic function in the variables $x,\xi \in S$ such that the equalities hold true for $x \in S$

$$\begin{gathered}\frac{\partial \mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = \ldots = \frac{\partial \Delta_\xi^{m-2}\mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = 0,\\ \frac{\partial \Delta_\xi^{m-1}\mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = (-1)^m,\end{gathered} \tag{4.5.7}$$

we call the Green's function of the Riquier-Neumann problem (4.5.1).

Remark 4.5. It should be noted that in the biharmonic case, in Definition 4.2, the Green's function of the Riquier-Neumann problem is defined differently. It should be kept in mind that $G_4^{rn}(x,\xi) = \mathcal{N}_4(x,\xi)$.

Theorem 4.13. *Function $\mathcal{N}_{2m}(x,\xi)$ for $\xi, x \in S$ and $m > 1$, defined recursively by the equality*

$$\mathcal{N}_{2m}(x,\xi) = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}_{2m-2}(x,y)\mathcal{N}_2(y,\xi)\,dy - \frac{1}{\tau_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy\cdot\int_S \mathcal{N}_2(y,\xi)\,dy\Big), \quad (4.5.8)$$

where $\tau_n = |S|$, and $\mathcal{N}_2(x,y)$ is the Green's function of the Neumann problem from (4.5.2)*, is the Green's function of the Riquier-Neumann problem* (4.5.1)*. The function $\mathcal{N}_{2m}(x,\xi)$ has the property*

$$\Lambda_x\mathcal{N}_{2k}(x,\xi)\big|_{x\in\partial S} = 0,\ k>1, \quad \Lambda_x\mathcal{N}_2(x,\xi)\big|_{x\in\partial S} = -1, \quad \xi\in S, \quad (4.5.9)$$

where $g^n_{2m}(x,\xi)$ from the representation (4.5.6) *has continuous derivatives with respect to ξ in $\bar S$ for $x\in S$.*

Proof. Let us prove the theorem using the method of mathematical induction on m. For $m=2$ the statement of the theorem is proved in Theorem 4.6 (see Remark 4.5). In this case, from the representation

$$\mathcal{N}_4(x,\xi) = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}_2(x,y)\mathcal{N}_2(y,\xi)\,dy - \frac{1}{\tau_n}\int_S \mathcal{N}_2(x,y)\,dy\cdot\int_S \mathcal{N}_2(y,\xi)\,dy\Big),$$

using (4.5.2) it is easy to obtain the equality

$$\mathcal{N}_4(x,\xi) = E^r_4(x,\xi) + \hat g^n_4(x,\xi),$$

where the biharmonic function $\hat g^n_4(x,\xi)$ is defined as

$$\hat g^n_4(x,\xi) = -\frac{1}{\omega_n}\Big(\int_S E_2(x,y)E_0(y,\xi)\,dy + \int_S E_0(x,y)E_2(y,\xi)\,dy - \int_S E_0(x,y)E_0(y,\xi)\,dy + \frac{1}{\tau_n}\int_S \mathcal{N}_2(x,y)\,dy\cdot\int_S \mathcal{N}_2(y,\xi)\,dy\Big).$$

Indeed, the first and third integrals in the resulting equality are harmonic functions in ξ, since the singularity in the first integral is integrable, and differentiation does not increase this singularity. The second and fourth integrals, by the property of the volume potential, are biharmonic functions with respect to ξ, since the harmonic function $E_0(x,\xi)$ has bounded derivatives with respect to ξ in $\bar S$ for $x\ inS$. In addition, $\hat g^n_4(x,\xi)$, due to the singularity of the order $|\xi-y|^{2-n}$ under the integrals, has continuous derivatives with respect to ξ in $\bar S$ at $x\in S$ [183]. If we remember that $E_4(x,\xi) = E^r_4(x,\xi) + h_4(x,\xi)$, then we have

$$\mathcal{N}_4(x,\xi) = E_4(x,\xi) + \hat g^n_4(x,\xi) - h_4(x,\xi) \equiv E_4(x,\xi) + g^n_4(x,\xi),$$

where $g_4^n(x,\xi)$, due to the properties of $\hat{g}_4^n$ and h_4, has bounded derivatives with respect to ξ in $\bar{S}$ at $x \in S$.

Let the theorem be true for some $m \geq 2$. Then, according to Definition 4.5, the Green's function $\mathcal{N}_{2m-2}(x,\xi)$ has the representation (4.5.6)

$$\mathcal{N}_{2m-2}(x,\xi) = E_{2m-2}(x,\xi) + g_{2m-2}^n(x,\xi),$$

where $g_{2m-2}^n(x,\xi)$ is some $(m-1)$-harmonic function in S in ξ for $x \in S$, which has bounded derivatives by ξ in $\bar{S}$. If we recall the function $E_{2k}^r(x,\xi)$ from (4.5.5) and the k-harmonic function $h_{2k}(x,\xi) = E_{2k}(x,\ xi) - E_{2k}^r(x,\xi)$, then we can write

$$\begin{aligned}\mathcal{N}_{2m-2}(x,\xi) &= E_{2m-2}^r(x,\xi)\\ &\quad + g_{2m-2}^n(x,\xi) + h_{2m-2}(x,\xi) \equiv E_{2m-2}^r(x,\xi) + \hat{g}_{2m-2}^n(x,\xi). \quad (4.5.10)\end{aligned}$$

Let us show that the function

$$\begin{aligned}&\mathcal{N}_{2m}(x,\xi)\\ &\quad = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}_{2m-2}(x,y)\mathcal{N}_2(y,\xi)\,dy - \frac{1}{\tau_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy \cdot \int_S \mathcal{N}_2(y,\xi)\,dy\Big)\end{aligned}$$

for $m \geq 2$ can be represented as (4.5.6). Using (4.5.10) and keeping in mind (4.5.2) it is not difficult to obtain that

$$\begin{aligned}\mathcal{N}_{2m}(x,\xi) = E_{2m}^r(x,\xi) - \frac{1}{\omega_n}\Big(\int_S E_{2m-2}^r(x,y)E_0(y,\xi)\,dy\\ - \int_S \hat{g}_{2m-2}^n(x,y)E_2(y,\xi)\,dy + \int_S \hat{g}_{2m-2}^n(x,y)E_0(y,\xi)\,dy\\ + \frac{1}{\tau_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy \cdot \int_S \mathcal{N}_2(y,\xi)\,dy\Big),\end{aligned}$$

where the function $E_{2k}^r(x,\xi)$ is defined in (4.5.5). Let us evaluate the integral terms in the resulting equality. The first and third integrals are harmonic functions in ξ (the singularity in the first integral is integrable, and differentiation does not increase this singularity). The second integral, by the property of the volume potential, is an m-harmonic function in ξ, since the $(m-1)$-harmonic function $\hat{g}_{2m-2}^n(x,\xi)$ has bounded derivatives with respect to ξ in $\bar{S}$ for $x \in S$. The fourth integral, containing the variable ξ, is a biharmonic function in S. From (4.1.29) it follows that $\omega_n^{-1}\int_S \mathcal{N}_2(y,\xi)\,dy = 1/(2n-4) - |\xi|^2/2n$. If the sum of all integral terms in this equality is denoted by $\hat{g}_{2m}^n(x,\xi)$, then due to the fact that the singularities under the integrals have an order no higher than $|\xi - y|^{3-n}$ (see Lemma 4.5), the function $\hat{g}^n_{2m}(x,\xi)$ has continuous derivatives with respect to ξ in $\bar{S}$ for $x \in S$. Finally, if we remember that $h_{2m}(x,\xi) = E_{2m}(x,\xi) - E^r_{2m}(x,\xi)$, then we have (4.5.6) with $g^n_{2m}(x,\xi) = \hat{g}^n_{2m}(x,\xi) - h_{2m}(x,\xi)$. The equality (4.5.6) is proved.

Let us check the boundary conditions for the function $\mathcal{N}_{2m}(x,\xi)$ from Definition 4.5. Due to the symmetry of the function $\mathcal{N}_2(x,\xi)$ we can write $\Lambda_\xi \mathcal{N}_2(x,\xi)|_{\xi\in\partial S} = -1$. That is why

$$\frac{\partial \mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = \Lambda_\xi \mathcal{N}_{2m}(x,\xi)\big|_{\xi\in\partial S}$$
$$= \frac{1}{\omega_n}\Big(-\int_S \mathcal{N}_{2m-2}(x,y)\,dy + \frac{1}{\tau_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy \cdot \tau_n \Big) = 0.$$

For $x \in S$, by the property of the volume potential and due to the continuous differentiability of the function $\mathcal{N}_{2m-2}(x,\xi)$ with respect to $\xi \in \bar{S}$, but $\xi \neq x$, we obtain

$$\Delta_\xi \mathcal{N}_{2m}(x,\xi) = -\mathcal{N}_{2m-2}(x,\xi) + \frac{1}{\tau_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy. \tag{4.5.11}$$

Therefore, by the induction hypothesis, for $x \in S$ we have

$$\frac{\partial \Delta^k_\xi \mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = -\frac{\partial \Delta^{k-1}_\xi \mathcal{N}_{2m-2}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = 0, \quad k = 1,\dots,m-2$$

and

$$\frac{\partial \Delta^{m-1}_\xi \mathcal{N}_{2m}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = -\frac{\partial \Delta^{m-2}_\xi \mathcal{N}_{2m-2}(x,\xi)}{\partial \nu_\xi}\Big|_{\xi\in\partial S} = -(-1)^{m-1} = (-1)^m,$$

which proves the equalities (4.5.7).

Let us prove the equality (4.5.9) for $k > 1$, since for $k = 1$ it follows from (4.5.3). Let $k = 2$, then, due to the weak singularity of the function $\mathcal{N}_2(x,\xi)$, for $x \in \partial S$ and $\xi \in S$ we write

$$\Lambda_x \mathcal{N}_4(x,\xi) = \frac{1}{\omega_n}\int_S \Lambda_x \mathcal{N}_2(x,y)\mathcal{N}_2(y,\xi)\,dy$$
$$- \frac{1}{\tau_n}\int_S \Lambda_x \mathcal{N}_2(x,y)\,dy \cdot \frac{1}{\omega_n}\int_S \mathcal{N}_2(y,\xi)\,dy = -\frac{1}{\omega_n}\int_S \mathcal{N}_2(y,\xi)\,dy$$
$$+ \frac{1}{\omega_n}\int_S \mathcal{N}_2(y,\xi)\,dy = 0.$$

Therefore, for $k > 2$ and $x \in \partial S$, $\xi \in S$, using the equality (4.5.8) we have

$$\Lambda_x \mathcal{N}_{2k}(x,\xi) = \frac{1}{\omega_n}\int_S \Lambda_x \mathcal{N}_{2k-2}(x,y)\mathcal{N}_2(y,\xi)\,dy - \frac{1}{\tau_n}\int_S \Lambda_x \mathcal{N}_{2k-2}(x,y)\,dy \cdot \frac{1}{\omega_n}\int_S \mathcal{N}_2(y,\xi)\,dy = 0,$$

hence follows (4.5.9). The theorem is proved. □

Example 4.3. To find a solution to the Riquier-Neumann problem with polynomials in the boundary conditions or on the right side of the equation, it may be necessary to use the following equality

$$\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,\xi)|\xi|^{2l}\,d\xi = -\frac{|x|^{2l+2}}{(2l+2)(2l+n)} + \frac{1}{(2l+2)(n-2)}, \tag{4.5.12}$$

where $l \in \mathbb{N}_0$. Let us prove it. In Theorem 3.29 a similar equality is obtained

$$\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,\xi)|\xi|^{2l}H_k(\xi)\,d\xi = -\frac{|x|^{2l+2} - (2l+2+k)/k}{(2l+2)(2l+2k+n)}H_k(x), \tag{4.5.13}$$

where $H_k(x)$ is a homogeneous harmonic polynomial of degree k, which does not work in the case under consideration since its right-hand side is not defined for $k = 0$.

Let $\{H_k^{(i)}(x) : i = 1, \dots, h_k,\ k \in \mathbb{N}_0\}$ be a complete system of homogeneous degree $k \in \mathbb{N}_0$ harmonic polynomial orthogonal on ∂S from Definition 1.5. In the equality (3.7.12) from Theorem 3.29 it is established that

$$\frac{1}{\omega_n}\int_S E_2(x,\xi)|\xi|^{2l}H_k(\xi)\,d\xi = -\frac{|x|^{2l+2}H_k(x)}{(2l+2)(2l+2k+n)} + \frac{H_k(x)}{(2l+2)(2k+n-2)},$$

where $k \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$. Hence, for $k = 0$ we get

$$\frac{1}{\omega_n}\int_S E_2(x,\xi)|\xi|^{2l}\,d\xi = -\frac{|x|^{2l+2}}{(2l+2)(2l+n)} + \frac{1}{(2l+2)(n-2)}.$$

From the equality (3.7.13) of Theorem 3.29 it follows that for $x \in S$ and $\xi \in \bar{S}$

$$E_0(x,\xi) = -\sum_{k=1}^{\infty}\frac{k+n-2}{k(2k+n-2)}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi),$$

moreover, this series converges uniformly in ξ. Therefore we have

$$\int_S E_0(x,\xi)|\xi|^{2l}\,d\xi = -\sum_{k=1}^{\infty}\frac{k+n-2}{k(2k+n-2)}\sum_{i=1}^{h_k} H_k^{(i)}(x)\int_S H_k^{(i)}(\xi)|\xi|^{2l}\,d\xi = 0$$

and therefore, taking into account (4.5.2), we obtain (4.5.12).

4.5.3 Solution of the Riquier-Neumann problem

Let us find an integral representation of the solution to the Riquier-Neumann problem.

Theorem 4.14. *Let the function $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$ be a solution to the Riquier-Neumann problem* (4.5.1)*, then it can be represented as*

$$u(x) = \frac{(-1)^{m-1}}{\omega_n} \int_{\partial S} \sum_{k=0}^{m-1} \Big(\Delta_\xi^{m-k-1} \mathcal{N}_{2m}(x,\xi)\Big) \varphi_k(\xi)\, ds_\xi + \frac{(-1)^m}{\omega_n} \int_S \mathcal{N}_{2m}(x,\xi) f(\xi)\, d\xi + C. \quad (4.5.14)$$

Proof. Let $u \in C^{2m}(S) \cap C^{2m-1}(\bar{S})$ be the solution to the Riquier-Neumann problem (4.5.1). Let us use the equality

$$\int_D (v\Delta^m u - u\Delta^m v)\, d\xi = \int_{\partial D} \sum_{k=0}^{m-1} \Big(\Delta^k v \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta^k v}{\partial \nu} \Delta^{m-k-1} u\Big)\, ds_\xi$$

for $D = \{\xi : |\xi| < 1 - \varepsilon\}$, where $x \in D$, $\varepsilon > 0$ is sufficiently small and $v(\xi) = E_{2m}(x,\xi)$. Similar to the proof of (4.2.12) we obtain

$$\int_D E_{2m}(x,\xi) f(\xi)\, d\xi = (-1)^m \omega_n u(x) + \int_{\partial D} \sum_{k=0}^{m-1} \Big(\Delta_\xi^k E_{2m}(x,\xi) \frac{\partial \Delta^{m-k-1} u}{\partial \nu} - \frac{\partial \Delta_\xi^k E_{2m}(x,\xi)}{\partial \nu} \Delta^{m-k-1} u\Big)\, ds_\xi.$$

If we now again use the previous equality for the same domain D, but with $v(\xi) = g_{2m}^n(x,\xi)$ (m-harmonic function in S from (4.5.6)), then we obtain a similar equality with $g_{2m}^n(x,\xi)$ instead of $E_{2m}(x,\xi)$ and without the first term on the right. Adding these equalities, grouping the integral terms and changing the summation index $k \to m-k-1$ we get

$$u(x) = \frac{(-1)^m}{\omega_n} \int_{|\xi|=1-\varepsilon} \sum_{k=0}^{m-1} \Big(\frac{\partial \Delta_\xi^{m-k-1} \mathcal{N}_{2m}(x,\xi)}{\partial \nu} \Delta^k u(\xi) - \Delta_\xi^{m-k-1} \mathcal{N}_{2m}(x,\xi) \frac{\partial \Delta^k u}{\partial \nu}\Big)\, ds_\xi + \frac{(-1)^m}{\omega_n} \int_{|\xi|<1-\varepsilon} \mathcal{N}_{2m}(x,\xi) f(\xi)\, d\xi. \quad (4.5.15)$$

Let us go to the limit at $\varepsilon \to +0$. In this case, we take into account that the functions $g_{2k}^n(x,\xi)$, and therefore $\mathcal{N}_{2m}(x,\xi)$ and its derivatives with respect to ξ, are continuous in ξ in $\{\xi : 1-\varepsilon < |\xi| \le 1\}$ for fixed $x \in S$. Therefore, due to the properties of the Green's function (4.5.7), surface integrals over $S = \{\xi : |\xi| = 1\}$ containing functions $\frac{\partial}{\partial \nu_\xi}(\Delta^{m-k-1}\mathcal{N}_{2m}(x,\xi))$ for $k = 1,\dots,m-1$ turn to 0, and the function $\frac{\partial}{\partial \nu_\xi}(\Delta^{m-1}\mathcal{N}_{2m}(x,\xi))$ becomes $(-1)^m$. In addition, due to the boundary conditions of the Riquier-Neumann problem, we also have $\frac{\partial}{\partial \nu}\Delta^k u(\xi) \to \varphi_k(\xi)$, $\varepsilon \to +0$ at $k = 0,\dots,m-1$. Thus, from (4.5.15) in the limit for $\varepsilon \to +0$ we obtain the representation (4.5.14) for $C = \frac{1}{\omega_n}\int_{\partial S} u(\xi)\, ds_\xi$. The theorem is proved. □

Lemma 4.6. *1. Let $f \in C^1(\bar S)$, then the function*

$$u_f(x) = \frac{(-1)^m}{\omega_n}\int_S \mathcal{N}_{2m}(x,\xi) f(\xi)\, ds_\xi \tag{4.5.16}$$

is a solution to the following Riquier-Neumann problem

$$\Delta^m u(x) = f(x),\ x \in S,$$
$$\frac{\partial \Delta^k u}{\partial \nu}\Big|_{\partial S} = 0,\ k = 0,\dots,m-2,\ \frac{\partial \Delta^{m-1} u}{\partial \nu}\Big|_{\partial S} = \frac{1}{\omega_n}\int_S f(\xi)\, d\xi.$$

2. Let $\psi \in C^{1+\varepsilon}(\partial S)$ ($\varepsilon > 0$), then the function

$$v_\psi(x) = \frac{(-1)^{m-1}}{\omega_n}\int_{\partial S} \mathcal{N}_{2m}(x,\xi)\psi(\xi)\, ds_\xi \tag{4.5.17}$$

is a solution to the following Riquier-Neumann problem

$$\Delta^m v(x) = 0,\ x \in S,$$
$$\frac{\partial \Delta^k v}{\partial \nu}\Big|_{\partial S} = 0,\ k = 0,\dots,m-2,\ \frac{\partial \Delta^{m-1} v}{\partial \nu}\Big|_{\partial S} = \psi(x) - \frac{1}{\omega_n}\int_{\partial S} \psi(\xi)\, ds_\xi.$$

Proof. 1. We study the function $u_f(x)$ without specifying the smoothness of the function $f(x)$. To do this, we introduce the functions

$$\hat{\mathcal{N}}_2(x,\xi) = \mathcal{N}_2(x,\xi) - \frac{1}{\tau_n}\int_S \mathcal{N}_2(y,\xi)\, dy,\ \hat{\mathcal{N}}_{2k}(x,\xi) = \frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_{2k-2}(x,y)\hat{\mathcal{N}}_2(y,\xi)\, dy,\ k > 1.$$

The function $\hat{\mathcal{N}}_2(x,\xi)$ has the property

$$\int_S \hat{\mathcal{N}}_2(x,\xi)\,dx = \int_S \mathcal{N}_2(x,\xi)\,dx - \frac{\tau_n}{\tau_n}\int_S \mathcal{N}_2(y,\xi)\,dy = 0.$$

It is easy to see that the equality holds true

$$\begin{aligned}\mathcal{N}_{2m}(x,\xi) &= \frac{1}{\omega_n}\int_S \mathcal{N}_{2m-2}(x,y)\Big(\mathcal{N}_2(y,\xi) - \frac{1}{\tau_n}\int_S \mathcal{N}_2(y_1,\xi)\,dy_1\Big)\,dy \\ &= \frac{1}{\omega_n}\int_S \mathcal{N}_{2m-2}(x,y)\hat{\mathcal{N}}_2(y,\xi)\,dy. \quad (4.5.18)\end{aligned}$$

Therefore, due to the properties of the volume potential, rearranging the integrals, we have

$$\begin{aligned}\Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_4(x,\xi)f(\xi)\,d\xi &= \Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_2(x,y)\,dy \\ &\times \frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)f(\xi)\,d\xi = -\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(x,\xi)f(\xi)\,d\xi,\end{aligned}$$

from which it follows similarly

$$\begin{aligned}\Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_6(x,\xi)f(\xi)\,d\xi &= \Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_4(x,y)\,dy\,\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)f(\xi)\,d\xi \\ = -\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(x,y)\,dy\,\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)f(\xi)\,d\xi &= -\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_4(y,\xi)f(\xi)\,d\xi.\end{aligned}$$

Using the previous equalities by induction, changing the order of integration, we obtain

$$\begin{aligned}\Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_{2m}(x,\xi)f(\xi)\,d\xi &= \Delta_x \frac{1}{\omega_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy \\ \times \frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)f(\xi)\,d\xi &= -\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_{2m-4}(x,y)\,dy\,\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)f(\xi)\,d\xi \\ &= -\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_{2m-2}(y,\xi)f(\xi)\,d\xi. \quad (4.5.19)\end{aligned}$$

From the last equality, using the property of the volume potential, it is easy to obtain

$$\Delta_x^m u_f(x) = \Delta_x^m \frac{(-1)^m}{\omega_n}\int_S \mathcal{N}_{2m}(x,\xi)f(\xi)\,d\xi = -\Delta_x \frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(x,\xi)f(\xi)\,d\xi$$

$$= -\Delta_x \frac{1}{\omega_n} \int_S \mathcal{N}_2(x,\xi) f(\xi)\, d\xi = f(x). \quad (4.5.20)$$

Further, due to the weak singularity of the function $\mathcal{N}_{2m}(x,\xi)$ (the singularity is the same as that of the elementary solution $E_{2m}(x,\xi)$ (4.5.6)) it can be differentiated under the integral sign and therefore from (4.5.9) it follows that for $m > 1$ the equalities hold true

$$\frac{\partial u_f}{\partial \nu}\Big|_{\partial S} = \Lambda_x u_f(x)\big|_{\partial S} = \frac{(-1)^m}{\omega_n} \int_S \Lambda_x \mathcal{N}_{2m}(x,\xi)\big|_{x\in\partial S} f(\xi)\, d\xi = 0.$$

In addition, due to (4.5.19) for $k = 1, \dots, m-2$ we obtain

$$\frac{\partial \Delta^k u_f}{\partial \nu}\Big|_{\partial S} = \Lambda_x \Delta_x^k u_f(x)\big|_{\partial S} = \frac{(-1)^{m-k}}{\omega_n} \int_S \Lambda_x \hat{\mathcal{N}}_{2m-2k}(x,\xi)\big|_{x\in\partial S} f(\xi)\, d\xi$$
$$= \frac{(-1)^{m-k}}{\omega_n} \int_S \Lambda_x \mathcal{N}_{2m-2k}(x,\xi)\big|_{x\in\partial S} f(\xi)\, d\xi = 0.$$

Similar to what is done above, in accordance with (4.5.19) and (4.5.9), we find

$$\frac{\partial \Delta^{m-1} u_f}{\partial \nu}\Big|_{\partial S} = -\frac{1}{\omega_n} \int_S \Lambda_x \hat{\mathcal{N}}_2(x,\xi)\big|_{x\in\partial S} f(\xi)\, d\xi = \frac{1}{\omega_n} \int_S f(\xi)\, d\xi.$$

This means that the function $u_f(x)$ from (4.5.16) is a solution to the Riquier-Neumann problem (4.5.1) with $\varphi_k = 0$, $k = 0, \dots, m-2$ and $\varphi_{m-1} = \frac{1}{\omega_n} \int_S f(\xi)\, d\xi$.

Let us determine what smoothness it is enough to require from the function $f(x)$ so that all arguments made regarding the function $u_f(x)$ are valid. To satisfy the last equality from (4.5.20) for the volume potential it is sufficient that $f \in C^1(\bar{S})$ [28]. In this case, the volume potential with density $f(x)$ is also be from $C^1(\bar{S})$ [183] and therefore the entire chain of equalities from (4.5.20) is correct.

2. Consider the function $v_\psi(x)$ from (4.5.17). Substituting the value of the function $\mathcal{N}_{2m}(x,\xi)$ from (4.5.18) and changing the order of integration we have

$$v_\psi(x) = \frac{(-1)^{m-1}}{\omega_n} \int_S \mathcal{N}_{2m-2}(x,y)\, dy\, \frac{1}{\omega_n} \int_{\partial S} \hat{\mathcal{N}}_2(y,\xi)\psi(\xi)\, ds_\xi.$$

Due to the properties of the functions $u_f(x)$ and $\hat{\mathcal{N}}_2(x,\xi)$ obtained above, the function $v_\psi(x)$ is a solution to the Riquier-Neumann problem (4.5.1) for $m = m-1$, $\varphi_k = 0$, $k = 0, \dots, m-3$,

$$\varphi_{m-2} = \frac{1}{\omega_n}\int_S \frac{1}{\omega_n}\int_{\partial S} \hat{\mathcal{N}}_2(y,\xi)\psi(\xi)\,ds_\xi\,dy$$
$$= \frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi \frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)\,dy = 0$$

and

$$f(x) = \Delta^{m-1}v(x) = \frac{1}{\omega_n}\int_{\partial S}\hat{\mathcal{N}}_2(x,\xi)\psi(\xi)\,ds_\xi.$$

Therefore, by the property of the potential of a simple layer, we have

$$\Delta^m v_\psi(x) = \Delta\Delta^{m-1}v_\psi(x) = \Delta\frac{1}{\omega_n}\int_{\partial S}\hat{\mathcal{N}}_2(x,\xi)\psi(\xi)\,ds_\xi = 0,$$

and in addition, by the property (4.5.4) of the Green's function $\mathcal{N}_2(x,\xi)$ we write

$$\frac{\partial\Delta^{m-1}v_\psi}{\partial\nu}\Big|_{\partial S} = \psi(x) - \frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi.$$

Thus, the function $v_\psi(x)$ is a solution to the Riquier-Neumann problem (4.5.1) for $f=0$, $\varphi_k=0$, $k=0,\dots,m-2$ and $\varphi_{m-1}(x) = \psi(x) - \frac{1}{\omega_n}\int_{\partial S}\psi(\xi)\,ds_\xi$, as stated.

Finally, let us find out what smoothness of the function $\psi(x)$ is sufficient for the validity of the above reasoning? Since the function $v_\psi(x)$ is represented in a form similar to the form of the function $u_f(x)$ from (4.5.16) with the density $\frac{1}{\omega_n}\int_S \hat{\mathcal{N}}_2(y,\xi)\psi(\xi)\,ds_\xi$, then, as in the previous case, it is sufficient that this density be from $C^1(\bar{S})$, and this is true if $\psi\in C^{1+\varepsilon}(\partial S)$ [4]. The lemma is proved. □

Remark 4.6. For $m=1$, for the function ψ it is sufficient to require that $\psi\in C(\partial S)$.

Theorem 4.15. *Let $\varphi_0\in C(\partial S)$, $\varphi_k\in C^{1+\varepsilon}(\partial S)$ for $k=1,\dots,m-1$ and $f\, inC^1(\bar{S})$. Then the function*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S}\sum_{k=0}^{m-1}(-1)^k\Big(\mathcal{N}_{2k+2}(x,\xi) - \frac{1}{\tau_n}\int_S\mathcal{N}_{2k+2}(x,y)\,dy\Big)\varphi_k(\xi)\,ds_\xi$$
$$+\frac{(-1)^m}{\omega_n}\int_S\mathcal{N}_{2m}(x,\xi)f(\xi)\,d\xi + C \quad (4.5.21)$$

is a solution to the Riquier-Neumann problem (4.5.1) *provided that*

$$\int_{\partial S}\varphi_{m-1}(\xi)\,ds_\xi = \int_S f(\xi)\,d\xi. \quad (4.5.22)$$

Proof. Let us prove that the function $u(x)$, defined by the equality (4.5.14), is a solution to the following problem

$$\begin{aligned} \Delta^m u(x) &= f(x),\ x\in S,\\ \frac{\partial\Delta^k u}{\partial\nu}\Big|_{\partial S} &= 0,\ k=0,\dots,m-2,\\ \frac{\partial\Delta^{m-1}u}{\partial\nu}\Big|_{\partial S} &= \varphi_{m-1}(x)-\frac{1}{\omega_n}\int_{\partial S}\varphi_{m-1}(\xi)\,ds_\xi+\frac{1}{\omega_n}\int_S f(\xi)\,d\xi. \end{aligned} \tag{4.5.23}$$

It is easy to see that in this case, if the condition (4.5.22) is satisfied, the statement of the theorem follows. We prove this statement by induction on m. For $m=2$, Theorem 4.7 on page 337 proves that for a function of the form

$$\begin{aligned} u(x) = -\frac{1}{\omega_n}\int_{\partial S}\Delta_\xi\mathcal{N}_4(x,\xi)\varphi_0(\xi)\,ds_\xi-\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_4(x,\xi)\varphi_1(\xi)\,ds_\xi \\ +\frac{1}{\omega_n}\int_S\mathcal{N}_4(x,\xi)f(\xi)\,d\xi+C\equiv v_1(x)+v_2(x)+v_3(x)+C \end{aligned}$$

the equalities hold true

$$\begin{aligned} \Delta^2u(x) &= 0+0+f(x)=f(x),\\ \frac{\partial u}{\partial\nu}\Big|_{x\in\partial S} &= \varphi_0(x)+0+0=\varphi_0(x),\\ \frac{\partial\Delta u}{\partial\nu}\Big|_{x\in\partial S} &= 0+\left(\varphi_1(x)-\frac{1}{\omega_n}\int_{\partial S}\varphi_1(\xi)\,d\xi\right)+\frac{1}{\omega_n}\int_S f(\xi)\,d\xi. \end{aligned}$$

Here, in the equality on the right, the result of applying the corresponding operators to the functions $v_1(x)$, $v_2(x)$ and $v_3(x)$ is indicated. This corresponds to the statement being proved for $m=2$. Let us assume that the statement (4.5.23) holds true for $m=m-1$. Let us represent $u(x)$ from (4.5.14) as

$$u(x)=u_m(x)+u_f^{(m)}(x)+C,$$

where the function $u_f^{(m)}(x)$ is defined in (4.5.16) (the superscript m is additionally introduced here to avoid confusion) and

$$u_m(x)=\frac{(-1)^{m-1}}{\omega_n}\int_{\partial S}\sum_{k=0}^{m-1}\left(\Delta_\xi^{m-k-1}\mathcal{N}_{2m}(x,\xi)\right)\varphi_k(\xi)\,ds_\xi.$$

Note that the function $u_f(x)$ for $f\in C^1(\bar S)$ by Lemma 4.6 is a solution to the problem (4.5.23) for $\varphi_k=0$, $k=0,\dots,m-1$. Therefore, it is enough to prove that the function $u_m(x) = u(x) - u_f^{(m)}(x) + C$ is a solution to the problem (4.5.23) for $f = 0$. Let us transform the function $u_m(x)$. From the equality (4.5.11) for $k > 1$ it is easy to obtain

$$\Delta_\xi^k \mathcal{N}_{2m}(x,\xi) = -\Delta_\xi^{k-1}\mathcal{N}_{2m-2}(x,\xi),$$

and therefore

$$\begin{aligned}
u_m(x) \\
&= \frac{(-1)^{m-1}}{\omega_n}\Big(\int_{\partial S}\mathcal{N}_{2m}(x,\xi)\varphi_{m-1}(\xi)\,ds_\xi + \int_{\partial S}\Delta_\xi\mathcal{N}_{2m}(x,\xi)\varphi_{m-2}(\xi)\,ds_\xi\Big) \\
&\quad + \frac{(-1)^{m-2}}{\omega_n}\Big(\int_{\partial S}\sum_{k=0}^{m-2}\Big(\Delta_\xi^{m-k-2}\mathcal{N}_{2m-2}(x,\xi)\Big)\varphi_k(\xi)\,ds_\xi \\
&\quad - \int_{\partial S}\mathcal{N}_{2m-2}(x,\xi)\varphi_{m-2}(\xi)\,ds_\xi\Big) = u_{m-1}(x) \\
&+ \frac{(-1)^{m-1}}{\omega_n}\Big(\int_{\partial S}\mathcal{N}_{2m}(x,\xi)\varphi_{m-1}(\xi)\,ds_\xi + \int_{\partial S}\Delta_\xi\mathcal{N}_{2m}(x,\xi)\varphi_{m-2}(\xi)\,ds_\xi \\
&\quad + \int_{\partial S}\mathcal{N}_{2m-2}(x,\xi)\varphi_{m-2}(\xi)\,ds_\xi\Big).
\end{aligned}$$

Using the equality (4.5.11) and the notations from (4.5.16) and (4.5.17) we can write

$$\begin{aligned}
u_m(x) &= u_{m-1}(x) \\
&\quad + v_{\varphi_{m-1}}(x) + \frac{(-1)^{m-1}}{\omega_n}\int_S \mathcal{N}_{2m-2}(x,y)\,dy\cdot\frac{1}{\tau_n}\int_{\partial S}\varphi_{m-2}(\xi)\,ds_\xi \\
&= u_{m-1}(x) + v_{\varphi_{m-1}}(x) + u_1^{(m-1)}(x)\cdot\frac{1}{\tau_n}\int_{\partial S}\varphi_{m-2}(\xi)\,ds_\xi. \qquad (4.5.24)
\end{aligned}$$

For the legality of using the function $v_{\varphi_{m-1}}(x)$ by Lemma 4.6 it is sufficient to require $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ for $k = 1, \ldots, m-1$. Since, by the induction hypothesis, the function $u_{m-1}(x)$ is $(m-1)$-harmonic, then according to Lemma 4.6 we can write

$$\Delta^m u_m(x) = 0 + 0 + \Delta\frac{1}{\tau_n}\int_{\partial S}\varphi_{m-2}(\xi)\,ds_\xi = 0.$$

In addition, from (4.5.24) it also follows that on ∂S, due to Lemma 4.6 and the induction hypothesis, the equalities hold true

$$\frac{\partial\Delta^k u_m}{\partial\nu} = \frac{\partial\Delta^k u_{m-1}}{\partial\nu} + 0 + 0 = \varphi_k(x), \quad k = 0, \ldots, m-3,$$

$$
\begin{aligned}
\frac{\partial \Delta^{m-2} u_m}{\partial \nu} &= \frac{\partial \Delta^{m-2} u_{m-1}}{\partial \nu} + 0 + \frac{1}{\omega_n} \int_S \frac{1}{\tau_n} \int_{\partial S} \varphi_{m-2}(\xi)\, ds_\xi\, dy \\
&= \varphi_{m-2}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-2}(\xi)\, ds_\xi + \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-2}(\xi)\, ds_\xi = \varphi_{m-2}(x), \\
\frac{\partial \Delta^{m-1} u_m}{\partial \nu} &= 0 + \Big(\varphi_{m-1}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-1}(\xi)\, ds_\xi \Big) + 0 \\
&= \varphi_{m-1}(x) - \frac{1}{\omega_n} \int_{\partial S} \varphi_{m-1}(\xi)\, ds_\xi,
\end{aligned}
$$

which means the function $u_m(x)$ is a solution to the problem (4.5.23) for $f = 0$. The induction step is proved and therefore the function from (4.5.14) is a solution to the problem (4.5.1).

From the formula (4.5.11) it is easy to obtain the equality

$$
\Delta_\xi^k \mathcal{N}_{2m}(x,\xi) = (-1)^k \Big(\mathcal{N}_{2m-2k}(x,\xi) - \frac{1}{\tau_n} \int_S \mathcal{N}_{2m-2k}(x,y)\, dy \Big),
$$

with the help of which the equality (4.5.14) is converted to the form (4.5.21). The theorem is proved. □

4.5.4 A special case of the Riquier-Neumann problem

Let us consider one special case of boundary conditions in the Riquier-Neumann problem.

Theorem 4.16. *Let $\varphi_0 \in C(\partial S)$, $\varphi_k \in C^{1+\varepsilon}(\partial S)$ and $\int_{\partial S} \varphi_k(\xi)\, ds_\xi = 0$ for $k = 1, \dots, m-1$, and $f = 0$. Then a solution to the Riquier-Neumann problem* (4.5.1) *exists and can be written in the form*

$$
u(x) = \sum_{k=0}^{m-1} u_k[\varphi_k](x) + C, \tag{4.5.25}
$$

where is denoted

$$
u_k[\varphi](x) = \frac{(-1)^k}{\omega_n} \int_{\partial S} \mathcal{N}_{2k+2}^0(x,\xi) \varphi(\xi)\, ds_\xi,
$$

$$
\mathcal{N}_{2k+2}^0(x,\xi) = \frac{1}{\omega_n} \int_S \mathcal{N}_2(x,y) \mathcal{N}_{2k}^0(y,\xi)\, dy,\ k > 0; \quad \mathcal{N}_2^0(x,\xi) = \mathcal{N}_2(x,\xi).
$$

For $(k+1)$-harmonic functions $u_k[\varphi_k](x)$ the following conditions are satisfied:

$$
\begin{aligned}
\Delta u_k[\varphi](x) = u_{k-1}[\varphi](x), \quad \frac{\partial u_k[\varphi]}{\partial \nu}\Big|_{\partial S} = 0,\ k \in \mathbb{N};\\
\Delta u_0[\varphi](x) = 0, \quad \frac{\partial u_0[\varphi]}{\partial \nu}\Big|_{\partial S} = \varphi(x).
\end{aligned}
\tag{4.5.26}
$$

Proof. It is easy to see that under the conditions required in the theorem, all the conditions of Theorem 4.15 are also satisfied, which means that a solution to the Riquier-Neumann problem exists. Let us prove that the formula (4.5.25) defines this solution. To do this, it is enough to make sure that the equalities (4.5.26) hold true. According to the definition of the function $\mathcal{N}_{2k}^0(x,\xi)$ and according to the property of the volume potential, for $k>0$ we write

$$
\begin{aligned}
\Delta u_k[\varphi](x) &= \Delta_x \frac{(-1)^k}{\omega_n}\int_{\partial S} \mathcal{N}_{2k+2}^0(x,\xi)\varphi(\xi)\,ds_\xi \\
&= \frac{(-1)^{k-1}}{\omega_n}\int_{\partial S} \mathcal{N}_{2k}^0(x,\xi)\varphi(\xi)\,ds_\xi = u_{k-1}[\varphi](x),
\end{aligned}
$$

and also note that $\Delta u_0[\varphi](x) = 0$. In addition, for $k>0$, using (4.5.4), we obtain

$$
\begin{aligned}
\frac{\partial u_k[\varphi]}{\partial \nu}\Big|_{\partial S} &= \frac{(-1)^k}{\omega_n}\int_S \Lambda_x \mathcal{N}_2(x,y)\big|_{x\in\partial S}\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_{2k}^0(y,\xi)\varphi(\xi)\,ds_\xi\,dy \\
&= \frac{(-1)^{k-1}}{\omega_n}\int_{\partial S}\Big(\int_S \mathcal{N}_{2k}^0(y,\xi)\,dy\Big)\varphi(\xi)\,ds_\xi \equiv \int_{\partial S} F(\xi)\varphi(\xi)\,ds_\xi.
\end{aligned}
$$

Using the equality

$$
\frac{1}{\omega_n}\int_S \mathcal{N}_{2k}^0(y_1,\xi)\,dy_1 = \frac{1}{\omega_n}\int_S \mathcal{N}_2(y_1,y_2)\,dy_1 \ldots \frac{1}{\omega_n}\int_S \mathcal{N}_2(y_k,\xi)\,dy_k,
$$

symmetry of function $\mathcal{N}_2(x,\xi)$ and the equality (4.5.12) we are convinced that the function

$$
F(\xi) = \frac{(-1)^{k-1}}{\omega_n}\int_S \mathcal{N}_{2k}^0(y,\xi)\,dy
$$

is a polynomial of degree k in $|\xi|^2$ which means $F(\xi)|_{\partial S} = C$, and therefore by the conditions of the theorem

$$
\frac{\partial u_k[\varphi_k]}{\partial \nu}\Big|_{\partial S} = \int_{\partial S} F(\xi)\varphi_k(\xi)\,ds_\xi = C\int_{\partial S}\varphi_k(\xi)\,ds_\xi = 0.
$$

If $k = 0$, then according to (4.5.4)

$$\frac{\partial u_0[\varphi_0]}{\partial \nu}\Big|_{\partial S} = \frac{1}{\omega_n}\int_{\partial S}\frac{\partial \mathcal{N}_2(x,\xi)}{\partial \nu_x}\big|_{\partial S}\varphi_0(\xi)\,ds_\xi$$
$$= \varphi_0(x)\big|_{\partial S} - \frac{1}{\omega_n}\int_{\partial S}\varphi_0(\xi)\,ds_\xi = \varphi_0(x).$$

The theorem is proved. □

Example 4.4. Let us calculate the functions $u_p[H_k](x)$ from the equality (4.5.25), where $H_k(x)$ is a homogeneous harmonic polynomial of degree $k \in \mathbb{N}$ for $p \in \mathbb{N}_0$. In this case, the conditions of Theorem 4.16 are satisfied since the equality $\int_{\partial S} H_k(\xi)\,ds_\xi = 0$ holds true.

From the equality (4.1.28) of ection 4.1 for $x \in S$ and $\xi \in \partial S$ the equalities follow

$$\mathcal{N}_2(x,\xi) = E(x,\xi) - E_0(x,\xi) = \frac{1}{n-2}$$
$$+\sum_{k=1}^{\infty}\Big(\frac{1}{2k+n-2} + \frac{k+n-2}{k(2k+n-2)}\Big)\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi)$$
$$= \frac{1}{n-2} + \sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{h_k} H_k^{(i)}(x)H_k^{(i)}(\xi),$$

where the harmonic polynomials $H_k^{(i)}(x)$ are defined in Example 4.3. Therefore, due to the orthonormality of the polynomials $\frac{1}{\sqrt{\omega_n}}H_k^{(i)}(x)$ on ∂S and the uniform convergence of the series in $\xi \in \partial S$ for $x \in S$, we have

$$u_0[H_k](x) = \frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_2(x,\xi)H_k(\xi)\,ds_\xi = \frac{1}{(n-2)\omega_n}\int_{\partial S}H_k(\xi)\,ds_\xi$$
$$+\sum_{m=1}^{\infty}\frac{1}{m}\sum_{i=1}^{h_m}H_m^{(i)}(x)\frac{1}{\omega_n}\int_{\partial S}H_m^{(i)}(\xi)H_k(\xi)\,ds_\xi = \frac{1}{k}H_k(x).$$

Let us calculate $u_1[H_k](x)$. Using (4.5.13) for $l = 0$ and the previous calculations we find

$$u_1[H_k](x) = -\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_4^0(x,\xi)H_k(\xi)\,ds_\xi$$
$$= -\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,y)\frac{1}{\omega_n}\int_{\partial S}\mathcal{N}_2(y,\xi)H_k(\xi)\,ds_\xi\,dy$$

$$= -\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,y)u_0[H_k](y)\,dy = -\frac{1}{k}\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,y)H_k(y)\,dy$$
$$= \frac{1}{k}\frac{|x|^2-1-2/k}{2(2k+n)}H_k(x).$$

Similarly, in the general case, for $u_p[H_k](x)$ with $p \in \mathbb{N}$ we obtain

$$u_p[H_k](x) = \frac{1}{\omega_n}\int_{\partial S}\mathcal{N}^0_{2p+2}(x,\xi)H_k(\xi)\,ds_\xi$$
$$= -\frac{1}{\omega_n}\int_S \mathcal{N}_2(x,y)u_{p-1}[H_k](y)\,dy. \quad (4.5.27)$$

Hence, using the function $u_1[H_k](x)$ found above and the equality (4.5.13), we have

$$u_2[H_k](x) = \Big(\frac{|x|^4-1-4/k}{8k(2k+n)(2k+2+n)} - (k+2)\frac{|x|^2-1-2/k}{4k^2(2k+n)^2}\Big)H_k(x).$$

It is easy to verify that the 3-harmonic function $u_2[H_k](x)$ satisfies the condition

$$\frac{\partial u_2[H_k]}{\partial \nu}\Big|_{\partial S}$$
$$= \Big(\frac{(k+4)|x|^4-k-4}{8k(2k+n)(2k+2+n)} - (k+2)\frac{(k+2)|x|^2-k-2}{4k^2(2k+n)^2}\Big)H_k(x)\Big|_{\partial S} = 0,$$

and since

$$\Delta u_2[H_k] = \Big(\frac{|x|^2}{2k(2k+n)} - \frac{(k+2)}{2k^2(2k+n)}\Big)H_k(x) = u_1[H_k],$$

then

$$\frac{\partial \Delta u_2[H_k]}{\partial \nu}\Big|_{\partial S} = \frac{(k+2)|x|^2-k-2}{2k(2k+n)}H_k(x)\Big|_{\partial S} = 0.$$

Moreover, the equalities hold true

$$\Delta^2 u_2[H_k] = \frac{1}{k}H_k(x) = u_0[H_k], \qquad \frac{\partial \Delta^2 u_2[H_k]}{\partial \nu}\Big|_{\partial S} = H_k(x).$$

Using (4.5.27) and (4.5.13) one can sequentially find any function $u_p[H_k](x)$.

Chapter 5

Nonlocal boundary value problems with involution

This chapter studies nonlocal boundary value problems with involution. From the author's works devoted to nonlocal boundary value problems, this chapter includes results on four problems for a nonlocal biharmonic equation with involution and on eigenfunctions for the nonlocal Laplace operator with double involution.

In [145] the authors considered equations containing fractional derivatives of the desired function and equations with deviating arguments, in other words, equations that include an unknown function and its derivatives, generally speaking, for different values of arguments. Such equations are called nonlocal differential equations.

Among the nonlocal differential equations a special place is occupied by equations in which deviation of the arguments has an involutive character. A mapping $\mathcal{S}$ is called an involution, if $\mathcal{S}^2(x) = \mathcal{S}(\mathcal{S}(x)) = x$. It is known that differential equations containing an involutive deviation of an unknown function or its derivative are some model equations with an alternating deviation of the argument. In the general case, such equations can be attributed to the class of functional differential equations.

Researchers have been studying differential equations with involution for a long time. In 1816 Babbage considered algebraic and differential equations with involution [14]. Monographs by Przeworska-Rolewicz [152] and Wiener [185] have been devoted to the solvability's theory of various differential equations with involution. In [16, 37], spectral problems for a first order differential equation with involution are studied. In [134, 135, 137, 160, 173] spectral problems for differential operators of the first and second orders with involution are investigated. Some results of studies of spectral problems with involution are used in [2, 128, 169] to solve inverse problems.

The series of papers by Cabada and Tojo (see [40, 168] and a list of citations) are devoted to creation of the theory of Green's function of one-dimensional differential equations with involution. The method of representing solutions to boundary value problems based on the construction of the Green's function of the problem is one of the efficient methods for investigation of solutions to boundary value problems for elliptic equations. This method is also used in the present chapter. The Green's functions of the Dirichlet, Neumann, Robin, and others biharmonic problems in a two-dimensional disc are constructed by means of the Green's harmonic functions of the Dirichlet problem in [20].

Some aspects of the solvability theory of partial differential equations with involution are described in [9,12,13]. In [174], boundary value problems with involution in the boundary conditions for second and fourth order elliptic equations are studied.

Section 5.1 is organized as follows. In Section 5.1.1 we present the statements of the problems under consideration, and then in Lemmas 5.1-5.2 and Theorem 5.1 some auxiliary statements are given which are necessary in the further presentation. In Theorem 5.2 of Section 5.1.2 uniqueness of solutions to all considering problems is proved. In Section 5.1.3, based on Theorem 5.3, which gives a representation of the solution to the Dirichlet problem for a biharmonic equation in terms of the Green's function $G_4(x, \xi)$, the existence Theorem 5.4 for a solution to the Dirichlet boundary value problem for a nonlocal biharmonic equation is presented. In Section 5.1.4 using Lemma 5.7, the necessary and sufficient conditions for the solvability of the Neumann boundary value problem are obtained. In Theorem 5.5 an integral representation of the problem's solution is presented. In Section 5.1.5 the Navier boundary value problem is studied. In Theorem 5.6 using the Green's function of the corresponding Navier boundary value problem for biharmonic equation, an integral representation of a solution to the considered problem is obtained. Finally, in Theorem 5.7 of Section 5.1.6, conditions for the existence of a solution to the Riquier-Neumann boundary value problem are found, and an integral representation of the problem's solution is obtained.

In Section 5.2 we study the eigenfunctions of the nonlocal Laplace operator with double involution of arbitrary orders. First, matrices of a special form arising in this problem are studied. In Theorems 5.8-5.9 the block structure of these matrices is clarified, in Theorem 5.10 the multiplication of matrices is studied, and in theorem 5.11 their eigenvalues and eigenvectors are found. Then, in Theorems 5.12-5.13, using Lemma 5.3 the structure of eigenfunctions and eigenvalues of the Dirichlet problem is studied.

In Theorems 5.14-5.15, using Lemma 5.9, the eigenfunctions and eigenvalues of the Dirichlet problem for the considered nonlocal Laplace equation in the unit ball are determined. These eigenfunctions are presented explicitly. The completeness of the found system of eigenfunctions in $L_2(S)$ is proved. The introduced concepts and the results obtained are illustrated in seven examples.

More detailed information about the author's results on non-local problems can be found in the articles [100, 103, 108, 109, 115, 121, 176, 180].

5.1 Boundary value problems for nonlocal biharmonic equation

The purpose of this section is to apply the results obtained in Chapter 4 on the construction of Green's functions to the study of boundary value problems for nonlocal equations. Here we consider four boundary value problems for the nonlocal biharmonic equation.

5.1.1 Main problems and supporting statements

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, $n \geq 2$, ∂S be the unit sphere, and $\mathcal{S}$ be a real orthogonal matrix $\mathcal{S} \cdot \mathcal{S}^T = I$. Suppose also that there exists a natural number $l \in \mathbb{N}$ such that $\mathcal{S}^l = I$. Let us call l the order of $\mathcal{S}$.

Note that if $x \in S$, or $x \in \partial S$, then for any k the inclusions $\mathcal{S}^k x \in S$, or $\mathcal{S}^k x \in \partial S$ hold true. This is so because the transformation of the space $\mathbb{R}^n$ with the matrix $\mathcal{S}$ saves the norm $|x|^2 = (x, x) = (\mathcal{S}^T\mathcal{S}x, x) = (\mathcal{S}x, \mathcal{S}x) = |\mathcal{S}x|^2$. Let us give one simple example of such a matrix $\mathcal{S}$.

Example 5.1. It is obvious that the transformation carried out by the matrix $\mathcal{S}$ can also be a rotation in the space $\mathbb{R}^n$, for example, if $\mathcal{S} = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}}$, where

$$C^i_\varphi = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & I_{n-i-2} \end{pmatrix},$$

I_i is the identity $i \times i$ matrix and $i = 1, \dots, n-2$. This is so because $\mathcal{S}^T = C^{n-2}_{-\varphi_{n-2}} \cdots C^2_{-\varphi_2} C^1_{-\varphi_1}$ and $C^i_\varphi C^i_\psi = C^i_{\varphi+\psi}$, and therefore $\mathcal{S}\mathcal{S}^T = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}} \cdot C^{n-2}_{-\varphi_{n-2}} \cdots C^2_{-\varphi_2} C^1_{-\varphi_1} = I$. In addition, it is necessary to take $\varphi_i = l_i \frac{2\pi}{l}$, where $l_i, l \in \mathbb{N}$.

Let $a_1, a_2, \dots, a_l$ be some real numbers, $f(x)$ and $g(s)$ be functions, given on S and ∂S, respectively. Introduce the nonlocal operator

$$Lu(x) \equiv \sum_{k=1}^{l} a_k \Delta^2 u\left(\mathcal{S}^{k-1}x\right). \tag{5.1.1}$$

For example, if the matrix $\mathcal{S}$ is taken in the form $\mathcal{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then the operator L from (5.1.1) looks like

$$Lu(x) \\ = a_1\Delta^2 u(x_1, x_2) + a_2\Delta^2 u(-x_2, x_1) + a_4\Delta^2 u(-x_1, -x_2) + a_4\Delta^2 u(x_2, -x_1).$$

Consider the following boundary value problems in the unit ball S.

Dirichlet problem (see Section 2.2.1). *Find a function* $u(x) \in C^4(S) \cap C^1(\bar{S})$, *satisfying the conditions*

$$Lu(x) = f(x),\ x \in S, \tag{5.1.2}$$

$$u|_{\partial S} = g_0(x), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial S} = g_1(x),\ x \in \partial S, \tag{5.1.3}$$

where ν is the unit outer normal to sphere ∂S.

Neumann problem (see Section 3.3). *Find a function* $u(x) \in C^4(S) \cap C^2(\bar{S})$, *satisfying the equation* (5.1.2) *and the boundary conditions*

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = h_1(x), \quad \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = h_2(x),\ x \in \partial S. \tag{5.1.4}$$

Navier problem (see Section 4.1.2). *Find a function* $u(x) \in C^4(S) \cap C^2(\bar{S})$, *satisfying the equation* (5.1.2) *and the boundary conditions*

$$u|_{\partial S} = r_0(x), \quad \Delta u|_{\partial S} = r_1(x),\ x \in \partial S. \tag{5.1.5}$$

Riquier-Neumann problem (see Section 4.1.3). *Find a function* $u(x) \in C^4(S) \cap C^3(\bar{S})$, *satisfying the equation* (5.1.2) *and the boundary conditions*

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = p_0(x), \quad \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = p_1(x),\ x \in \partial S. \tag{5.1.6}$$

In the case when $a_1 \neq 0,\ a_k = 0,\ k = 2, 3, \ldots, l$ we obtain the classical Dirichlet, Neumann, Navier and Riquier-Neumann boundary value problems for the biharmonic equation. It should be noted that in the case $n = 2$ in [153] for the classical Laplace equation nonlocal boundary value problems with some matrix $\mathcal{S}$ are studied.

Now, we present some auxiliary statements regarding a special form of matrices.

Consider the matrix A, consisting of the coefficients $a_1, a_2, \ldots, a_l$ of the operator L

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} = (a_{j-i+1})_{i,j=1,\ldots,l},$$

where the indices of the coefficients a_{j-i+1} are taken modulo l.

Lemma 5.1. *Let $\lambda_1 = e^{i\frac{2\pi}{l}}$ be the primitive lth root of unity. Then*

$$\det A = \prod_{k=1}^{l} \left(a_1 \lambda_1^k + \ldots + a_l \lambda_l^k \right),$$

where $\lambda_k = e^{i\frac{2\pi k}{l}} = \lambda_1^k$, $k = 1, \ldots, l$.

Proof. It is not hard to see that $\lambda_k = e^{\left(i\frac{2\pi}{l}\right)k} = \lambda_1^k$ and $\lambda_l = \lambda_0 = 1$. Let us make sure that the number

$$\mu_k = a_1 \lambda_0^k + \ldots + a_l \lambda_{l-1}^k = \sum_{q=1}^{l} a_q \lambda_{q-1}^k$$

for $k = 1, \ldots, l$ is an eigenvalue of the matrix A, and the vector $B_k = \left(1, \lambda_1^k, \ldots, \lambda_{l-1}^k\right)^T$ is an eigenvector corresponding to the eigenvalue μ_k. Since the indices of the numbers λ_k can be calculated modulo l, then in the calculations below we consider the indices of the numbers a_k also modulo l. Then, for example, $a_0 = a_l$, $a_{-1} = a_{l-1}$ and $a_{l+1} = a_1$. Find the element of the mth row of the vector

$$C_k \equiv AB_k = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1^k \\ \vdots \\ \lambda_{l-1}^k \end{pmatrix},$$

where $m = 1, \ldots, l$. Since the mth row of the matrix A has the form $(a_{2-m}, a_{3-m}, \ldots, a_{l-m+1})$, then

$$(AB_k)_m = \sum_{j=1}^{l} a_{j-m+1} \lambda_{j-1}^k = \lambda_{m-1}^k \sum_{j=1}^{l} a_{j-m+1} \lambda_{j-m}^k = \mu_k \lambda_{m-1}^k,$$

where the equality $\lambda_m^k = \lambda_s^k \lambda_{m-s}^k$ is used. Therefore $AB_k = C_k = \mu_k B_k$. Now, using the equality

$$\det A = \prod_{k=1}^{l} \mu_k = \prod_{k=1}^{l} \left(a_1\lambda_0^k + ... + a_l\lambda_{l-1}^k\right)$$

we obtain the statement of the lemma. The lemma is proved. □

Clearly, $\lambda_k^n = (\lambda_1^k)^n = (\lambda_1^n)^k = \lambda_n^k$. Denote

$$\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k = \sum_{q=1}^{l} a_q\lambda_{q-1}^k = \sum_{q=1}^{l} a_q\lambda_k^{q-1}. \tag{5.1.7}$$

Example 5.2. Let $l = 3$. Then $\lambda_1 = e^{i\frac{2\pi}{3}}$, and hence $\lambda_k = e^{i\frac{2\pi k}{3}}$. In this case we have

$$\begin{aligned}\det A = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}\\ &= \left(a_1 + a_2 e^{i\frac{2\pi}{3}} + a_3 e^{i\frac{4\pi}{3}}\right)\left(a_1 + a_2 e^{i2\frac{2\pi}{3}} + a_3 e^{i2\frac{4\pi}{3}}\right)(a_1 + a_2 + a_3)\\ &= \left(a_1 + a_2 e^{i\frac{2\pi}{3}} + a_3 e^{i\frac{4\pi}{3}}\right)\left(a_1 + a_2 e^{i\frac{4\pi}{3}} + a_3 e^{i\frac{2\pi}{3}}\right)(a_1 + a_2 + a_3)\\ &= (a_1 + a_2 + a_3)\,(a_1^2 + a_2^2 + a_3^2 - a_2a_3 - a_1a_2 - a_1a_3) = a_1^3 + a_2^3 + a_3^3 - 3a_1a_2a_3.\end{aligned}$$

Lemma 5.2. *Let the numbers μ_k in (5.1.7) be nonzero $\mu_k \neq 0$, $k = 1, \ldots, l$. Then there exists an inverse to the matrix A, which is given by the formula*

$$A^{-1} = \frac{1}{l}\mathrm{M}_+\mathrm{diag}^{-1}(\mu_1, \ldots, \mu_l)\,\mathrm{M}_-{}^T,$$

where

$$\mathrm{M}_+ = \left(\lambda_{i-1}^{j}\right)_{i,j=1,l}, \quad \mathrm{M}_- = \left(\lambda_{i-1}^{-j}\right)_{i,j=1,l}.$$

Proof. It is not hard to see that $\mathrm{M}_+ = (B_1, \ldots, B_l)$, where $B_k = \left(1, \lambda_1^k, \ldots, \lambda_{l-1}^k\right)^T$ is the eigenvector of the matrix A corresponding to the eigenvalue μ_k (see Lemma 5.1). Then $A\mathrm{M}_+ = (\mu_1 B_1, \ldots, \mu_l B_l)$ and that means

$$\begin{aligned}A\mathrm{M}_+\mathrm{diag}^{-1}(\mu_1, \ldots, \mu_l) = (\mu_1 B_1, \ldots, \mu_l B_l)\,\mathrm{diag}\left(\mu_1^{-1}, \ldots, \mu_l^{-1}\right)\\ = (B_1, \ldots, B_l) = \mathrm{M}_+,\end{aligned}$$

where

$$A\mathrm{M}_{+}\mathrm{diag}^{-1}\left(\mu_1,\ldots,\mu_l\right)\mathrm{M}_{-}{}^{T}=\mathrm{M}_{+}\mathrm{M}_{-}{}^{T}.$$

Let us calculate the product of the resulting matrices

$$\mathrm{M}_{+}\mathrm{M}_{-}{}^{T}\equiv\left(m_{i,j}\right)_{i,j=1,\ldots,l}.$$

It is not hard to see that

$$m_{i,j}=\sum_{k=1}^{l}\lambda_{i-1}^{k}\lambda_{j-1}^{-k}=\sum_{k=1}^{l}\left(\frac{\lambda_{i-1}}{\lambda_{j-1}}\right)^{k}=\sum_{k=1}^{l}\lambda_{i-j}^{k},$$

where it is taken into account that $\lambda_k/\lambda_j=\lambda_{k-j}$ and $\lambda_0=e^{i\frac{2\pi 0}{l}}=e^0=1$. Obviously, $\lambda_{i-j}=\lambda_1^{i-j}$ is lth root of unity for any i and j.

Let us make sure that if λ is some lth root of unity, then

$$\sum_{k=1}^{l}\lambda^k=\begin{cases}l, & \lambda=1\\ 0, & \lambda\neq 1\end{cases}.$$

Indeed, for $\lambda=1$ the equality is obvious, and for $\lambda\neq 1$ we have

$$\begin{aligned}\lambda+\lambda^2+\ldots+\lambda^{l-1}+\lambda^l&=\frac{1}{1-\lambda}\left(\lambda+\lambda^2+\ldots+\lambda^{l-1}+\lambda^l\right)(1-\lambda)\\&=\frac{1}{1-\lambda}\left(\lambda+\lambda^2+\ldots+\lambda^{l-1}+\lambda^l-\lambda^2-\lambda^3-\ldots-\lambda^l-\lambda^{l+1}\right)\\&=\frac{1}{1-\lambda}(\lambda-\lambda)=0.\end{aligned}$$

Therefore

$$m_{i,j}=\begin{cases}l, & i=j\\ 0, & i\neq j\end{cases}$$

and consequently

$$A\mathrm{M}_{+}\,\mathrm{diag}\left(\mu_1^{-1},\ldots,\mu_l^{-1}\right)\mathrm{M}_{-}{}^{T}=lI.$$

This implies the statement of the lemma. □

Based on these two lemmas it is not hard to prove the statement.

Theorem 5.1. *Let $\mu_k = a_1\lambda_k^0 + \ldots + a_l\lambda_k^{l-1} \neq 0$, $k = 1, \ldots, l$, where $\{\lambda_k\}$ are the lth roots of unity. Then the solution to the system of algebraic equations $Ab = g$ can be written in the form*

$$b = (b_i)_{i=1,\ldots,l} = \frac{1}{l}\left(\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_k^{i-j}g_j\right)_{i=1,\ldots,l}. \tag{5.1.8}$$

Proof. Using Lemma 5.2, we can write

$$\begin{aligned}
b &= A^{-1}g = \frac{1}{l}\mathrm{M}_+\mathrm{diag}^{-1}\left(\mu_1, \ldots, \mu_l\right)\mathrm{M}_-{}^T \cdot (g_j)_{j=1,\ldots,l} \\
&= \frac{1}{l}\mathrm{M}_+\mathrm{diag}^{-1}\left(\mu_1, \ldots, \mu_l\right)\left(\sum_{j=1}^{l}\lambda_{j-1}^{-i}g_j\right)_{i=1,\ldots,l} = \frac{1}{l}\mathrm{M}_+\left(\sum_{j=1}^{l}\frac{1}{\mu_i}\lambda_{j-1}^{-i}g_j\right)_{i=1,\ldots,l} \\
&= \left(\frac{1}{l}\sum_{k=1}^{l}\lambda_{i-1}^{k}\sum_{j=1}^{l}\frac{1}{\mu_k}\lambda_{j-1}^{-k}g_j\right)_{i=1,\ldots,l} = \left(\frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}(\lambda_{i-1}/\lambda_{j-1})^k g_j\right)_{i=1,\ldots,l} \\
&= \left(\frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_{i-j}^{k}g_j\right)_{i=1,\ldots,l}.
\end{aligned}$$

The theorem is proved. □

Corollary 5.1. If $\mu_k \neq 0$ for $k = 1, \ldots, l$, then the matrix A^{-1} exists and has a structure similar to the matrix A

$$A^{-1} = C \equiv (c_{j-i+1})_{i,j=1,\ldots,l}, \quad c_j = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\lambda_k^{j-1}\mu_k}.$$

Moreover, for $k = 1, 2, \ldots, l$, the equality $\mu_k(C) = 1/\mu_k(A)$ holds true, where $\mu_k(A) = a_1\lambda_k^0 + \ldots + a_l\lambda_k^{l-1}$ and $\mu_k(C) = c_1\lambda_k^0 + \ldots + c_l\lambda_k^{l-1}$.

Proof. The equality for A^{-1} follows immediately from (5.1.8). Further, the equalities are true

$$\mu_k(C) = \sum_{j=1}^{l}c_j\lambda_{j-1}^{k} = \frac{1}{l}\sum_{j=1}^{l}\lambda_{j-1}^{k}\sum_{p=1}^{l}\frac{1}{\lambda_p^{j-1}\mu_p(A)}$$

$$= \sum_{p=1}^{l} \frac{1}{\mu_p(A)} \frac{1}{l} \sum_{j=1}^{l} \frac{\lambda_{j-1}^k}{\lambda_{j-1}^p} = \sum_{p=1}^{l} \frac{1}{\mu_p(A)} \frac{1}{l} \sum_{j=1}^{l} \lambda_{j-1}^{k-p} = \frac{1}{\mu_k(A)}.$$

The corollary is proved. □

5.1.2 Uniqueness of solutions

In order to study the uniqueness of the solution to the problem (5.1.2), (5.1.3), we start with the statement.

Lemma 5.3. *The operator $I_{\mathcal{S}}u(x) = u(\mathcal{S}x)$ and the Laplace operator Δ commute: $\Delta I_{\mathcal{S}}u(x) = I_{\mathcal{S}}\Delta u(x)$. Operators $\Lambda u = \sum_{i=1}^{n} x_i u_{x_i}(x)$ and $I_{\mathcal{S}}$ also commute: $\Lambda I_{\mathcal{S}}u(x) = I_{\mathcal{S}}\Lambda u(x)$ and the equality $\nabla I_{\mathcal{S}} = I_{\mathcal{S}}\mathcal{S}^T\nabla$ holds true.*

Proof. Let us write the matrix $\mathcal{S}$ in the form $\mathcal{S} = \left(s_{ij}\right)_{i,j=1,\ldots,l}$. Since

$$\frac{\partial}{\partial x_i} I_{\mathcal{S}}u(x) = \frac{\partial}{\partial x_i} u(\mathcal{S}x) = \frac{\partial}{\partial x_i} u((\mathcal{S}_{row}^1, x), \cdots, (\mathcal{S}_{row}^n, x))$$
$$= \sum_{j=1}^{n} s_{ji} I_{\mathcal{S}} u_{x_j}(x) = (\mathcal{S}_{col}^i, I_{\mathcal{S}}\nabla u(x)) = I_{\mathcal{S}}(\mathcal{S}_{col}^i, \nabla)u(x),$$

then

$$\frac{\partial^2}{\partial x_i^2} I_{\mathcal{S}}u(x) = \frac{\partial}{\partial x_i} I_{\mathcal{S}}(\mathcal{S}_{col}^i, \nabla)u(x) = I_{\mathcal{S}}(\mathcal{S}_{col}^i, \nabla)^2 u(x).$$

That is why

$$\Delta I_{\mathcal{S}}u(x) = \sum_{i=1}^{n} I_{\mathcal{S}}(\mathcal{S}_{col}^i, \nabla)^2 u(x) = I_{\mathcal{S}}\left|\left((\mathcal{S}_{col}^1, \nabla), \ldots, (\mathcal{S}_{col}^n, \nabla)\right)\right|^2 u(x)$$
$$= I_{\mathcal{S}}\left|\mathcal{S}^T\nabla\right|^2 u(x) = I_{\mathcal{S}}(\mathcal{S}^T\nabla, \mathcal{S}^T\nabla)u(x) = I_{\mathcal{S}}(\mathcal{S}\mathcal{S}^T\nabla, \nabla)u(x) = I_{\mathcal{S}}\Delta u(x).$$

The lemma is proved. □

Corollary 5.2. If a function $u(x)$ is biharmonic in S, then the function $u(\mathcal{S}x) = I_{\mathcal{S}}u(x)$ is also biharmonic in S.

Corollary 5.3. If a function $u(x)$ is biharmonic in S, then it satisfies the homogeneous equation (5.1.2) in S.

Indeed, according to Lemma 5.3, for any $x \in S$, we have

$$Lu(x) = \sum_{k=1}^{l} a_k \Delta^2 u \left(\mathcal{S}^{k-1}x\right) = \sum_{k=1}^{l} a_k \Delta^2 I_{\mathcal{S}^{k-1}} u\,(x)$$

$$= \sum_{k=1}^{l} a_k I_{\mathcal{S}^{k-1}} \Delta^2 u\,(x) = 0.$$

The converse statement is also true. Let

$$c_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1} \mu_k}, \tag{5.1.9}$$

where $j = 1, 2, \ldots, l$. Introduce the operators

$$\begin{gathered} I_L v(x) = \sum_{k=1}^{l} a_k I_{\mathcal{S}^{k-1}} v(x) = \sum_{k=1}^{l} a_k v(\mathcal{S}^{k-1}x), \\ J_L v(x) = \sum_{k=1}^{l} c_k v(\mathcal{S}^{k-1}x). \end{gathered} \tag{5.1.10}$$

Then the operator L of the equation (5.1.2) can be written as $Lu = I_L \Delta^2 u$.

Lemma 5.4. *Let the function $u \in C^4(S)$ satisfy the homogeneous equation* (5.1.2) *and $\mu_k \neq 0$, $k = 1, \ldots, l$. Then the function $u(x)$ is biharmonic in S.*

Proof. Let the function $u \in C^4(S)$ satisfy the homogeneous equation (5.1.2). Denote

$$v(x) = \sum_{k=1}^{l} a_k u(\mathcal{S}^{k-1}x) = I_L u(x). \tag{5.1.11}$$

It is obvious that $v(x) \in C^4(S)$ and

$$\Delta^2 v(x) = \Delta^2 I_L u = I_L \Delta^2 u = Lu = 0, \quad x \in S,$$

i.e. the function $v(x)$ is biharmonic in S. Due to Corollary 5.2, functions $v(\mathcal{S}^k x)$ are also biharmonic in the domain S. On the other hand, based on (5.1.11) we can get the equalities

$$\begin{gathered} v(\mathcal{S}x) = a_l u(x) + a_1 u(\mathcal{S}x) + \ldots + a_{l-1} u(\mathcal{S}^{l-1}x) \\ v(\mathcal{S}^2 x) = a_{l-1} u(x) + a_l u(\mathcal{S}x) + \ldots + a_{l-2} u(\mathcal{S}^{l-1}x) \\ \ldots \quad \ldots \quad \ldots \\ v(\mathcal{S}^{l-1}x) = a_2 u(x) + a_3 u(\mathcal{S}x) + \ldots + a_1 u(\mathcal{S}^{l-1}x), \end{gathered} \tag{5.1.12}$$

where the condition $\mathcal{S}^l = I$ is used. Hence, the system of algebraic equations (5.1.11), (5.1.12) for the functions $u(x), u(\mathcal{S}x), \ldots, u(\mathcal{S}^{l-1}x)$ can be rewritten in matrix form using the matrix A

$$\begin{pmatrix} v(x) \\ v(\mathcal{S}x) \\ \vdots \\ v(\mathcal{S}^{l-1}x) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \ldots & a_1 \end{pmatrix} \begin{pmatrix} u(x) \\ u(\mathcal{S}x) \\ \vdots \\ u(\mathcal{S}^{l-1}x) \end{pmatrix}.$$

By virtue of the lemma's conditions, the determinant of this system is nonzero $\det A \neq 0$. If we use Theorem 5.1 for

$$b = \Big(u(x), u(\mathcal{S}x), \ldots, u(\mathcal{S}^{l-1}x)\Big)^T, \quad g = \Big(v(x), v(\mathcal{S}x), \ldots, v(\mathcal{S}^{l-1}x)\Big)^T,$$

then we obtain

$$u(x) = b_1 = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_k^{1-j}g_j = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_k^{1-j}v(\mathcal{S}^{j-1}x) =$$
$$= \sum_{j=1}^{l} v(\mathcal{S}^{j-1}x)\frac{1}{l}\sum_{k=1}^{l}\frac{1}{\lambda_k^{j-1}\mu_k} = \sum_{j=1}^{l} c_j v(\mathcal{S}^{j-1}x).$$

Consequently, remembering the definition of the operator J_L from (5.1.10), we get

$$u(x) = c_1 v(x) + c_2 v(\mathcal{S}x) + \ldots + c_l v(\mathcal{S}^{l-1}x) = J_L v(x). \tag{5.1.13}$$

As noted above, the functions $v(\mathcal{S}^k x)$ are biharmonic in S for $k = 0, 1, \ldots, l-1$, and therefore the function $u(x)$ from (5.1.13) is also biharmonic in S. The lemma is proved. □

By virtue of Lemma 5.4, we are able to prove the statement.

Theorem 5.2. *Suppose that for all $k = 1, \ldots, l$ the equalities $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ hold true and solutions to the Dirichlet, Neumann, Navier and Riquier-Neumann boundary value problems exist. Then*

1) the solution to the Dirichlet problem (5.1.2), (5.1.3) *is unique;*

2) the solution to the Neumann problem (5.1.2), (5.1.4) *is unique up to a constant term;*

3) the solution to the Navier problem (5.1.2), (5.1.5) *is unique;*

4) the solution to the Riquier-Neumann problem (5.1.2), (5.1.6) *is unique up to a constant term.*

Proof. 1) Let us prove that the homogenous problem (5.1.2), (5.1.3) has only zero solution, and then solution of the non-homogenous problem (5.1.2), (5.1.3) is unique. Let $u(x)$ be a solution of the homogenous problem (5.1.2), (5.1.3). If $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ for $k = 1, \ldots, l$, then by Lemma 5.1 $\det A \neq 0$. By Lemma 5.4 the function $u(x)$ is biharmonic in S and satisfies the homogeneous conditions (5.1.3). Hence the function $u(x)$ is a solution to the Dirichlet problem

$$\Delta^2 u(x) = 0,\ x \in S; \quad u(x)\big|_{\partial S} = \frac{\partial u(x)}{\partial \nu}\Big|_{\partial S} = 0.$$

By virtue of the uniqueness of the solution to the Dirichlet problem (Theorem 2.16), we have $u(x) = 0$.

2) Let $u(x)$ be a solution to the homogenous problem (5.1.2), (5.1.4), then by Lemmas 5.4 function $u(x)$ is biharmonic in S and therefore satisfies the boundary conditions of the Neumann problem

$$\Delta^2 u(x) = 0,\ x \in S; \quad \frac{\partial u(x)}{\partial \nu}\Big|_{\partial S} = \frac{\partial^2 u(x)}{\partial \nu^2}\Big|_{\partial S} = 0.$$

By Theorem 3.7 the solution to this problem is only a constant $u(x) = C$.

3) Let a function $u(x)$ satisfies the homogeneous boundary conditions of the problem (5.1.2), (5.1.5), then by Lemma 5.4

$$\Delta^2 u(x) = 0,\ x \in S; \quad u|_{\partial S} = \Delta u|_{\partial S} = 0,\ x \in \partial S.$$

By Theorem 1.4 the solution to this problem is only the function $u(x) = 0$.

4) Let the function $u(x)$ satisfy the homogeneous boundary conditions of the problem (5.1.2), (5.1.6), then by Lemma 5.4

$$\Delta^2 u(x) = 0,\ x \in S; \quad \frac{\partial u(x)}{\partial \nu}\Big|_{\partial S} = \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial S} = 0,\ x \in \partial S.$$

By Theorem 4.7 the solution to this problem is only a constant. □

5.1.3 Dirichlet conditions

In Section 2.2.1, an elementary solution to the biharmonic equation is defined

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n>4,\ n=3 \\ -\dfrac{1}{4}\ln|x-\xi|, & n=4 \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi|-1\big), & n=2 \end{cases}, \tag{5.1.14}$$

and in Theorem 2.4 we can find that for $n \geq 3$ the function of the form

$$G_4(x,\xi) = E_4(x,\xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big), \tag{5.1.15}$$

where $E(x,\xi) = |x-\xi|^{2-n}/(n-2)$, is the Green's function of the Dirichlet problem for the biharmonic equation in the unit ball. From Theorem 4.1, using Remark 4.1 and similar to Theorem 4.3 the following statement follows.

Theorem 5.3. *Let $\varphi_0 \in C^{2+\varepsilon}(\partial S)$, $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ $(\varepsilon > 0)$ and $f \in C^1(\bar{S})$, then the solution to the Dirichlet problem for the biharmonic equation for $n \geq 3$ can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S} g_0(\xi)\frac{\partial}{\partial\nu}\Delta_\xi G_4(x,\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial S} g_1(\xi)\Delta_\xi G_4(x,\xi)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4(x,\xi)f(\xi)\,d\xi, \tag{5.1.16}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

In what follows, we need the statement.

Lemma 5.5. *Let the function $g(x)$ be continuous on ∂S or S. Then for any $k \in \mathbb{N}$*

$$\int_{\partial S} g(\mathcal{S}^k y)\,ds_y = \int_{\partial S} g(y)\,ds_y, \qquad \int_S g(\mathcal{S}^k y)\,dy = \int_S g(y)\,dy.$$

Proof. Let us prove the first equality. Let the function $w(x)$ be a solution to the Dirichlet problem for the Laplace equation in S with the condition $w(x) = g(x)$ on the boundary ∂S. Then the function $w(\mathcal{S}^k x)$ will be a solution to the Dirichlet problem for the Laplace equation in S (Corollary 5.2) with the condition $w(\mathcal{S}^k x) = g(\mathcal{S}^k x)$

to ∂S. It is known that solutions to these problems are represented in the form of Poisson integrals

$$w(x) = \int_{\partial S} P(x,y)g(y)\,ds_y, \quad w(\mathcal{S}^k x) = \int_{\partial S} P(x,y)g(\mathcal{S}^k y)\,ds_y,$$

where $P(x,y) = \frac{1}{\omega_n}\frac{1-|x|^2}{|x-y|^n}$ is the Poisson kernel. Since $P(0,y) = \frac{1}{\omega_n}\frac{1}{|y|^n} = \frac{1}{\omega_n}$ for $y \in \partial S$, then

$$\frac{1}{\omega_n}\int_{\partial S} g(y)\,ds_y = w(0) = \frac{1}{\omega_n}\int_{\partial S} g(\mathcal{S}^k y)\,ds_y.$$

This implies the equality to be proved. The second equality follows from the rule for changing variables in a multiple integral

$$\int_S g(\mathcal{S}y)\,dy = \int_S g(z)|\det \mathcal{S}^T|\,dz = \int_S g(y)\,dy$$

since $\mathcal{S}S = S$. The lemma is proved. □

Let us prove the existence theorem for the solution to the problem (5.1.2), (5.1.3).

Theorem 5.4. *Let the coefficients $\{a_k : k = 1,\dots,l\}$ of the operator L be such that $\mu_k \neq 0$ for $k = 1,\dots,l$ and $g_0 \in C^{2+\varepsilon}(\partial S)$, $g_1 \in C^{1+\varepsilon}(\partial S)$ ($\varepsilon > 0$), and $f \in C^1(\bar{S})$. Then the solution to the problem* (5.1.2)*,* (5.1.3) *exists and can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial S} g_0(\xi)\Lambda\Delta_\xi G_4(x,\xi)\,ds_\xi \\ - \frac{1}{\omega_n}\int_{\partial S} g_1(\xi)\Delta_\xi G_4(x,\xi)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4(x,\xi)J_L f(\xi)\,d\xi, \quad (5.1.17)$$

where the operator J_L is defined in (5.1.10)*.*

Proof. Let the function $u(x)$ be a solution to the problem (5.1.2), (5.1.3). Denote $v(x) = I_L u(x)$, then using (5.1.13) we get $u(x) = J_L v(x)$. Due to Lemma 5.3 on the commutativity of the operators I_S and Δ, I_S and Λ, taking into account the equalities $\frac{\partial}{\partial \nu}u|_{\partial S} = \Lambda u|_{\partial S}$ and $Lu = \Delta^2 v$, we get the followin problem for the function $v(x)$

$$\begin{gathered} \Delta^2 v(x) = f(x),\ x \in S; \\ v\big|_{\partial S} = I_L g_0(s),\ \Lambda u\big|_{\partial S} = I_L g_1(s),\ s \in \partial S. \end{gathered} \quad (5.1.18)$$

It is clear that $g_0 \in C^{2+\varepsilon}(\partial S) \Rightarrow I_L g_0 \in C^{2+\varepsilon}(\partial S)$, $g_1 \in C^{1+\varepsilon}(\partial S) \Rightarrow I_L g_1 \in C^{1+\varepsilon}(\partial S)$ and hence the solution to the Dirichlet problem (5.1.18) exists and is unique.

Next, let the function $v(x)$ be a solution to the problem (5.1.18). Then for the function $u(x) = J_L v(x)$ we obtain the equation (5.1.2) and after applying the operator J_L to the boundary conditions of the problem (5.1.18) we obtain the boundary conditions (5.1.3).

By virtue of Theorem 5.3 and using (5.1.16), the solution to the problem (5.1.18) can be represented as

$$v(x) = \frac{1}{\omega_n}\int_{\partial S} I_L g_0(\xi)\Lambda\Delta_\xi G_4(x,\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial S} I_L g_1(\xi)\Delta_\xi G_4(x,\xi)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4(x,\xi)f(\xi)\,d\xi.$$

Applying the operator J_L to both sides of the equality and taking into account (5.1.13), we obtain

$$u(x) = \frac{1}{\omega_n}J_L\int_{\partial S} I_L g_0(\xi)\Lambda\Delta_\xi G_4(x,\xi)\,ds_\xi - \frac{1}{\omega_n}J_L\int_{\partial S} I_L g_1(\xi)\Delta_\xi G_4(x,\xi)\,ds_\xi + \frac{1}{\omega_n}J_L\int_S G_4(x,\xi)f(\xi)\,d\xi. \quad (5.1.19)$$

In the resulting expression, we first transform the last integral. It is easy to see that $|\mathcal{S}^k x - \mathcal{S}^k\xi| = |\mathcal{S}^k(x-\xi)| = |x-\xi|$, which means that according to (5.1.14) $E_4(\mathcal{S}^k x, \mathcal{S}^k\xi) = E_4(x,\xi)$ and hence given (5.1.15) we can find $G_4(\mathcal{S}^k x, \mathcal{S}^k\xi) = G_4(x,\xi)$. Further, by virtue of Lemma 5.5

$$I_{\mathcal{S}^k}\int_S G_4(x,\xi)f(\xi)\,d\xi = \int_S G_4(\mathcal{S}^k x, \mathcal{S}^k\xi)f(\mathcal{S}^k\xi)\,d\xi = \int_S G_4(x,\xi)I_{\mathcal{S}^k}f(\xi)\,d\xi.$$

Therefore, according to (5.1.10), we have

$$J_L\int_S G_4(x,\xi)f(\xi)\,d\xi = \sum_{k=1}^{l} c_k I_{\mathcal{S}^{k-1}}\int_S G_4(x,\xi)f(\xi)\,d\xi = \int_S G_4(x,\xi)J_L f(\xi)\,d\xi.$$

Similarly, by Lemma 5.5, we get

$$J_L\int_{\partial S}\hat{g}_0(\xi)\frac{\partial}{\partial\nu}\Delta_\xi G_4(x,\xi)\,ds_\xi = \int_{\partial S} J_L\hat{g}_0(\xi)\frac{\partial}{\partial\nu}\Delta_\xi G_4(x,\xi)\,ds_\xi,$$

$$J_L \int_{\partial S} \hat{g}_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi = \int_{\partial S} J_L \hat{g}_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi,$$

where $\hat{g}_0 = I_L g_0$, $\hat{g}_1 = I_L g_1$. Thus, the function $u(x)$ from (5.1.19) can be rewritten as

$$\begin{aligned} u(x) = \frac{1}{\omega_n} \int_{\partial S} J_L I_L g_0(\xi) \Lambda \Delta_\xi G_4(x,\xi)\, ds_\xi \\ - \frac{1}{\omega_n} \int_{\partial S} J_L I_L g_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_S G_4(x,\xi) J_L f(\xi)\, d\xi. \quad (5.1.20) \end{aligned}$$

By virtue of (5.1.11) and (5.1.13), the following equalities are also true

$$\hat{g}_i(\xi) = I_L g_i(\xi), \quad g_i(\xi) = J_L \hat{g}_i(\xi), \quad i = 1,2,$$

from which it follows that $J_L I_L g_i(\xi) = g_i(\xi)$ for an arbitrary function $g_i(\xi)$ on ∂S. So (5.2.14) is converted to (5.1.17). This proves the theorem. □

Corollary 5.4. Let $v_0(x)$ and $v_1(x)$ be functions harmonic in S such that $v_0(x)|_{\partial S} = g_0$ and $v_1(x)|_{\partial S} = g_1$. Then the solution of the Dirichlet problem (5.1.2), (5.1.3) can be written as

$$\begin{aligned} u(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2} v_1(x) \\ + \frac{1}{\omega_n} \int_S G_4(x,\xi) J_L f(\xi)\, d\xi. \quad (5.1.21) \end{aligned}$$

The representation (5.1.21) of the function $u(x)$ follows from Theorem 5.4 and from the representation of the solution to the Dirichlet problem for the homogeneous biharmonic equation

$$\begin{aligned} \Delta^2 u_0(x) = 0,\ x \in S; \\ u_0|_{\partial S} = g_0(s),\ \left.\frac{\partial u_0}{\partial \nu}\right|_{\partial S} = g_1(s),\ s \in \partial S \end{aligned} \quad (5.1.22)$$

in the form (2.4.1)

$$u_0(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2} v_1(x).$$

By the way, in Theorem 2.13 on page 142 a method for representing the solution of an inhomogeneous Dirichlet problem without using the Green's function is given.

Example 5.3. Let $\mathcal{S}$ be a real orthogonal matrix such that $\mathcal{S}^2 = I$. The problem (5.1.2), (5.1.3) in this case takes the form

$$\begin{aligned} &a_1\Delta^2 u(x) + a_2\Delta^2 u(\mathcal{S}x) = f(x),\ x \in S; \\ &u|_{\partial S} = g_0(s),\ \frac{\partial u}{\partial \nu}\Big|_{\partial S} = g_1(s),\ s \in \partial S. \end{aligned} \tag{5.1.23}$$

Then $l = 2$, $\lambda_1 = e^{i\pi} = -1$, $\lambda_2 = e^{2i\pi} = 1$, $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$ and

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \quad \det A = \mu_1 \cdot \mu_2 = a_1^2 - a_2^2.$$

Let $a_1^2 - a_2^2 \neq 0 \Leftrightarrow a_1 \neq \pm a_2$. Then by (5.1.9) we have

$$c_1 = \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\lambda_k^0\mu_k} = \frac{1}{2}\Big(\frac{1}{\mu_1} + \frac{1}{\mu_2}\Big) = \frac{1}{2}\Big(\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\Big) = \frac{a_1}{a_1^2 - a_2^2},$$
$$c_2 = \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\lambda_k^1\mu_k} = \frac{1}{2}\Big(\frac{1}{-\mu_1} + \frac{1}{\mu_2}\Big) = \frac{1}{2}\Big(-\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\Big) = \frac{-a_2}{a_1^2 - a_2^2}$$

and that means

$$J_L f = c_1 f(x) + c_2 f(\mathcal{S}x) = \frac{a_1 f(x) - a_2 f(\mathcal{S}x)}{a_1^2 - a_2^2}.$$

In accordance with Corollary 5.4, the solution to the problem (5.2.15) can be written as

$$\begin{aligned} u(x) = v_0(x) + \frac{1 - |x|^2}{2}\Lambda v_0(x) - \frac{1 - |x|^2}{2}v_1(x) \\ + \frac{1}{\omega_n}\int_S G_4(x,\xi)\frac{a_1 f(\xi) - a_2 f(\mathcal{S}\xi)}{a_1^2 - a_2^2}\,d\xi, \end{aligned}$$

where $v_0(x)$ and $v_1(x)$ are harmonic functions in S such that $v_0|_{\partial S} = g_0$, $v_1|_{\partial S} = g_1$.

5.1.4 Neumann conditions

Let us investigate the necessary and sufficient conditions for solvability of the Neumann boundary value problem (5.1.2), (5.1.4). Let $G_4(x,\xi)$ be the Green's function (5.1.15) of the classical Dirichlet problem for biharmonic equation in the unit ball.

Theorem 5.5. *Let the coefficients* $\{a_k : k = 1, \dots, l\}$ *of the operator* L *be such that* $\mu_k = a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k \neq 0$*, for* $k = 1, \dots, l$ *and* $f \in C^2(\bar{S})$*,* $g_0(x) \in C^{3+\varepsilon}(\partial S)$*,* $g_1(x) \in C^{2+\varepsilon}(\partial S)$*,* $\varepsilon > 0$*. For solvability of the problem* (5.1.2)*,* (5.1.4) *the following condition is necessary and sufficient*

$$\int_{\partial S} (g_0(x) - g_1(x))\, ds_x + \frac{1}{2\mu_l}\int_S (1 - |x|^2) f(x)\, dx = 0. \tag{5.1.24}$$

If solution of the problem exists, then it is unique up to constant term and it can be represented in the form

$$u(x) = \int_0^1 w(tx)\frac{dt}{t}, \tag{5.1.25}$$

where

$$\begin{aligned} w(x) = \frac{1}{\omega_n}\int_{\partial S} & g_0(\xi)(\Lambda - 1)\Delta_\xi G_4(x,\xi)\, ds_\xi \\ & - \frac{1}{\omega_n}\int_{\partial S} g_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n}\int_S G_4(x,\xi) J_L(\Lambda + 4) f(\xi)\, d\xi, \end{aligned}$$

the operator J_L *is defined in* (5.1.10)*, and the function* $G_4(x,\xi)$ *is given in* (5.1.15)*.*

Proof. Suppose that a solution $u(x)$ to the problem (5.1.2), (5.1.4) exists, and the operator Λ is defined in Lemma 5.3. For convenience, assume that the function $u(x)$ belongs to the class $C^4(S) \cap C^2(\bar{S})$. For this it is enough to require $f(x) \in C^2(\bar{S})$, $g_0(x) \in C^{3+\varepsilon}(\partial S)$, $g_1(x) \in C^{2+\varepsilon}(\partial S)$, $\varepsilon > 0$. Apply the operator Λ to the function $u(x)$ and denote $w(x) = \Lambda u(x)$. Then, taking into account the equalities $\Delta^2\Lambda u(x) = \Delta(\Lambda+2)\Delta u(x) = (\Lambda+4)\Delta^2 u(x)$, where $x \in S$, and the commutability of the operators Λ and I_S (see Lemma 5.3) we have

$$Lw(x) = I_L\Delta\Lambda u(x) = (\Lambda + 4)I_L\Delta u(x) = (\Lambda + 4)Lu(x) = (\Lambda + 4)f(x).$$

From the boundary conditions (5.1.4), by using the property of the operator Λ, it follows that

$$\begin{gathered} w(x)\big|_{\partial S} = \Lambda u(x)\big|_{\partial S} = \frac{\partial u(x)}{\partial \nu}\Big|_{\partial S} = g_0(x), \\ \Lambda w(x)\big|_{\partial S} - w(x)\big|_{\partial S} = (\Lambda - 1)w(x)\big|_{\partial S} = \Lambda(\Lambda - 1)u(x)\big|_{\partial S} \\ = \frac{\partial^2 u(x)}{\partial \nu^2}\Big|_{\partial S} = g_1(x). \end{gathered}$$

Thus, if $u(x)$ is a solution to the problem (5.1.2), (5.1.4), then for the function $w(x) = \Lambda u(x)$ we obtain the Dirichlet type boundary value problem (5.1.2), (5.1.3)

$$\begin{gathered} Lw(x) = (\Lambda + 4) f(x),\ x \in S; \\ w(x)\big|_{\partial S} = g_0(x),\ \frac{\partial w(x)}{\partial \nu}\Big|_{\partial S} = g_0(x) + g_1(x). \end{gathered} \tag{5.1.26}$$

Moreover, the equality $w(x) = \Lambda u(x)$ implies necessity of the condition $w(0) = 0$ because $(\Lambda u)(0) = 0$. Further, the functions $F(x) = (\Lambda + 4) f(x) \in C^1(\bar{S})$ and $g_0(x) \in C^{3+\varepsilon}(\partial S)$, $g_1(x) \in C^{2+\varepsilon}$ satisfy the conditions of Theorem 5.4, which means that the solution to the problem (5.1.26) exists and is unique and can be represented in the form (5.1.21)

$$w(x) = w_0(x) + \frac{1}{\omega_n} \int_S G_4(x, \xi)(\Lambda_\xi + 4) J_L f(\xi)\, d\xi, \tag{5.1.27}$$

where the function $w_0(x)$ is a solution to the Dirichlet problem (5.1.22) with boundary functions $g_0(x)$ and $g_0(x) + g_1(x)$, as in problem (5.1.26), and the operator J_L is defined in (5.1.10). Here, under the integral, the commutativity of the operators Λ_ξ and J_L is used.

Now let $w(x)$ be a solution to the problem (5.1.26) and $w(0) = 0$. Then the function from (5.1.25)

$$u(x) = \int_0^1 w(tx) \frac{dt}{t},$$

satisfies the equality $\Lambda u(x) = w(x)$ since

$$\Lambda u(x) = \int_0^1 \sum_{i=1}^n x_i w_{x_i}(tx)\, dt = \int_0^1 \frac{dw(tx)}{dt}\, dt = w(x) - w(0) = w(x).$$

As noted above, $Lw(x) = (\Lambda + 4) Lu(x)$, which means that the equation from (5.1.26) implies

$$0 = (\Lambda + 4) Lu(x) - (\Lambda + 4) f(x) = (\Lambda + 4)(Lu(x) - f(x)). \tag{5.1.28}$$

Lemma 5.6. *The general solution to the equation* $(\Lambda + 4)v(x) = 0$ *has the form*

$$v(x) = C\big(\ln(x_2/x_1), \ldots, \ln(x_n/x_1)\big) x_1^{-4},$$

where $C(\tau_2, \ldots, \tau_n)$ *is an arbitrary differentiable function.*

Proof. Making the change of variables $t_i = \ln x_i$, $i = 1, \dots, n$ in the equation under consideration, we obtain the equation of the form $(v_{t_1}, \dots, v_{t_n})^T \cdot (1, \dots, 1)^T + 4v = 0$, and then again after the replacement $t = B\tau$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \dots & & & & \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix},$$

the differential part of the equation takes the form

$$\big(B^T (v_{\tau_1}, \dots, v_{\tau_n})^T\big) \cdot (1, \dots, 1)^T = B(1, \dots, 1)^T \cdot (v_{\tau_1}, \dots, v_{\tau_n})^T = v_{\tau_1}.$$

Therefore, the original equation is reduced to the form $v_{\tau_1} + 4v = 0$. The general solution to this equation have the form $v = C(\tau_2, \dots, \tau_n)e^{-4\tau_1}$, where $C(\tau_2, \dots, \tau_n)$ is some differentiable function. Returning to the old variables and taking into account that $B^{-1} = B$, we find

$$v(x) = C(t_2 - t_1, \dots, t_n - t_1)e^{-4t_1} = C\big(\ln(x_2/x_1), \dots, \ln(x_n/x_1)\big)x_1^{-4}.$$

The lemma is proved. □

According to Lemma 5.6, the equality (5.1.28) implies the equality $Lu(x) - f(x) = v(x)$. Since the left-hand side of this equality is a function differentiable at the origin, and the function $v(x)$ has no limit at the point $x = 0$ if $C \neq const$ and has an infinite limit if $C = const \neq 0$, then this equality is possible only if $v(x) = 0$.

Therefore, the function $u(x)$ satisfies the equation (5.1.2). Substituting $w(x) = \Lambda u(x)$ into the boundary conditions (5.1.26) we obtain the boundary conditions (5.1.4) and hence the function $u(x)$ is a solution to the problem (5.1.2), (5.1.4).

Thus, the equality $w(0) = 0$ is the necessary and sufficient condition for the solvability of the problem (5.1.2), (5.1.4).

Let us find the conditions under which the equality $w(0) = 0$ is satisfied. From the representation (5.1.27) we find

$$w(0) = w_0(0) + \frac{1}{\omega_n} \int_S G_4(0, \xi)(\Lambda + 4)J_L f(\xi)\, d\xi. \tag{5.1.29}$$

In the obtained equality, the index ξ of the operator Λ_ξ is omitted, since all functions in it depend only on ξ. Let us calculate the value of $w_0(0)$. If we use (2.4.1), we can get

$$w_0(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2}v_1(x),$$

where $v_0(x)$ and $v_1(x)$ are harmonic functions in S such that $v_0(x)|_{\partial S} = g_0$ and $v_1(x)|_{\partial S} = g_0 + g_1$. Using the mean value property for harmonic functions we can write

$$v_0(0) = \frac{1}{\omega_n}\int_{\partial S} g_0(\xi)\,ds_\xi, \quad v_1(0) = \frac{1}{\omega_n}\int_{\partial S}\big(g_0(\xi) + g_1(\xi)\big)\,ds_\xi,$$

and therefore, since $\Lambda v_0(x)|_{x=0} = 0$, we obtain

$$w_0(0) = v_0(0) - \frac{1}{2}v_1(0) = \frac{1}{2\omega_n}\int_{\partial S}\big(g_0(\xi) - g_1(\xi)\big)\,ds_\xi. \tag{5.1.30}$$

Let us move on to the second term in (5.1.29). Denote $\hat{f} = J_L f$.

Lemma 5.7. *For $n \geq 3$ the equality*

$$\int_S G_4(0,\xi)(\Lambda + 4)\hat{f}(\xi)\,d\xi = \int_S \frac{1-|\xi|^2}{4}\hat{f}(\xi)\,d\xi \tag{5.1.31}$$

holds true, where $G_4(x,\xi)$ is the Green's function of the Dirichlet problem for the biharmonic equation in the unit ball (5.1.15).

Proof. Let us start with $n > 4$ or $n = 3$. From (5.1.15), using the symmetry of the functions $E_4(x,\xi)$ and $E(x,\xi)$, we get

$$G_4(0,\xi) = E_4(\xi,0) - E_4(\xi/|\xi|,0) + \frac{|\xi|^2-1}{4}E(\xi/|\xi|,0)$$
$$= \frac{|\xi|^{4-n}-1}{2(n-2)(n-4)} + \frac{|\xi|^2-1}{4(n-2)}.$$

To prove (5.1.31), based on the obtained equality, we calculate successively two integrals

$$I_1 = \frac{1}{2(n-2)(n-4)}\int_S \big(|\xi|^{4-n} - 1\big)(\Lambda + 4)\hat{f}(\xi)\,d\xi,$$
$$I_2 = \frac{1}{4(n-2)}\int_S \big(|\xi|^2 - 1\big)(\Lambda + 4)\hat{f}(\xi)\,d\xi.$$

Let us find the auxiliary integral

$$
\begin{aligned}
J_k &= \int_S \big(|\xi|^k - 1\big)\Lambda \hat{f}(\xi)\, d\xi \\
&= \int_S \big(|\xi|^k - 1\big) \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i} \hat{f}(\xi)\, d\xi = \int_S \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\Big(\xi_i \big(|\xi|^k - 1\big)\hat{f}(\xi)\Big)\, d\xi \\
&\qquad - \int_S \sum_{i=1}^n \big(|\xi|^k - 1\big)\hat{f}(\xi)\, d\xi - \int_S \hat{f}(\xi) \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}\big(|\xi|^k - 1\big)\, d\xi.
\end{aligned}
$$

Applying the divergence theorem to the first integral above, we get

$$
\int_S \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\Big(\xi_i \big(|\xi|^k - 1\big)\hat{f}(\xi)\Big)\, d\xi = \int_{\partial S} |\xi|^2 \big(|\xi|^k - 1\big)\hat{f}(\xi)\, d\xi = 0.
$$

Therefore

$$
\begin{aligned}
J_k &= -n \int_S \big(|\xi|^k - 1\big)\hat{f}(\xi)\, d\xi - k \int_S |\xi|^k \hat{f}(\xi)\, d\xi \\
&= -(n+k) \int_S |\xi|^k \hat{f}(\xi)\, d\xi + n \int_S \hat{f}(\xi)\, d\xi.
\end{aligned}
$$

Hence using the value of J_{4-n} we have

$$
\begin{aligned}
I_1 &= \frac{1}{2(n-2)(n-4)} \int_S \big(4|\xi|^{4-n} - 4 - 4|\xi|^{4-n} + n\big)\hat{f}(\xi)\, d\xi \\
&= \frac{1}{2(n-2)} \int_S \hat{f}(\xi)\, d\xi.
\end{aligned}
$$

Let us find I_2. Using the value of J_2 we get

$$
\begin{aligned}
I_2 &= \frac{1}{4(n-2)} \int_S \big(4|\xi|^2 - 4 - (n+2)|\xi|^2 + n\big)\hat{f}(\xi)\, d\xi \\
&= \frac{1}{4(n-2)} \int_S \big((2-n)(|\xi|^2 - 1) - 2\big)\hat{f}(\xi)\, d\xi \\
&= \frac{1}{4} \int_S (1 - |\xi|^2)\hat{f}(\xi)\, d\xi - \frac{1}{2(n-2)} \int_S \hat{f}(\xi)\, d\xi.
\end{aligned}
$$

Thus

$$\int_S G_4(0,\xi)(\Lambda+4)\hat f(\xi)\,d\xi = (I_1+I_2) = \frac{1}{2(n-2)}\int_S \hat f(\xi)\,d\xi$$
$$+\frac{1}{4}\int_S (1-|\xi|^2)\hat f(\xi)\,d\xi - \frac{1}{2(n-2)}\int_S \hat f(\xi)\,d\xi = \int_S \frac{1-|\xi|^2}{4}\hat f(\xi)\,d\xi.$$

Let $n=4$, then we have

$$G_4(0,\xi) = -\frac{1}{4}\ln|\xi| + \frac{|\xi|^2-1}{4}.$$

As in the previous case, we write

$$\int_S \ln|\xi|\Lambda\hat f(\xi)\,d\xi = -4\int_S \ln|\xi|\hat f(\xi)\,d\xi - \int_S \hat f(\xi)\,d\xi,$$

and hence

$$-\frac{1}{4}\int_S \ln|\xi|(\Lambda+4)\hat f(\xi)\,d\xi$$
$$= \int_S \ln|\xi|\hat f(\xi)\,d\xi + \frac{1}{4}\int_S \hat f(\xi)\,d\xi - \int_S l|\xi|\hat f(\xi)\,d\xi = \frac{1}{4}\int_S \hat f(\xi)\,d\xi.$$

Using the value of I_2 for $n=4$ we get

$$\int_S G_4(0,\xi)(\Lambda+4)\hat f(\xi)\,d\xi = \frac{1}{4}\int_S \hat f(\xi)\,d\xi + \frac{1}{4}\int_S (1-|\xi|^2)\hat f(\xi)\,d\xi$$
$$-\frac{1}{4}\int_S \hat f(\xi)\,d\xi = \frac{1}{4}\int_S (1-|\xi|^2)\hat f(\xi)\,d\xi,$$

which is the same as (5.1.31). This proves the lemma. □

Using (5.1.30) and (5.1.31) obtained above and remembering the notation $\hat f = J_L f$ from (5.1.29) we find

$$w(0) = \frac{1}{2\omega_n}\int_{\partial S}\big(g_0(\xi)-g_1(\xi)\big)\,ds_\xi + \frac{1}{4\omega_n}\int_S (1-|\xi|^2)J_L f(\xi)\,d\xi.$$

This implies that the condition $w(0)=0$ is equivalent to the condition

$$\int_{\partial S}\big(g_0(\xi)-g_1(\xi)\big)\,ds_\xi + \int_S \frac{1-|\xi|^2}{2}J_L f(\xi)\,d\xi = 0. \tag{5.1.32}$$

Let us transform the obtained condition a little more. First, note that, according to the properties of the matrix S and by virtue of Lemma 5.5

$$\int_S \frac{1-|\xi|^2}{2} f(\mathcal{S}\xi)\,d\xi = \int_S \frac{1-|S\xi|^2}{2} f(\mathcal{S}\xi)\,d\xi = \int_S \frac{1-|\xi|^2}{2} f(\xi)\,d\xi.$$

According to the definition of the operator J_L (5.1.13), we get

$$\int_S \frac{1-|\xi|^2}{2} J_L f(\xi)\,d\xi = \sum_{k=1}^{l} c_k \int_S \frac{1-|\xi|^2}{2} f(\mathcal{S}^{k-1}\xi)\,d\xi$$
$$= \int_S \frac{1-|\xi|^2}{2} f(\xi)\,d\xi \sum_{k=1}^{l} c_k.$$

From Corollary 5.1 for $k = l$ it follows that

$$\mu_l(A)\mu_l(C) = \sum_{k=1}^{n} a_k \sum_{k=1}^{n} c_k = 1$$

and hence $\sum_{k=1}^{n} c_k = 1/\mu_l$. Therefore, the condition (5.1.32) is reduced to the form (5.1.24). The theorem is proved. □

Example 5.4. Let $\mathcal{S} = -I$. Consider the homogeneous problem (5.1.2), (5.1.4) of the form

$$\begin{gathered} Lu(x) \equiv a_1\Delta^2 u(x) + a_2\Delta^2 u(-x) = x_i,\ x \in S; \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial S} = \left.\frac{\partial^2 u}{\partial \nu^2}\right|_{\partial S} = 0, \end{gathered} \tag{5.1.33}$$

where $1 \le i \le n$. Similar to Example 5.3 we have $l = 2$, $\lambda_1 = -1$, $\lambda_2 = 1$, $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$ and

$$c_1 = \frac{a_1}{a_1^2 - a_2^2}, \quad c_2 = \frac{-a_2}{a_1^2 - a_2^2},$$

if $a_1^2 - a_2^2 \ne 0$. Therefore

$$J_L f = c_1 f(x) + c_2 f(-x) = \frac{a_1}{a_1^2 - a_2^2} f(x) - \frac{a_2}{a_1^2 - a_2^2} f(-x).$$

The solvability condition (5.1.24) is satisfied because $g_0 = g_1 = 0$ and $\int_S (1 - |x|^2) x_i\,dx = 0$. We compute the function

$$w(x) = \frac{1}{\omega_n} \int_S G_4(x,\xi) J_L(\Lambda + 4)\xi_i\,d\xi.$$

It is easy to see that

$$J_L(\Lambda+4)\xi_i = 5J_L\xi_i = \frac{5a_1}{a_1^2-a_2^2}\xi_i + \frac{5a_2}{a_1^2-a_2^2}\xi_i = \frac{5}{a_1-a_2}\xi_i.$$

In Theorem 2.6 we can find that

$$\frac{1}{\omega_n}\int_S G_4(x,\xi)|\xi|^{2k}H_m(\xi)\,d\xi$$
$$= \frac{|x|^{2k+4}-1-(k+2)(|x|^2-1)}{(2k+2)(2k+4)(2k+2m+n)(2k+2m+n+2)}H_m(x),$$

where $H_m(x)$ is homogeneous harmonic polynomial of degree m. Hence it follows that

$$\frac{1}{\omega_n}\int_S G_4(x,\xi)\xi_i\,d\xi = \frac{|x|^4-1-2(|x|^2-1)}{2\cdot 4(n+2)(n+4)}x_i,$$

which means

$$w(x) = \frac{5}{a_1-a_2}\frac{|x|^4-1-2(|x|^2-1)}{8(n+2)(n+4)}x_i.$$

So, according to (5.1.25), the solution to the problem (5.1.33) has the form

$$u(x) = \int_0^1 w(tx)\frac{dt}{t} = \frac{5x_i}{8(a_1-a_2)(n+2)(n+4)}$$
$$\times\int_0^1 (|tx|^4-1-2(|tx|^2-1))\,dt = \frac{5x_i(\frac{1}{5}|x|^4-1-2(\frac{1}{3}|x|^2-1)}{8(a_1-a_2)(n+2)(n+4)}$$
$$= \frac{|x|^4-\frac{10}{3}|x|^2+5}{8(a_1-a_2)(n+2)(n+4)}x_i.$$

Let us check the found solution. It is not hard to check that

$$\Delta^2 u(x) = \frac{\Delta^2|x|^4x_i}{8(a_1-a_2)(n+2)(n+4)} = \frac{x_i}{a_1-a_2},\quad \Delta^2 u(-x) = -\frac{x_i}{a_1-a_2},$$

that is why the equation from (5.1.33) is satisfied $Lu(x) = x_i$. Boundary conditions are also satisfied

$$\frac{\partial u}{\partial \nu}\Big|_{\partial S} = \frac{x_i(5|x|^4-10|x|^2+5)}{8(a_1-a_2)(n+2)(n+4)}\Big|_{\partial S} = 0,$$
$$\frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial S} = \frac{x_i(25|x|^4-30|x|^2+5)}{8(a_1-a_2)(n+2)(n+4)}\Big|_{\partial S} = 0.$$

5.1.5 Navier conditions

Let $G_2(x,\xi)$ be the Green's function (2.2.1) to the Dirichlet problem for the Poisson equation in the unit ball S. Using the function $G_2(x,\xi)$, in Theorem 4.2 it is shown that the Green's function $G_4^r(x,\xi)$ of the Navier boundary value problem for the biharmonic equation

$$\begin{gathered}\Delta^2 u(x) = f(x),\\ u|_{\partial S} = r_0(x), \quad \Delta u|_{\partial S} = r_1(x),\ x \in \partial S\end{gathered} \tag{5.1.34}$$

satisfies the equality

$$G_4^r(x,\xi) = \frac{1}{\omega_n}\int_S G_2(x,y)G_2(y,\xi)\,dy. \tag{5.1.35}$$

Then Theorem 4.3 on page 332 shows that if the function $u \in C^4(S)\cap C^3(\bar S)$ is a solution to the Navier problem (5.1.2), (5.1.5) then it can be represented in the form

$$u(x) = u_0^n(x) + \frac{1}{\omega_n}\int_S G_4^r(x,\xi)f(\xi)\,d\xi, \tag{5.1.36}$$

where $u_0^n(x)$ is a solution to the problem (5.1.34) when $f=0$ and

$$u_0^n(x) = \frac{1}{\omega_n}\int_{\partial S}\Lambda\Delta_\xi G_4^r(x,\xi)r_0(\xi)\,ds_\xi + \frac{1}{\omega_n}\int_{\partial S}\Lambda G_4^r(x,\xi)r_1(\xi)\,ds_\xi. \tag{5.1.37}$$

If $r_0\in C(\partial S)$, $r_1\in C^{1+\varepsilon}(\partial S)$ and $f\in C^1(\bar S)$, then $u(x)$ from (5.1.36) is a solution to the Navier boundary value problem (5.1.34).

Theorem 5.6. *Let the coefficients $\{a_k : k=1,\dots,l\}$ of the operator L be such that $\mu_k\neq 0$ for $k=1,\dots,l$ and $r_0\in C(\partial S)$, $r_1\in C^{1+\varepsilon}(\partial S)$, and $f\in C^1(\bar S)$. Then the solution to the problem* (5.1.2), (5.1.5) *exists and can be represented as*

$$u(x) = u_0^n(x) + \frac{1}{\omega_n}\int_S G_4^r(x,\xi)J_L f(\xi)\,d\xi, \tag{5.1.38}$$

where the operator J_L is defined in (5.1.10) *and $u_0^n(x)$ is taken from* (5.1.37).

Proof. Similar to the proof of Theorem 5.4, using Lemma 5.3 on the commutativity of the operators I_S and Δ, it is easy to show that the problem (5.1.2), (5.1.5) is equivalent to the Navier problem of type (5.1.34)

$$\begin{gathered}\Delta^2 v(x) = f(x),\ x\in S;\\ v|_{\partial S} = I_L r_0(x),\ \Delta v|_{\partial S} = I_L r_1(x),\ x\in\partial S.\end{gathered} \tag{5.1.39}$$

Moreover, if the function $v(x)$ is a solution to this problem, then the function $u(x) = J_L v(x)$ is a solution to the problem (5.1.2), (5.1.5). The converse is also true. If the function $u(x)$ is a solution to the problem (5.1.2), (5.1.5), then the function $v(x) = I_L u(x)$ is a solution to the problem (5.1.39).

It is clear that $r_0 \in C(\partial S) \Rightarrow I_L r_0 \in C(\partial S)$, $r_1 \in C^{1+\varepsilon}(\partial S) \Rightarrow I_L r_1 \in C^{1+\varepsilon}(\partial S)$ and, therefore, according to Theorem 4.3, the solution to the Navier problem (5.1.39) exists, is unique, and can be written as

$$v(x) = \frac{1}{\omega_n}\int_{\partial S} I_L r_0(\xi)\Lambda\Delta_\xi G_4^r(x,\xi)\,ds_\xi \\ + \frac{1}{\omega_n}\int_{\partial S} I_L r_1(\xi)\Lambda G_4^r(x,\xi)\,ds_\xi + \frac{1}{\omega_n}\int_S G_4^r(x,\xi) f(\xi)\,d\xi,$$

where the equality $\frac{\partial}{\partial \nu} v|_{\partial S} = \Lambda v|_{\partial S}$ is used. Applying the operator J_L to both parts of the resulting equality and taking into account that $u(x) = J_L v(x)$, we obtain

$$u(x) = \frac{1}{\omega_n} J_L \int_{\partial S} I_L r_0(\xi)\Lambda\Delta_\xi G_4^r(x,\xi)\,ds_\xi \\ + \frac{1}{\omega_n} J_L \int_{\partial S} I_L r_1(\xi)\Lambda G_4^r(x,\xi)\,ds_\xi + \frac{1}{\omega_n} J_L \int_S G_4^r(x,\xi) f(\xi)\,d\xi. \quad (5.1.40)$$

In the obtained expression, we transform the last integral. It is easy to see that $|\mathcal{S}^k x - \mathcal{S}^k\xi| = |\mathcal{S}^k(x-\xi)| = |x-\xi|$, and hence, according to (2.2.1) $G_2(\mathcal{S}^k x, \mathcal{S}^k\xi) = G_2(x,\xi)$ and, therefore, taking into account (5.1.35) and Lemma 5.5, we find

$$G_4^r(\mathcal{S}^k x, \mathcal{S}^k\xi) = \frac{1}{\omega_n}\int_S G_2(\mathcal{S}^k x, y) G_2(y, \mathcal{S}^k\xi)\,dy \\ = \frac{1}{\omega_n}\int_S G_2(\mathcal{S}^k x, \mathcal{S}^k y) G_2(\mathcal{S}^k y, \mathcal{S}^k\xi)\,dy = G_4^r(x,\xi).$$

Further, again by virtue of Lemma 5.5

$$I_{\mathcal{S}^k}\int_S G_4^r(x,\xi) f(\xi)\,d\xi = \int_S G_4^r(\mathcal{S}^k x, \mathcal{S}^k\xi) f(\mathcal{S}^k\xi)\,d\xi = \int_S G_4^r(x,\xi) I_{\mathcal{S}^k} f(\xi)\,d\xi,$$

whence by (5.1.10) we find

$$J_L \int_S G_4^r(x,\xi) f(\xi)\,d\xi = \sum_{k=1}^{l} c_k I_{\mathcal{S}^{k-1}} \int_S G_4^r(x,\xi) f(\xi)\,d\xi$$

$$= \int_S G_4^r(x,\xi) J_L f(\xi)\, d\xi.$$

Similarly, by Lemma 5.5, using the commutativity Lemma 5.3, we obtain

$$J_L \int_{\partial S} \hat{r}_0(\xi) \Lambda \Delta_\xi G_4^r(x,\xi)\, ds_\xi = \int_{\partial S} J_L \hat{r}_0(\xi) \Lambda \Delta_\xi G_4^r(x,\xi)\, ds_\xi,$$
$$J_L \int_{\partial S} \hat{r}_1(\xi) \Lambda G_4^r(x,\xi)\, ds_\xi = \int_{\partial S} J_L \hat{r}_1(\xi) \Lambda G_4^r(x,\xi)\, ds_\xi,$$

where $\hat{r}_0 = I_L r_0$, $\hat{r}_1 = I_L r_1$ are denoted. Thus, the solution $u(x)$ from (5.1.40) can be rewritten as

$$u(x) = \frac{1}{\omega_n} \int_{\partial S} J_L I_L r_0(\xi) \Lambda \Delta_\xi G_4^r(x,\xi)\, ds_\xi$$
$$+ \frac{1}{\omega_n} \int_{\partial S} J_L I_L r_1(\xi) \Lambda G_4^r(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_S G_4^r(x,\xi) J_L f(\xi)\, d\xi. \quad (5.1.41)$$

By virtue of (5.1.11) and (5.1.13), we have the equalities $\hat{r}_i(\xi) = I_L r_i(\xi)$, $r_i(\xi) = J_L \hat{r}_i(\xi)$, $i = 1, 2$ from which it follows that $J_L I_L r_i(\xi) = r_i(\xi)$ for an arbitrary function $r_i(\xi)$ on ∂S. Therefore (5.1.41) implies (5.1.38). The theorem is proved. □

Example 5.5. Let us find the solution to the homogeneous problem (5.1.2), (5.1.5) if $\mathcal{S} = -I$ and $f(x) = x_i|x|^2$, $1 \le i \le n$. In this case we have

$$\begin{gathered} a_1 \Delta^2 u(x) + a_2 \Delta^2 u(-x) = x_i |x|^2,\ x \in S; \\ u|_{\partial S} = \Delta u|_{\partial S} = 0,\ s \in \partial S. \end{gathered} \quad (5.1.42)$$

Similar to example 5.4 for $a_1^2 - a_2^2 \neq 0$ we obtain

$$J_L f = c_1 f(x) + c_2 f(-x) = \frac{x_i|x|^2}{a_1 - a_2}.$$

In accordance with (5.1.37), the solution to the problem (5.1.42) has the form

$$u(x) = \frac{1}{\omega_n} \int_S G_4^r(x,\xi) J_L f(\xi)\, d\xi,$$

where by (5.1.35)

$$G_4^r(x,\xi) = \frac{1}{\omega_n} \int_S G_2(x,y) G_2(y,\xi)\, dy.$$

Similar to Example 5.4

$$\frac{1}{\omega_n}\int_S G_2(x,\xi)|\xi|^{2k}H_m(\xi)\,d\xi = \frac{(|x|^{2k+2}-1)H_m(x)}{(2k+2)(2k+2m+n)},$$

whence it follows

$$\frac{1}{\omega_n}\int_S G_2(y,\xi)|\xi|^2\xi_i\,d\xi = \frac{|y|^4-1}{4(n+4)}y_i,$$

$$\frac{1}{\omega_n}\int_S G_2(x,y)\frac{|y|^4-1}{4(n+4)}y_i\,dy = \frac{(|x|^6-1)x_i}{4\cdot 6(n+4)(n+6)} - \frac{(|x|^2-1)x_i}{4\cdot 2(n+4)(n+2)}.$$

So we have

$$u(x) = \frac{x_i}{a_1-a_2}\left(\frac{|x|^6-1}{24(n+4)(n+6)} - \frac{|x|^2-1}{8(n+4)(n+2)}\right).$$

Using the equality $\Delta(|x|^{2k}H_m(x)) = 2k(2k+2m+n-2)|x|^{2k-2}H_m(x)$, it is not hard to check that the resulting function is the solution to (5.1.42).

5.1.6 Riquier-Neumann conditions

Consider the Riquier-Neumann boundary value problem for the biharmonic equation in S corresponding to the problem (5.1.2), (5.1.6)

$$\begin{gathered}\Delta^2 u(x) = f(x),\\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial S} = p_0(x),\ \left.\frac{\partial \Delta u}{\partial \nu}\right|_{\partial S} = p_1(x),\ x\in\partial S.\end{gathered} \tag{5.1.43}$$

Explicit form of the Green's function to the Neumann problem for the Poisson equation in the unit ball $\mathcal{N}(x,\xi)$ is given in (3.7.7)

$$\mathcal{N}(x,\xi) = E(x,\xi) - E_0(x,\xi),$$

where the harmonic function $E_0(x,\xi)$ is written in the form

$$E_0(x,\xi) = \int_0^1\left(\hat{E}\left(\frac{x}{|x|}, t|x|\xi\right) + 1\right)\frac{dt}{t}$$

and $\hat{E}(x,\xi) = \Lambda_x E(x,\xi)$. For the cases $n=2$ and $n=3$ see [133]. Using the Green's function $\mathcal{N}(x,\xi)$, in Theorem 4.6, the Green's function of the Riquier-Neumann problem for the biharmonic equation (5.1.43) is constructed

$$G_4^{rn}(x,\xi) = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}(x,y)\mathcal{N}(y,\xi)\,dy \\ - \frac{1}{|S|}\int_S \mathcal{N}(x,y)\,dy \cdot \int_S \mathcal{N}(y,\xi)\,dy\Big), \quad (5.1.44)$$

In Theorem 4.7 it is it is established that if $p_0 \in C(\partial S)$, $p_1 \in C^{1+\varepsilon}(\partial S)$ and $f \in C^1(\bar{S})$, then the solution to the Riquier-Neumann problem (5.1.43) exists and is unique up to a constant term under the condition

$$\int_{\partial S} p_1(\xi)\,ds_\xi = \int_S f(\xi)\,d\xi. \quad (5.1.45)$$

This solution can be written as

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi, \quad (5.1.46)$$

where the function $u_0(x)$ denotes the solution to the Riquier-Neumann problem for the homogeneous biharmonic equation

$$u_0(x) = -\frac{1}{\omega_n}\int_{\partial S} \Delta_\xi G_4^{rn}(x,\xi)p_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial S} G_4^{rn}(x,\xi)p_1(\xi)\,ds_\xi. \quad (5.1.47)$$

Theorem 5.7. *Let the coefficients $\{a_k : k = 1,\dots,l\}$ of the operator L be such that $\mu_k \neq 0$ for $k = 1,\dots,l$ and $p_0 \in C(\partial S)$, $p_1 \in C^{1+\varepsilon}(\partial S)$ $(\varepsilon > 0)$ and $f \in C^1(\bar{S})$. Then under the condition*

$$\int_{\partial S} p_1(\xi)\,ds_\xi = \frac{1}{\mu_l}\int_S f(\xi)\,d\xi, \quad (5.1.48)$$

the solution of the Riquier-Neumann problem (5.1.2)*,* (5.1.6) *exists up to a constant term and can be represented as*

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)J_L f(\xi)\,d\xi, \quad (5.1.49)$$

where the function $u_0(x)$ is defined in (5.1.47).

Proof. Similar to the proof of Theorems 5.4 and 5.6, using Lemma 5.3 on the commutativity of the operators I_S and Δ, I_S and Λ and the equality $\frac{\partial}{\partial \nu}v|_{\partial S} = \Lambda v|_{\partial S}$ it is not hard to see that the problem (5.1.2), (5.1.6) is equivalent to the Riquier-Neumann problem of type (5.1.43)

$$\begin{gathered} \Delta^2 v(x) = f(x),\ x \in S; \\ \Lambda v\big|_{\partial S} = I_L p_0(x), \quad \Lambda\Delta v\big|_{\partial S} = I_L p_1(x),\ x \in \partial S. \end{gathered} \quad (5.1.50)$$

If the function $v(x)$ is a solution to this problem, then the function $u(x) = J_L v(x)$ is a solution to the problem (5.1.2), (5.1.6) and vice versa, if the function $u(x)$ is a solution to the problem (5.1.2), (5.1.6), then the function $v(x) = I_L u(x)$ is a solution to the problem (5.1.50).

It is clear that $p_0 \in C(\partial S) \Rightarrow I_L p_0 \in C(\partial S)$, $p_1 \in C^{1+\varepsilon}(\partial S) \Rightarrow I_L p_1 \in C^{1+\varepsilon}(\partial S)$ and hence, according to Theorem 4.7, the solution of the Riquier-Neumann problem (5.1.50) exists under the condition (5.1.45)

$$\int_{\partial S} I_L p_1(\xi)\, ds_\xi = \int_S f(\xi)\, d\xi. \tag{5.1.51}$$

This solution can be written as (5.1.46)

$$v(x) = v_0(x) + \frac{1}{\omega_n} \int_S G_4^{rn}(x,\xi) f(\xi)\, d\xi,$$

where according to (5.1.47)

$$v_0(x) = -\frac{1}{\omega_n} \int_{\partial S} \Delta_\xi G_4^{rn}(x,\xi) I_L p_0(\xi)\, ds_\xi - \frac{1}{\omega_n} \int_{\partial S} G_4^{rn}(x,\xi) I_L p_1(\xi)\, ds_\xi.$$

Applying the operator J_L to the function $v(x)$ and taking into account that $u(x) = J_L v(x)$ we get

$$u(x) = J_L v_0(x) + \frac{1}{\omega_n} J_L \int_S G_4^{rn}(x,\xi) f(\xi)\, d\xi. \tag{5.1.52}$$

We transform the integral in the resulting expression. It is easy to see that, similarly to the equalities obtained in the proof of Theorems 5.4 and 5.6 with respect to the function $G_2(x,\xi)$, we get

$$\begin{aligned}\mathcal{N}(\mathcal{S}^k x, \mathcal{S}^k \xi) &= E(\mathcal{S}^k x, \mathcal{S}^k \xi) - E_0(\mathcal{S}^k x, \mathcal{S}^k \xi) = E(x,\xi) \\ &\quad - \int_0^1 \Big(\Lambda_1 E\Big(\mathcal{S}^k \frac{x}{|x|}, t|x|\mathcal{S}^k \xi\Big) + 1\Big)\frac{dt}{t} = E(x,\xi) - E_0(x,\xi) = \mathcal{N}(x,\xi),\end{aligned}$$

where Λ_1 is operator Λ on first argument. Hence, by virtue of Lemma 5.5, taking into account (5.1.44), we find

$$\begin{aligned}G_4^{rn}(\mathcal{S}^k x, \mathcal{S}^k \xi) &= \frac{1}{\omega_n}\Big(\int_S \mathcal{N}(\mathcal{S}^k x, y)\mathcal{N}(y, \mathcal{S}^k \xi)\, dy - \frac{1}{|S|}\int_S \mathcal{N}(\mathcal{S}^k x, y)\, dy \\ &\quad \cdot \int_S \mathcal{N}(y, \mathcal{S}^k \xi)\, dy\Big) = \frac{1}{\omega_n}\Big(\int_S \mathcal{N}(\mathcal{S}^k x, \mathcal{S}^k y)\mathcal{N}(\mathcal{S}^k y, \mathcal{S}^k \xi)\, dy\end{aligned}$$

$$-\frac{1}{|S|}\int_S \mathcal{N}(\mathcal{S}^k x, \mathcal{S}^k y)\, dy \cdot \int_S \mathcal{N}(\mathcal{S}^k y, \mathcal{S}^k \xi)\, dy\Big) = G_4^{rn}(x,\xi).$$

Further, again by virtue of Lemma 5.5

$$I_{\mathcal{S}^k}\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi$$
$$= \int_S G_4^{rn}(\mathcal{S}^k x, \mathcal{S}^k\xi)f(\mathcal{S}^k\xi)\,d\xi = \int_S G_4^{rn}(x,\xi)I_{\mathcal{S}^k}f(\xi)\,d\xi,$$

whence, using (5.1.10), we get

$$J_L\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi$$
$$= \sum_{k=1}^{l} c_k I_{\mathcal{S}^{k-1}}\int_S G_4^{rn}(x,\xi)f(\xi)\,d\xi = \int_S G_4^{rn}(x,\xi)J_L f(\xi)\,d\xi.$$

Now let us transform $J_L v_0(x)$ from (5.1.52). By Lemma 5.5, using the commutativity Lemma 5.3 and the equalities $J_L I_L p_i(\xi) = p_i(\xi)$, $i = 1, 2$, we get

$$J_L v_0(x) = -\frac{1}{\omega_n}J_L\int_{\partial S}\Delta_\xi G_4^{rn}(x,\xi)I_L p_0(\xi)\,ds_\xi$$
$$-\frac{1}{\omega_n}J_L\int_{\partial S}G_4^{rn}(x,\xi)I_L p_1(\xi)\,ds_\xi = -\frac{1}{\omega_n}\int_{\partial S}\Delta_\xi G_4^{rn}(x,\xi)J_L I_L p_0(\xi)\,ds_\xi$$
$$-\frac{1}{\omega_n}\int_{\partial S}G_4^{rn}(x,\xi)J_L I_L p_1(\xi)\,ds_\xi = u_0(x),$$

where the function $u_0(x)$ is defined in (5.1.46). Thus, the solution $u(x)$ from (5.1.52) becomes

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_S G_4^{rn}(x,\xi)J_L f(\xi)\,d\xi,$$

matching (5.1.49).

Finally, transform the condition (5.1.51) for the existence of the function $v(x)$. It is not hard to see that

$$\int_{\partial S} I_L p_1(\xi)\,ds_\xi = \sum_{k=1}^{l} a_k\int_{\partial S}p_1(\mathcal{S}^{k-1}\xi)\,ds_\xi = \mu_l\int_{\partial S}p_1(\xi)\,ds_\xi,$$

where the value of μ_k is defined in (5.1.7). Since $\mu_l \neq 0$, then (5.1.51) implies the condition (5.1.48). The theorem is proved. □

5.2 Eigenfunctions of a nonlocal Laplace operator with double involution

5.2.1 Introduction and problem statement

Among the nonlocal differential equations, which are the subject of many works, a special place is occupied by equations with involutive deviations of the argument. An involution is a mapping $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{S}^2 x = x$. It should be noted monographs [39, 152, 185] from a variety of research papers in this direction. Papers [1,2,6,10,11,38,108,129,155,159,186] are devoted to the questions of solvability of boundary and initial-boundary value problems for differential equations with involution. Spectral questions of differential equations with involution are studied in [16–18, 46, 50, 136, 140, 161, 175, 177, 181]. For example, in [140], the following boundary value problem for ODE is studied

$$y''(t) + ay''(-t) = \lambda y(t), \quad -\pi < t < \pi, \quad y(-\pi) = y(\pi) = 0.$$

The eigenfunctions and eigenvalues of this problem are constructed in an explicit form. The system of eigenfunctions is complete in $L_2[-\pi, \pi]$.

In [108] the following nonlocal analog of the Laplace operator is considered

$$L_1[u](x) \equiv a_0 \Delta u(x) + a_1 \Delta u(\mathcal{S}x) + \ldots + a_{l-1} \Delta u(\mathcal{S}^{l-1} x),$$

where Δ is the Laplace operator, a_i for $i = 0, 1, \ldots, l-1$ are real numbers, $\mathcal{S}$ is an $n \times n$ orthogonal matrix for which there exists a number $l \in \mathbb{N}$ such that $\mathcal{S}^l = I$, where I is the identity matrix. In the paper [108] cited above for the corresponding nonlocal Poisson equation $L_1[u] = f(x)$ in the unit ball S, the solvability questions for the main boundary value problems with the Dirichlet, Neumann, and Robin boundary conditions are studied. The corresponding spectral problem for the Dirichlet boundary value problem is considered in [181].

In [175] boundary value problems of Dirichlet and Neumann type for the biharmonic equation with involution are studied. The existence and uniqueness solvability conditions are obtained, and an explicit representation of solutions to the problems under consideration is given. Next, in [177], a nonlocal Laplace operator with a multiple involution of the following form is introduced

$$L_m[u](x) \equiv \sum_{(i_m \ldots i_1)_2 = 0}^{2^m - 1} a_{(i_m \ldots i_1)_2} \Delta u \left(\mathcal{S}_m^{i_m} \ldots \mathcal{S}_1^{i_1} x \right),$$

where $a_{(i_m\ldots i_1)_2}$ are real numbers, $(i_m\ldots i_1)_2$ is a representation of the index i in the binary number system, $\mathcal{S}_i$ are orthogonal $n\times n$ matrices satisfying the condition $\mathcal{S}_i^2 = I, i = 1,\ldots,m$. In the article [177] an explicit form of eigenfunctions and eigenvalues of the corresponding Dirichlet problem is found

$$L_m[u](x) + \lambda u(x) = 0,\ x \in S;\quad u(x) = 0,\ x \in \partial S$$

and the completeness of the system of eigenfunctions in the space $L_2(S)$ is proved. Finally, in [174] boundary value problems for the biharmonic equation with multiple involution are studied and a boundary value problem containing modified integro-differential Hadamard operators in the boundary conditions is considered.

In this section, continuing the research begun above on the solvability of boundary value problems for harmonic and biharmonic equations with both one-dimensional involution and multiple involution, we are going to study similar questions for the Laplace operator with double involution of arbitrary orders.

Section 5.2.2 examines matrices of a special form that arise in this problem. The block structure of these matrices is determined (Theorems 5.8-5.9), their multiplication is clarified (Theorem 5.10) and eigenvalues and eigenvectors are found (Theorem 5.11). Then in Section 5.2.3 (see Theorems 5.12-5.13), using Lemma 5.3 the structure of eigenfunctions and eigenvalues of the Dirichlet problem is studied. In Section 5.2.4 (see Theorems 5.14-5.15), using Lemma 5.9, the eigenfunctions and eigenvalues of the Dirichlet problem for the considered nonlocal Laplace equation in the unit ball are determined. These eigenfunctions are presented explicitly. The completeness of the found system of eigenfunctions in $L_2(S)$ is proved. All concepts and results obtained are illustrated with 7 examples. The main results of Section 5.2 are published in papers [179, 181].

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$ and $\partial S = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. Let also $\mathcal{S}_1, \mathcal{S}_2$ be two real commutative orthogonal $n\times n$ matrices such that $\mathcal{S}_i^{l_i} = I$, $l_i \in \mathbb{N}$, $i = 1, 2$. Recall that since $|x|^2 = (\mathcal{S}_i^T\mathcal{S}_i x, x) = (\mathcal{S}_i x, \mathcal{S}_i x) = |\mathcal{S}_i x|^2$, then $x \in S \Rightarrow \mathcal{S}_i x \in S$ and $y \in \partial S \Rightarrow \mathcal{S}_i y \in \partial S$. An example of the matrix $\mathcal{S}_i$ can be found in Example 5.1.

Let $l_1, l_2 \in \mathbb{N}_0$ and $a_0,\ldots,a_{l_1-1},a_{l_1},\ldots,a_{2l_1-1},\ldots,a_{(l_2-1)l_1-1},\ldots,a_{l_2l_1-1}$ is a sequence of real numbers, which we denote by $\mathbf{a}$. If the summation index i is written in the form $i = (i_2, i_1) \equiv i_2\cdot l_1 + i_1$, where $i_k = 0, 1, \ldots l_k - 1$ for $k = 1, 2$, then elements $\mathbf{a}$ can be written as $a_{(0,0)},\ldots,a_{(0,l_1-1)},a_{(1,0)},\ldots,a_{(1,l_1-1)},\ldots,a_{(l_2-2,l_1-1)},\ldots,a_{(l_2-1,l_1-1)}$. It is clear that if $0 \leq i < l_1l_2$, then $i_1 = \{i/l_1\}l_1$, $i_2 = [i/l_1]$, where $[a]$ is the integer part of the number a and $\{a\} = a - [a]$. In what follows, we consider the sequence $\mathbf{a}$ also as a vector.

Let us introduce the following nonlocal differential operator formed by the sequence $\mathbf{a}$ and the Laplace operator Δ

$$L_{\mathbf{a}}u \equiv \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)}\Delta u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)$$

and consider the following boundary value problem.

Problem $\mathcal{S}_2$. *Find a function $u(x) \neq 0$ from the class $u \in C^2(S) \cap C(\bar{S})$ satisfying the conditions*

$$L_{\mathbf{a}}u(x) + \lambda u(x) = 0,\ x \in S, \tag{5.2.1}$$

$$u(x) = 0,\ x \in \partial S \tag{5.2.2}$$

for $\lambda \in \mathbb{R}$.

In the special case when $l_1 = l_2 = 2$, this problem coincides with the spectral Dirichlet problem studied in [177].

5.2.2 Properties of matrices $A_{(2)}$

To study the above problem (5.2.1)-(5.2.2) we need some auxiliary statements. Let us introduce the function

$$v(x) = \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)}u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x), \tag{5.2.3}$$

where the summation is carried out in ascending order by index $i = (i_2, i_1) \equiv i_2 \cdot l_1 + i_1$ in the form $(0,0), \ldots, (0, l_1 - 1), (1, 0), \ldots, (1, l_1 - 1), \ldots, (l_2 - 2, l_1 - 1), \ldots, (l_2 - 1, l_1 - 1)$. From the equality (5.2.3), taking into account that $\mathcal{S}_2^{l_2} = \mathcal{S}_1^{l_1} = I$, it is easy to conclude that functions of the form $v(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)$, where $j = 0, \ldots, l_2 l_1 - 1$ can be linearly expressed through the functions $u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)$. If we consider the following vectors

$$U(x) = \left(u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)\right)^T_{i=0,\ldots,l_2l_1-1}, \quad V(x) = \left(v(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)\right)^T_{i=0,\ldots,l_2l_1-1}$$

of order $l_2 l_1$, then this dependence can be represented in matrix form

$$V(x) = A_{(2)}U(x), \tag{5.2.4}$$

where $A_{(2)} = \left(a_{i,j}\right)_{i,j=0,\ldots,l_2l_1-1}$ is the corresponding $l_2l_1 \times l_2l_1$ matrix.

Let us study the structure of matrices of the form $A_{(2)}$. To do this, we introduce an operation in the index of the matrix coefficients in the following sense $i \oplus j = (i_2, i_1) \oplus (j_2, j_1) \equiv ((i_2 + j_2 \bmod l_2), (i_1 + j_1 \bmod l_1))$, where (i_2, i_1) is the representation of the index i, as stated above. It is clear that the operation $\oplus$ is a commutative and associative operation on the numbers $i \in \{0, \ldots, l_2l_1 - 1\}$ and $(i_2, i_1) \oplus (0, 0) = (i_2, i_1)$. Since $(i_2, i_1) \oplus (j_2, j_1) = (0, 0) \Leftrightarrow i_2 + j_2 = 0 \bmod l_2, i_1 + j_1 = 0 \bmod l_1 \Leftrightarrow j_2 = l_2 - i_2, j_1 = l_1 - i_1$, then we can determine the index $\ominus i \equiv (l_2 - i_2, l_1 - i_1)$ for which $i \oplus (\ominus i) = 0$. For example, if $l_1 = 2$, $l_2 = 3$, then $\ominus(2, 1) = (1, 1)$ or $\ominus 5 = 3$. If we further determine that $(-i_2, -i_1) \equiv \ominus i = (l_2 - i_2, l_1 - i_1)$, then we have

$$
\begin{aligned}
&(-i_2, -i_1) \oplus (j_2, j_1) = (l_2 - i_2, l_1 - i_1) \oplus (j_2, j_1) \\
&= (l_2 - i_2 + j_2 \bmod l_2, l_1 - i_1 + j_1 \bmod l_1) = (-i_2 + j_2 \bmod l_2, -i_1 + j_1 \bmod l_1),
\end{aligned}
$$

that is, the operation $\oplus$ is formally applicable to indices of the form $(-i_2, -i_1)$ defined above. Let us define the operation $\ominus$ by the equality

$$
i \ominus j \equiv i \oplus (\ominus j) = (i_2 - j_2 \bmod l_2, i_1 - j_1 \bmod l_1).
$$

Let us extend the operations $\oplus$ and $\ominus$ to all numbers of the form (i_2, i_1), assuming that $(i_2, i_1) \equiv (i_2 \bmod l_2, i_1 \bmod l_1)$. For example, if $l_1 = 2$, $l_2 = 3$, then $(1, -1) = (1, 1)$ and $(5, -3) = (2, 1)$.

Theorem 5.8. *The matrix $A_{(2)}$ from the equality* (5.2.4) *can be represented as*

$$
A_{(2)} \equiv \left(a_{i,j}\right)_{i,j=0,\ldots,l_2l_1-1} = \left(a_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1}. \tag{5.2.5}
$$

The linear combination of matrices of the form (5.2.5) *is a matrix of the form* (5.2.5).

Proof. Consider the function $v(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)$ from (5.2.3), whose coefficients at $u(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1})$ constitute the $i \equiv (i_2, i_1)$th row of the matrix $A_{(2)}$ from (5.2.4)

$$
\begin{aligned}
v(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x) &= \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}u(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x) \\
&= \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}u(\mathcal{S}_2^{j_2+i_2 \bmod l_2}\mathcal{S}_1^{j_1+i_1 \bmod l_1}x).
\end{aligned} \tag{5.2.6}
$$

Here the following properties $\mathcal{S}_2^{l_2}x = x$, $\mathcal{S}_1^{l_1}x = x$ and $\mathcal{S}_1\mathcal{S}_2x = \mathcal{S}_2\mathcal{S}_1x$ of matrices $\mathcal{S}_1$ and $\mathcal{S}_2$ are taken into account. Let us replace the index j by $k = i \oplus j$. Then $k \ominus i = i \oplus j \ominus i = j$ and the correspondence $j \leftrightarrow k$ is one-to-one. Replacement $j \to k$ of the index change only the order of summation in the sum (5.2.6). For example, if $l_1 = 2$, $l_2 = 3$ and $i = (0,1)$, then the sequence $j : 0,1,2,3,4,5$ goes to $k = 1 \oplus j : 1,0,3,2,5,4$. After replacing the index, we get

$$v(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x) = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus i}u(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x).$$

Comparing the resulting equality with (5.2.4) we are convinced that $a_{i,k} = a_{k\ominus i}$. It means the equality (5.2.5) is correct. It is clear that if $\alpha, \beta \in \mathbb{R}$ are constants, then

$$\alpha\left(a_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1} + \beta\left(b_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1} = \left(\alpha a_{j\ominus i} + \beta b_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1},$$

which completes the proof of the theorem. □

Example 5.6. Consider the matrix $A_{(2)}$ with $l_2 = 3$ and $l_1 = 2$

$$A_{(2)} = \begin{pmatrix} a_{(0,0)\ominus(0,0)} & a_{(0,1)\ominus(0,0)} & a_{(1,0)\ominus(0,0)} & a_{(1,1)\ominus(0,0)} & a_{(2,0)\ominus(0,0)} & a_{(2,1)\ominus(0,0)} \\ a_{(0,0)\ominus(0,1)} & a_{(0,1)\ominus(0,1)} & a_{(1,0)\ominus(0,1)} & a_{(1,1)\ominus(0,1)} & a_{(2,0)\ominus(0,1)} & a_{(2,1)\ominus(0,1)} \\ a_{(0,0)\ominus(1,0)} & a_{(0,1)\ominus(1,0)} & a_{(1,0)\ominus(1,0)} & a_{(1,1)\ominus(1,0)} & a_{(2,0)\ominus(1,0)} & a_{(2,1)\ominus(1,0)} \\ a_{(0,0)\ominus(1,1)} & a_{(0,1)\ominus(1,1)} & a_{(1,0)\ominus(1,1)} & a_{(1,1)\ominus(1,1)} & a_{(2,0)\ominus(1,1)} & a_{(2,1)\ominus(1,1)} \\ a_{(0,0)\ominus(2,0)} & a_{(0,1)\ominus(2,0)} & a_{(1,0)\ominus(2,0)} & a_{(1,1)\ominus(2,0)} & a_{(2,0)\ominus(2,0)} & a_{(2,1)\ominus(2,0)} \\ a_{(0,0)\ominus(2,1)} & a_{(0,1)\ominus(2,1)} & a_{(1,0)\ominus(2,1)} & a_{(1,1)\ominus(2,1)} & a_{(2,0)\ominus(2,1)} & a_{(2,1)\ominus(2,1)} \end{pmatrix}$$
$$= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 \\ a_4 & a_5 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_0 & a_1 \\ a_3 & a_2 & a_5 & a_4 & a_1 & a_0 \end{pmatrix}.$$

Let us present a corollary from Theorem 5.8 that is important for further analysis.

Corollary 5.5. The matrix $A_{(2)}$ is uniquely determined by its first row $\mathbf{a} = \left(a_0, a_1, \ldots, a_{l_2l_1-1}\right)$.

Proof. Indeed, by Theorem 5.8, the i-th row of the matrix $A_{(2)}$ can be written through its 1st row in the form $\left(a_{0\ominus i}, a_{1\ominus i}, \ldots, a_{(l_2l_1-1)\ominus i}\right)$, i.e. the ith row of the matrix $A_{(2)}$ is a permutation of its 1st row. □

We denote this property of the matrix $A_{(2)}$ by the equality $A_{(2)} \equiv A_{(2)}(\mathbf{a})$. For example, the matrix $A_{(2)}$ from Example 5.6 can be written as $A_{(2)}(\mathbf{a})$, where $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$.

In Section 5.1.1 matrices of a simpler form are studied

$$A_{(1)}(a_0, \dots, a_{l-1}) = (a_{j-i \bmod l})_{i,j=0,l-1} = \begin{pmatrix} a_0 & a_1 & \dots & a_{l-1} \\ a_{l-1} & a_0 & \dots & a_{l-2} \\ & \dots & & \\ a_1 & a_2 & \dots & a_0 \end{pmatrix}, \tag{5.2.7}$$

which coincide with the matrix $A_{(2)}$ in the case of $l_2 = 1$, $l_1 = l$, since in this case $(i_2, i_1) = (0, i_1) = i_1$ and $i_1 = 0, \dots, l-1$.

Theorem 5.9. *The matrix $A_{(2)}$ has a structure of a matrix consisting of $l_2 \times l_2$ square blocks, each of which is a $l_1 \times l_1$ matrix of type $A_{(1)}$. If we represent the sequence $\mathbf{a}$ in the form $\mathbf{a} = (\mathbf{a}_0, \dots, \mathbf{a}_{l_2-1})$, where $\mathbf{a}_k = (a_{kl_1}, \dots, a_{(k+1)l_1-1})$, $k = 0, \dots, l_2 - 1$ and denote $A_{(1)}^{(k)} = A_{(1)}(\mathbf{a}_k)$, then the equality is true*

$$A_{(2)}(\mathbf{a}) = A_{(1)}\big(A_{(1)}^{(0)}, \dots, A_{(1)}^{(l_2-1)}\big) \equiv \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \dots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \dots & A_{(1)}^{(l_2-2)} \\ & \dots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \dots & A_{(1)}^{(0)} \end{pmatrix}. \tag{5.2.8}$$

Proof. Obviously, the block matrix on the right side of (5.2.8) has size $l_2 l_1 \times l_2 l_1$. We denote an arbitrary element of it as $a_{i,j}$, where $i, j = 0, \dots, l_2 l_1 - 1$. If we write $i = (i_2, i_1)$, $j = (j_2, j_1)$, then we get $a_{i,j} = a_{(i_2,i_1),(j_2,j_1)}$. This means that the element $a_{i,j}$ is in the j_2th block column and in the i_2th block row of this block matrix. Therefore, in accordance with the structure of the matrix $A_{(1)}$ from (5.2.7) we have $a_{i,j} \in A_{(1)}^{(j_2 - i_2 \bmod l_2)}$. If we now take into account the values of the indices j_1 and i_1, which mean that the element $a_{i,j}$ is in the j_1th column and in the i_1th row of the matrix $A_{(1)}^{(j_2-i_2 \bmod l_2)}$, then $a_{i,j} = b_{j_1 - i_1 \bmod l_1}$, where b_k is the element of the 1st row of the matrix $A_{(1)}^{(j_2-i_2 \bmod l_2)}$. Since from the definition of $A_{(1)}^{(m)}$ it follows that $b_k = a_{ml_1+k}$, where $k = 0, \dots, l_1 - 1$, then we have

$$\begin{aligned} a_{i,j} = b_{j_1 - i_1 \bmod l_1} = a_{(j_2 - i_2 \bmod l_2)l_1 + (j_1 - i_1 \bmod l_1)} \\ = a_{((j_2-i_2 \bmod l_2),(j_1-i_1 \bmod l_1))} = a_{j \ominus i}. \end{aligned}$$

Taking into account the equality (5.2.5) from Theorem 5.8, this implies the equality of matrices (5.2.8). The theorem is proved. □

Example 5.7. The property (5.2.8) of matrices of the form $A_{(2)}$ is visible in the matrix $A_{(2)}(\mathbf{a})$ from Example 5.6, where $l_2 = 3$, $l_1 = 2$ and $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$. If we denote $\mathbf{a}_0 = (a_0, a_1)$, $\mathbf{a}_1 = (a_2, a_3)$, $\mathbf{a}_2 = (a_4, a_5)$ and

$$A_{(1)}(\mathbf{a}_0) = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} = A_{(1)}^{(0)}, \quad A_{(1)}(\mathbf{a}_1) = \begin{pmatrix} a_2 & a_3 \\ a_3 & a_2 \end{pmatrix} = A_{(1)}^{(1)},$$
$$A_{(1)}(\mathbf{a}_2) = \begin{pmatrix} a_4 & a_5 \\ a_5 & a_4 \end{pmatrix} = A_{(1)}^{(2)},$$

then $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ and the matrix $A_{(2)}(\mathbf{a})$ is written as

$$A_{(2)}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 \\ a_4 & a_5 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_0 & a_1 \\ a_3 & a_2 & a_5 & a_4 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & A_{(1)}^{(2)} \\ A_{(1)}^{(2)} & A_{(1)}^{(0)} & A_{(1)}^{(1)} \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & A_{(1)}^{(0)} \end{pmatrix}$$
$$= A_{(1)}\big(A_{(1)}^{(0)}, A_{(1)}^{(1)}, A_{(1)}^{(2)}\big).$$

Corollary 5.6. The transposed matrix $A_{(2)}^T(\mathbf{a})$ has the structure of the matrix $A_{(2)}$ and, in addition, $A_{(2)}^T(\mathbf{a}) = A_{(2)}(\mathbf{b})$, where

$$\mathbf{b} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)},$$

and both index components $-j_2$ and $-j_1$ are taken over modl_2 and modl_1, respectively, i.e. $a_{(-j_2,-j_1)} = a_{(l_2-j_2,l_1-j_1)}$.

Proof. By Theorem 5.8 we write

$$A_{(2)}^T(\mathbf{a}) = \big(a_{j\ominus i}\big)^T_{i,j=0,\dots,(l_2-1,l_1-1)} = \big(a_{i\ominus j}\big)_{i,j=0,\dots,(l_2-1,l_1-1)}$$
$$= \big(a_{\ominus(j\ominus i)}\big)_{i,j=0,\dots,(l_2-1,l_1-1)} = \big(b_{j\ominus i}\big)_{i,j=0,\dots,(l_2-1,l_1-1)}.$$

Therefore $\mathbf{b} = (a_{\ominus j})_{j=0,\dots,l_2l_1-1} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$. The corollary is proved. □

Example 5.8. For the matrix $A^T_{(2)}(\mathbf{a})$ from Example 5.7 we have

$$\begin{aligned}\mathbf{b} = (a_{\ominus j})_{j=0,\dots,5} &= (a_0, a_{-1}, a_{-2}, a_{-3}, a_{-4}, a_{-5})\\ &= (a_0, a_{(0,-1)}, a_{(-1,0)}, a_{(-1,-1)}, a_{(-2,0)}, a_{(-2,-1)})\\ &= (a_0, a_{(0,1)}, a_{(2,0)}, a_{(2,1)}, a_{(1,0)}, a_{(1,1)}) = (a_0, a_1, a_4, a_5, a_2, a_3).\end{aligned}$$

Let us study the product of matrices of the form (5.2.5).

Theorem 5.10. *The product of matrices of the form* (5.2.5) *is again a matrix of the form* (5.2.5). *Multiplication of matrices of the form* (5.2.5) *is commutative.*

Proof. Let $A_{(2)}(\mathbf{a})$ and $B_{(2)}(\mathbf{b})$ be matrices of the form (5.2.5), then

$$\begin{aligned}&A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b})\\ &\qquad= (a_{j\ominus i})_{i,j=0,\dots l_2l_1-1}(b_{j\ominus i})_{i,j=0,\dots l_2l_1-1} = \Big(\sum_{k=0}^{l_2l_1-1} a_{k\ominus i}b_{j\ominus k}\Big)_{i,j=0,\dots l_2l_1-1}.\end{aligned}$$

In the sum from the above equality, we replace the index $k \to s$, as in the proof of Theorem 5.8, according to the equality $k \ominus i = s$. Then $k = k \ominus i \oplus i = s \oplus i$ and this means that the correspondence $k \leftrightarrow s$ is one-to-one. Replacing the index $k \to s$ only changes the order of summation in the sum. Due to the commutativity and associativity of the operation $\oplus$, we have

$$\begin{aligned}&A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b})\\ &\qquad= \Big(\sum_{s=0}^{l_2l_1-1} a_s b_{j\ominus(s\oplus i)}\Big)_{i,j=0,\dots l_2l_1-1} = \Big(\sum_{s=0}^{l_2l_1-1} a_s b_{(j\ominus i)\ominus s}\Big)_{i,j=0,\dots l_2l_1-1}.\end{aligned}\tag{5.2.9}$$

The elements of the first row of the matrix obtained on the right have the form

$$c_j = \sum_{s=0}^{l_2l_1-1} a_s b_{j\ominus s}$$

and therefore $A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b}) = (c_{j\ominus i})_{i,j=0,\dots l_2l_1-1}$. Thus, the matrix $A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b})$ has the form (5.2.5).

The commutativity of the product $A_{(2)}B_{(2)}$ (we omit the arguments $\mathbf{a}$ and $\mathbf{b}$ for brevity) is obtained from the equality (5.2.9). Let us replace $(j \ominus i) \ominus s \to k$ in the last sum from (5.2.9). Then $s = (j \ominus i) \ominus k$ and we get

$$\sum_{s=0}^{l_2 l_1-1} a_s b_{(j\ominus i)\ominus s} = \sum_{k=0}^{l_2 l_1-1} a_{(j\ominus i)\ominus k} b_k = \sum_{k=0}^{l_2 l_1-1} b_k a_{(j\ominus i)\ominus k}.$$

The last sum in this equality is the common element of the matrix $(B_{(2)}A_{(2)})_{i,j}$. Therefore, the equality (5.2.9) means that $A_{(2)}B_{(2)} = B_{(2)}A_{(2)}$. The theorem is proved. □

Let $\lambda_k = \exp(i2\pi k/l)$, $k = 0, \ldots, l-1$ be the lth roots of unity. Lemma 5.1 shows that for a matrix of the form

$$A_{(1)}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & \ldots & a_{l-1} \\ a_{l-1} & a_0 & \ldots & a_{l-2} \\ & \ldots & & \\ a_1 & a_2 & \ldots & a_0 \end{pmatrix},$$

where $\mathbf{a} = (a_0, \ldots, a_{l-1})$, its eigenvectors $\mathbf{b}_k$ and eigenvalues μ_k for $k = 1, \ldots, l$ have the form

$$\mathbf{b}_k = (1, \lambda_k, \ldots, \lambda_k^{l-1})^T, \quad \mu_k = \sum_{s=0}^{l-1} a_s \lambda_k^s = \mathbf{a} \cdot \mathbf{b}_k. \tag{5.2.10}$$

Note that the eigenvectors of the matrices $A_{(1)}(\mathbf{a})$ do not depend on the vector $\mathbf{a}$. Is this property true for matrices of type $A_{(2)}(\mathbf{a})$?

The following theorem gives an idea about the eigenvectors and eigenvalues of the matrices $A_{(2)}(\mathbf{a})$ from (5.2.5).

Theorem 5.11. *The eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ can be chosen in the form*

$$\mathbf{a}_{(i_2,i_1)} = \left(I_{l_1}, \lambda_{i_2} I_{l_1}, \ldots, \lambda_{i_2}^{l_2-1} I_{l_1}\right)^T \mathbf{b}_{i_1}, \tag{5.2.11}$$

where I_{l_1} is the unit $l_1 \times l_1$ matrix, the vector $\mathbf{b}_{i_1}$ is taken from (5.2.10) *with $l = l_1$ and λ_{i_2} is l_2th root of unity, $i_1 = 0, \ldots, l_1 - 1$, $i_2 = 0, \ldots, l_2 - 1$. The eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ do not depend on the vector* **a**.

Proof. In accordance with Theorem 5.9, we represent the matrix $A_{(2)}(\mathbf{a})$ in the form

$$A_{(2)}(a_0, \ldots, a_{l_2 l_1-1}) = \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \ldots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \ldots & A_{(1)}^{(l_2-2)} \\ & \ldots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \ldots & A_{(1)}^{(0)} \end{pmatrix},$$

where $A_{(1)}^{(j_2)} = A_{(1)}(a_{j_2 l_1}, \ldots, a_{(j_2+1)l_1-1})$, $j_2 = 0, \ldots, l_2-1$. Consider the block multiplication of $l_2 l_1 \times l_2 l_1$ matrix by $l_2 l_1 \times l_1$ matrix of the form

$$\begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \ldots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \ldots & A_{(1)}^{(l_2-2)} \\ & \ldots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \ldots & A_{(1)}^{(0)} \end{pmatrix} \cdot \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ \ldots \\ B_{l_2-1} \end{pmatrix},$$

where the square blocks on the right B_m have size $l_1 \times l_1$ and need to be found. Let us extend the values of the upper indices of the matrices $A_{(1)}^{(i_2)}$ to $\mathbb{Z}$ and calculate them by modl_2. Then similarly to (5.2.7) the m-th row block of the matrix $A_{(2)}$ can be written as $\big(A_{(1)}^{(-m \bmod l_2)}, A_{(1)}^{(1-m \bmod l_2)}, \ldots, A_{(1)}^{(l_2-m \bmod l_2)}\big)$. Since the exponent of $\lambda_{i_2}^k$ can be calculated by modl_2, we write

$$B_m = \sum_{k=0}^{l_2-1} A_{(1)}^{(k-m \bmod l_2)} I_{l_1} \lambda_{i_2}^k = \sum_{k=0}^{l_2-1} \lambda_{i_2}^k A_{(1)}^{(k-m \bmod l_2)} = \sum_{s=0}^{l_2-1} \lambda_{i_2}^{s+m} A_{(1)}^{(s)} = \lambda_{i_2}^m B_0.$$

Here the index replacement $s = k - m \bmod l_2$ is made. Thus we have

$$A_{(2)}(a_0, \ldots, a_{l_2 l_1-1}) \cdot \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} = \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} B_0.$$

It is easy to see that in the resulting equality the matrix

$$B_0 = B_0(\lambda_{i_2}) = \sum_{s=0}^{l_2-1} \lambda_{i_2}^s A_{(1)}^{(s)}$$

has type $A_{(1)}$ and, therefore, the vectors $\mathbf{b}_{i_1}$ for $i_1 = 0, \ldots, l_1-1$ are eigenvectors of the matrix B_0. Let us multiply the above matrix equality on the right by the vector $\mathbf{b}_{i_1}$. Then we get

$$A_{(2)} \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} \mathbf{b}_{i_1} = \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} B_0 \mathbf{b}_{i_1} = \lambda_{(i_2,i_1)} \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \ldots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} \mathbf{b}_{i_1},$$

where $\lambda_{(i_2,i_1)}$ is the eigenvalue of the matrix $B_0(\lambda_{i_2})$, corresponding to the eigenvector $\mathbf{b}_{i_1}$. If we now recall the notation (5.2.11), we get

$$A_{(2)}(\mathbf{a})\mathbf{a}_{(i_2,i_1)} = \lambda_{(i_2,i_1)}\mathbf{a}_{(i_2,i_1)},$$

i.e., the vector $\mathbf{a}_{(i_2,i_1)}$ of dimension $l_2 l_1$ is an eigenvector of the matrix $A_{(2)}(\mathbf{a})$. The theorem is proved. □

Let us present important corollaries from Theorem 5.11 that allow us to construct eigenvectors and eigenvalues of the matrix $A_{(2)}(\mathbf{a})$.

Corollary 5.7. 1^0. The eigenvector of the matrix $A_{(2)}(\mathbf{a})$ with number (i_2, i_1) can be written as

$$\mathbf{a}_{(i_2,i_1)} = \left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}, \tag{5.2.12}$$

where λ_2 is l_2th root of unity, and λ_1 is l_1th root of unity. The eigenvalue corresponding to this eigenvector can be written in a similar form

$$\lambda_{(i_2,i_1)} = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} = \mathbf{a}\cdot\mathbf{a}^T_{(i_2,i_1)}. \tag{5.2.13}$$

2^0. The eigenvectors of the matrix $A^T_{(2)}(\mathbf{a})$ coincide with the eigenvectors $\mathbf{a}_{(i_2,i_1)}$ of the matrix $A_{(2)}(\mathbf{a})$, and the corresponding eigenvalues are $\lambda^t_{(i_2,i_1)} = \bar{\lambda}_{(i_2,i_1)} = \lambda_{(-i_2,-i_1)}$.

Proof. It is easy to see that the equality (5.2.11) can be written in the form

$$\begin{aligned}\mathbf{a}_{(i_2,i_1)} &= \left(\mathbf{b}^T_{i_1}, \lambda_{i_2}\mathbf{b}^T_{i_1}, \ldots, \lambda_{i_2}^{l_2-1}\mathbf{b}^T_{i_1}\right)^T \\ &= \Big(1, \lambda_{i_1}, \ldots, \lambda_{i_1}^{l_1-1}, \lambda_{i_2}, \lambda_{i_2}\lambda_{i_1}, \ldots \lambda_{i_2}\lambda_{i_1}^{l_1-1}, \ldots, \lambda_{i_2}^{l_2-1}, \lambda_{i_2}^{l_2-1}\lambda_{i_1}, \ldots, \\ &\qquad \lambda_{i_2}^{l_2-1}\lambda_{i_1}^{l_1-1}\Big)^T = \left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)},\end{aligned}$$

where the order of the vector elements corresponds to the order established for the numbers (j_2, j_1). This proves the equality (5.2.12).

Further, from Theorem 5.11 it follows that the eigenvalue $\lambda_{(i_2,i_1)}$ of the matrix $A_{(2)}(\mathbf{a})$ corresponding to the eigenvector $\mathbf{a}_{(i_2,i_1)}$ matches the eigenvalue of the matrix

$$B_0(\lambda_{i_2}) = \sum_{j_2=0}^{l_2-1} \lambda_{i_2}^{j_2} A^{(j_2)}_{(1)},$$

corresponding to the eigenvector $\mathbf{b}_{i_1}$. Here

$$A_{(1)}^{(j_2)} = A_{(1)}(a_{j_2 l_1}, \ldots, a_{(j_2+1)l_1-1}).$$

Since the matrix $B_0(\lambda_{i_2})$ is of type $A_{(1)}$, then in accordance with (5.2.10) we find the first row of the matrix $B_0(\lambda_{i_2})$. Let us denote it as $b_0(\lambda_{i_2})$

$$b_0(\lambda_{i_2}) = \Big(\sum_{j_2=0}^{l_2-1} \lambda_{i_2}^{j_2} a_{j_2 l_1}, \ldots, \sum_{j_2=0}^{l_2-1} \lambda_{i_2}^{j_2} a_{j_2 l_1 + l_1 - 1} \Big).$$

Using the equality (5.2.10) we find

$$\lambda_{(i_2,i_1)} = \sum_{j_1=0}^{l_1-1} \lambda_{i_1}^{j_1} \sum_{j_2=0}^{l_2-1} a_{j_2 l_1 + j_1} \lambda_{i_2}^{j_2} = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1},$$

which is the same as (5.2.13). Statement 1^0 is proved.

By Corollary 5.6 and Theorem 5.11 the eigenvectors of the matrix $A_{(2)}^T(\mathbf{a})$ coincide with the eigenvectors of the matrix $A_{(2)}(\mathbf{a})$. Let's find the eigenvalue corresponding to the vector $\mathbf{a}_{(i_2,i_1)}$. According to Corollary 5.6 $A_{(2)}^T(\mathbf{a}) = A_{(2)}(\mathbf{b})$, where $\mathbf{b} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$. That is why

$$\begin{aligned}\lambda_{(i_2,i_1)}^t = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(-j_2,-j_1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} &= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{(k_2,k_1)} \lambda_{i_2}^{-k_2} \lambda_{i_1}^{-k_1} \\ &= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{(k_2,k_1)} \bar{\lambda}_{i_2}^{k_2} \bar{\lambda}_{i_1}^{k_1} = \bar{\lambda}_{(i_2,i_1)}.\end{aligned}$$

From the last equality it is clear that $\bar{\lambda}_{(i_2,i_1)} = \lambda_{(-i_2,-i_1)}$. Statement 2^0 is proved, and therefore the corollary is also proved. □

Another property of the eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ is given below in Corollary 5.9.

Remark 5.1. The expression $\lambda_{i_2}\lambda_{i_1}$ is an ordered pair and the first place in it is λ_{i_2}, which is l_2th root of unity, and the second place in it is λ_{i_1} – l_1th root of unity. Therefore, in the general case $\lambda_1\lambda_1 \neq \lambda_1^2$.

Remark 5.2. In [177, Corollary 3] we obtain eigenvectors and eigenvectors of matrices, which for $n = 2$ are a special case of matrices $A_{(2)}$. The eigenvectors are written as

$$\mathbf{a}_2^{(i_2,i_1)} = \left((-1)^{j_2 i_2 + j_1 i_1}\right)_{(j_2,j_1)=0,\ldots,3},$$

which is the same as (5.2.12) for $l_1 = l_2 = 2$. Indeed, in this case $\lambda_{i_2} = (-1)^{i_2}$, $\lambda_{i_1} = (-1)^{i_1}$, $(l_2-1, l_1-1) = (1,1) = 3$ and therefore the common eigenvector term from (5.2.12) is $\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} = (-1)^{i_2 j_2 + i_1 j_1}$, which coincides with the general term of the vector $\mathbf{a}_2^{(i_2,i_1)}$. The eigenvalues obtained in (5.2.13) also coincide with those found in [177] for $n = 2$.

Example 5.9. For the matrix $A_{(2)}(\mathbf{a})$ from example 5.7 we have $\lambda_{i_2} = \lambda^{i_2}$, $\lambda_{i_1} = (-1)^{i_1}$, where $\lambda = \exp\left(i\frac{2\pi}{3}\right)$. Therefore, according to equality (5.2.12) $\mathbf{a}_{(i_2,i_1)} = (\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1})^T_{(j_2,j_1)=0,\ldots,(2,1)}$ we get

$$\mathbf{a}_{(0,0)} = (1,1,1,1,1,1),\ \mathbf{a}_{(0,0)} = (1,-1,1,-1,1,-1),\ \mathbf{a}_{(1,0)} = (1,1,\lambda,\lambda,\bar{\lambda},\bar{\lambda}),$$
$$\mathbf{a}_{(1,1)} = (1,-1,\lambda,-\lambda,\bar{\lambda},-\bar{\lambda}),\ \mathbf{a}_{(2,0)} = (1,1,\bar{\lambda},\bar{\lambda},\lambda,\lambda),$$
$$\mathbf{a}_{(2,1)} = (1,-1,\bar{\lambda},-\bar{\lambda},\lambda,-\lambda)$$

and using (5.2.13) $\lambda_{(i_2,i_1)} = \mathbf{a}\cdot\mathbf{a}_{(i_2,i_1)}$ we calculate

$$\lambda_{(0,0)} = a_0 + a_1 + a_2 + a_3 + a_4 + a_5,\ \lambda_{(0,1)} = a_0 - a_1 + a_2 - a_3 + a_4 - a_5,$$
$$\lambda_{(1,0)} = a_0 + a_1 + \lambda(a_2 + a_3) + \bar{\lambda}(a_4 + a_5),$$
$$\lambda_{(1,1)} = a_0 - a_1 + \lambda(a_2 - a_3) + \bar{\lambda}(a_4 - a_5),$$
$$\lambda_{(2,0)} = a_0 + a_1 + \bar{\lambda}(a_2 + a_3) + \lambda(a_4 + a_5),$$
$$\lambda_{(2,1)} = a_0 - a_1 + \bar{\lambda}(a_2 - a_3) + \lambda(a_4 - a_5).$$

5.2.3 Main problem $\mathcal{S}_2$

To study the problem $\mathcal{S}_2$ we also need Lemma 5.3 from the previous section and the following statement.

Lemma 5.8. *The equation* (5.2.1) *generates an equivalent matrix equation*

$$A_{(2)}(\mathbf{a})\Delta U(x) + \lambda U(x) = 0, \tag{5.2.14}$$

where $U(x) = \left(u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)\right)^T_{(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)}$ *and* $\lambda \in \mathbb{R}$.

Proof. Let the function $u(x)$ satisfy the equation (5.2.1). Let us denote

$$v(x) = \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} u(\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}x)$$

and $V(x) = \big(v(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$. The function $v(x)$ generates the equality (5.2.4). Let's apply the Laplace operator Δ to (5.2.4). Since matrices of the form $\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}$ are orthogonal, by virtue of Lemma 5.3 we can write

$$\begin{aligned}
&\Delta V(x) \\
&\quad = \Big(\Delta I_{\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}} v(x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} = \Big(I_{\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}} \Delta v(x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \\
&\quad = \Big(I_{\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}} \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} I_{\mathcal{S}_2^{i_2}\mathcal{S}_1^{i_1}} \Delta u(x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \\
&\quad = \Big(\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} I_{\mathcal{S}_2^{j_2+i_2 \bmod l_2}\mathcal{S}_1^{j_1+i_1 \bmod l_1}} \Delta u(x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \\
&\quad = \Big(\sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus j} I_{\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}} \Delta u(x)\Big)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)} \\
&\quad = \Big(\sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus j} \Delta u(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x)\Big)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)} \\
&\quad = A_{(2)}(\mathbf{a})\Delta U(x).
\end{aligned}$$

In the transformations performed, the replacement of the summation index $j\oplus i = k$ is used. Using equality

$$\Delta v(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x) + \lambda u(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x) = 0$$

(see (5.2.1)), from which it follows that $\Delta V(x) + \lambda U(x) = 0$, we easily obtain (5.2.14). It should be noted that the first equation in (5.2.14) is the same as (5.2.1). The lemma is proven. □

Basing on Lemmas 5.3 and 5.8, we prove the following statement about necessary conditions for the existence of eigenvalues of Problem $\mathcal{S}_2$.

Theorem 5.12. *Let the function $u(x) \neq 0$ be an eigenfunction of problem $\mathcal{S}_2$, and λ be its eigenvalue corresponding to $u(x)$. Then the function*

$$w(x) = U(x) \cdot \mathbf{a}_{(i_2,i_1)},$$

where $U(x) = \big(u(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$ and $\mathbf{a}_{(i_2,i_1)}$ is the eigenvector of the matrix $A_{(2)}(\mathbf{a})$ such that $\lambda_{(i_2,i_1)} \neq 0$, is a solution to the boundary value problem

$$\Delta w(x) + \mu w(x) = 0,\ x \in S, \tag{5.2.15}$$

$$w\ (x) = 0,\ x \in \partial S, \tag{5.2.16}$$

where $\mu = \lambda/\bar{\lambda}_{(i_2,i_1)}$.

Proof. Let λ be the eigenvalue of problem $\mathcal{S}_2$, and $u(x) \neq 0$ be the corresponding eigenfunction. By Lemma 5.8 the equality (5.2.14) holds true. Multiply it scalarly by the vector $\mathbf{a}_{(i_2,i_1)}$

$$A_{(2)}(\mathbf{a})\Delta U(x) \cdot \mathbf{a}_{(i_2,i_1)} + \lambda U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0,$$

from here we get

$$\Delta\big(U(x) \cdot A^T_{(2)}(\mathbf{a})\mathbf{a}_{(i_2,i_1)}\big) + \lambda U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0.$$

Since by Corollary 5.7 the vector $\mathbf{a}_{(i_2,i_1)}$ is also an eigenvector of the matrix $A^T_{(2)}$, and $\bar{\lambda}_{(i_2,i_1)}$ is its eigenvalue, then, taking into account the definition of $w(x)$, we find

$$\bar{\lambda}_{(i_2,i_1)}\Delta w(x) + \lambda w(x) = 0.$$

According to the conditions of the lemma $\lambda = \bar{\lambda}_{(i_2,i_1)}\mu$ and $\lambda_{(i_2,i_1)} \neq 0$. Therefore we get

$$0 = \bar{\lambda}_{(i_2,i_1)}\ (\Delta w(x) + \mu w(x)) \Rightarrow \Delta w(x) + \mu w(x) = 0,\ x \in S.$$

The equality (5.2.15) is proved. Finally, since $u(x) = 0$ for $x \in \partial S$ and $x \in \partial S \Rightarrow \mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x \in \partial S$, then $U(x) = 0$ for $x \in \partial S$ and, therefore, $w(x) = U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0$, $x \in \partial S$. The theorem is proved. □

The converse of Theorem 5.12 is important and allows us to construct solutions to the problem $\mathcal{S}_2$.

Theorem 5.13. *Let the function* $w(x) \neq 0$ *be a solution to the problem* (5.2.15)-(5.2.16)

$$\Delta w(x) + \mu w(x) = 0,\ x \in S,$$
$$w(x) = 0,\ x \in \partial S$$

for some $\mu > 0$. *Then the function*

$$u_{(i_2,i_1)}(x) = W(x) \cdot \mathbf{a}_{(i_2,i_1)}, \tag{5.2.17}$$

where

$$W(x) = \big(w(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$$

and $\mathbf{a}_{(i_2,i_1)}$ *is the eigenvector of the matrix* $A_{(2)}(\mathbf{a})$ *from* (5.2.12) *with eigenvalue* $\lambda_{(i_2,i_1)} \neq 0$ *from* (5.2.13)*, is a solution to the problem* (5.2.1)-(5.2.2) *for* $\lambda = \mu\bar{\lambda}_{(i_2,i_1)}$.

Proof. Let $w(x) \neq 0$ be a solution to problem (5.2.15)-(5.2.16). Consider the vector $W(x) = \big(w(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$ and compose the function

$$u_{(i_2,i_1)}(x) = W(x) \cdot \mathbf{a}_{(i_2,i_1)},$$

where $x \in S$. It is easy to see that according to Corollary 5.7, the following equalities hold true in S

$$\begin{aligned}
u_{(i_2,i_1)}(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x) &= W(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x) \cdot \mathbf{a}_{(i_2,i_1)} \\
&= \big(w(\mathcal{S}_2^{j_2+k_2}\mathcal{S}_1^{j_1+k_1}x)\big)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)} \cdot \big(\lambda_{i_2}^{k_2}\lambda_{i_1}^{k_1}\big)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)} \\
&= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} w(\mathcal{S}_2^{j_2+k_2}\mathcal{S}_1^{j_1+k_1}x)\lambda_{i_2}^{k_2}\lambda_{i_1}^{k_1} = \sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)} w(\mathcal{S}_2^{m_2}\mathcal{S}_1^{m_1}x)\lambda_{i_2}^{m_2-j_2}\lambda_{i_1}^{m_1-j_1} \\
&= \Big(\sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)} w(\mathcal{S}_2^{m_2}\mathcal{S}_1^{m_1}x)\lambda_{i_2}^{m_2}\lambda_{i_1}^{m_1}\Big)\lambda_{i_2}^{-j_2}\lambda_{i_1}^{-j_1} = \bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}W(x) \cdot \mathbf{a}_{(i_2,i_1)}.
\end{aligned}$$

Therefore, again by Corollary 5.7, we find

$$\begin{aligned}
U_{(i_2,i_1)}(x) &= \big(u_{(i_2,i_1)}(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \\
&= W(x) \cdot \mathbf{a}_{(i_2,i_1)}\big(\bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} = u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)}.
\end{aligned}$$

Here we used the substitution of indices $j \oplus k = m \Leftrightarrow k = m \ominus j$. Thus the equality

$$\Delta U_{(i_2,i_1)}(x) = \Delta u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)}.$$

is true. Hence, since by Lemma 5.3 we have

$$\Delta W(x) = \big(\Delta w(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$$
$$= \big(-\mu w(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\big)_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} = -\mu W(x),$$

then we can write

$$A_{(2)}(\mathbf{a})\Delta U_{(i_2,i_1)}(x) = \Delta u_{(i_2,i_1)}(x)A_{(2)}(\mathbf{a})\bar{\mathbf{a}}_{(i_2,i_1)} = \Delta u_{(i_2,i_1)}(x)\overline{A_{(2)}(\mathbf{a})\mathbf{a}}_{(i_2,i_1)}$$
$$= (\Delta W(x)\cdot \mathbf{a}_{(i_2,i_1)})\bar{\lambda}_{(i_2,i_1)}\bar{\mathbf{a}}_{(i_2,i_1)} = -\mu(W(x)\cdot \mathbf{a}_{(i_2,i_1)})\bar{\lambda}_{(i_2,i_1)}\bar{\mathbf{a}}_{(i_2,i_1)}$$
$$= -\mu\bar{\lambda}_{(i_2,i_1)}u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)} = -\mu\bar{\lambda}_{(i_2,i_1)}U_{(i_2,i_1)}(x).$$

Extracting the first components of this vector equality, we find

$$\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}\Delta u_{(i_2,i_1)}(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x) = -\mu\bar{\lambda}_{(i_2,i_1)}u_{(i_2,i_1)}(x), \quad x\in S,$$

which means that the function $u_{(i_2,i_1)}(x)$ is a solution to the equation (5.2.1).

Let us check the boundary conditions (5.2.2). Since $x\in\partial S \Rightarrow \mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x\in\partial S$, then for $x\in\partial S$ we have

$$u_{(i_2,i_1)}(x)$$
$$= W(x)\cdot \mathbf{a}_{(i_2,i_1)} = \Big(w(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}\cdot \mathbf{a}_{(i_2,i_1)} = \mathbf{0}\cdot \mathbf{a}_{(i_2,i_1)} = 0.$$

Therefore, the function $u_{(i_2,i_1)}(x)$ is a solution to the problem $\mathcal{S}_2$. The theorem is proved. □

Example 5.10. Consider the problem (5.2.1)-(5.2.2) with $l_2 = 3$, $l_1 = 2$ and $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$. Let us use Theorem 5.13. To do this, take the eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ from Example 5.9 in the form (5.2.12). Let μ be the eigenvalue of the boundary value problem (5.2.15)-(5.2.16), and $w_\mu(x)$ is the corresponding eigenfunction. Then the eigenfunctions of the problem (5.2.1)-(5.2.2) corresponding to μ can be taken in the form $u_{(i_2,i_1)}(x) = W(x)\cdot \mathbf{a}_{(i_2,i_1)}$ (5.2.17):

$$u_{(0,0)}(x) = w_\mu(x)$$
$$+w_\mu(\mathcal{S}_1x) + w_\mu(\mathcal{S}_2x) + w_\mu(\mathcal{S}_2\mathcal{S}_1x) + w_\mu(\mathcal{S}_2^2x) + w_\mu(\mathcal{S}_2^2\mathcal{S}_1x),$$
$$u_{(0,1)}(x) = w_\mu(x)$$

$$
\begin{aligned}
&\qquad -w_\mu(\mathcal{S}_1x)+w_\mu(\mathcal{S}_2x)-w_\mu(\mathcal{S}_2\mathcal{S}_1x)+w_\mu(\mathcal{S}_2^2x)-w_\mu(\mathcal{S}_2^2\mathcal{S}_1x),\\
&u_{(1,0)}(x)=w_\mu(x)\\
&\qquad +w_\mu(\mathcal{S}_1x)+\lambda w_\mu(\mathcal{S}_2x)+\lambda w_\mu(\mathcal{S}_2\mathcal{S}_1x)+\bar\lambda w_\mu(\mathcal{S}_2^2x)+\bar\lambda w_\mu(\mathcal{S}_2^2\mathcal{S}_1x),\\
&u_{(1,1)}(x)=w_\mu(x)\\
&\qquad -w_\mu(\mathcal{S}_1x)+\lambda w_\mu(\mathcal{S}_2x)-\lambda w_\mu(\mathcal{S}_2\mathcal{S}_1x)+\bar\lambda w_\mu(\mathcal{S}_2^2x)-\bar\lambda w_\mu(\mathcal{S}_2^2\mathcal{S}_1x),\\
&u_{(2,0)}(x)=w_\mu(x)\\
&\qquad +w_\mu(\mathcal{S}_1x)+\bar\lambda w_\mu(\mathcal{S}_2x)+\bar\lambda w_\mu(\mathcal{S}_2\mathcal{S}_1x)+\lambda w_\mu(\mathcal{S}_2^2x)+\lambda w_\mu(\mathcal{S}_2^2\mathcal{S}_1x),\\
&u_{(2,1)}(x)=w_\mu(x)\\
&\qquad -w_\mu(\mathcal{S}_1x)+\bar\lambda w_\mu(\mathcal{S}_2x)-\bar\lambda w_\mu(\mathcal{S}_2\mathcal{S}_1x)+\lambda w_\mu(\mathcal{S}_2^2x)-\lambda w_\mu(\mathcal{S}_2^2\mathcal{S}_1x).
\end{aligned}
$$

If we use the eigenvalues of the matrix $A_{(2)}(\mathbf{a})$ from (5.2.12), then the eigenvalues of the problem (5.2.1)-(5.2.2), corresponding to the eigenfunctions written above, look like $\mu_{(i_2,i_1)}=\mu\bar\lambda_{(i_2,i_1)}=\mu\lambda_{(-i_2,-i_1)}$:

$$
\begin{aligned}
&\mu_{(0,0)}=\mu\lambda_{(0,0)}, \quad \mu_{(0,1)}=\mu\lambda_{(0,1)}, \quad \mu_{(1,0)}=\mu\lambda_{(2,0)},\\
&\mu_{(1,1)}=\mu\lambda_{(2,1)}, \quad \mu_{(2,0)}=\mu\lambda_{(1,0)}, \quad \mu_{(2,1)}=\mu\lambda_{(1,1)}.
\end{aligned}
$$

In further research it is necessary to expand the polynomials into a sum of polynomials of generalized parity.

Definition 5.1. Let $H(x)$ be some function defined on S. Let us call the function

$$
F_{(i_2,i_1)}[H](x)=\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}H\big(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x\big), \quad x\in S \tag{5.2.18}
$$

a component of the function $H(x)$ of generalized parity (i_2,i_1).

Lemma 5.9. *The function $F_{(i_2,i_1)}[H](x)$ has the "generalized parity" property*

$$
F_{(i_2,i_1)}[H]\big(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x\big)=\bar\lambda_{i_2}^{k_2}\bar\lambda_{i_1}^{k_1}F_{(i_2,i_1)}[H](x) \tag{5.2.19}
$$

and the following equality is true

$$
F_{(k_2,k_1)}\big[F_{(i_2,i_1)}[H]\big](x)=\begin{cases}F_{(i_2,i_1)}[H](x) & (i_2,i_1)=(k_2,k_1),\\ 0 & (i_2,i_1)\neq(k_2,k_1).\end{cases} \tag{5.2.20}
$$

In addition, the function $H(x)$ can be represented as the sum of all its components

$$
H(x)=\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}F_{(i_2,i_1)}[H](x), \quad x\in S. \tag{5.2.21}
$$

Proof. It is not hard to see that

$$F_{(i_2,i_1)}[H]\big(\mathcal{S}_2^{k_2}\mathcal{S}_1^{k_1}x\big) = \frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} H\big(\mathcal{S}_2^{k_2+j_2}\mathcal{S}_1^{k_1+j_1}x\big)$$

$$= \frac{1}{l_2 l_1}\sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{m_2-k_2}\lambda_{i_1}^{m_1-k_1} H\big(\mathcal{S}_2^{m_2}\mathcal{S}_1^{m_1}x\big) = \lambda_{i_2}^{-k_2}\lambda_{i_1}^{-k_1}F_{(i_2,i_1)}[H](x)$$

$$= \bar\lambda_{i_2}^{k_2}\bar\lambda_{i_1}^{k_1}F_{(i_2,i_1)}[H](x),$$

where under the sum sign the index $k \oplus j = m$ is replaced as in Theorem 5.10. The equality (5.2.19) is proved.

Now consider the equality (5.2.21). It is easy to see that

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} F_{(i_2,i_1)}[H](x) = \frac{1}{l_2 l_1}\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} H\big(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x\big)$$

$$= \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} H\big(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x\big)\frac{1}{l_2 l_1}\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}. \qquad (5.2.22)$$

Let us calculate the internal sum from the right side of the equality (5.2.22). Let $(j_2, j_1) \neq 0$, then, in particular, $j_2 \neq 0$ is possible (the case $j_1 \neq 0$ is considered similarly), which means $\lambda_{j_2} \neq 1$. Taking into account that $\lambda_{j_2}^{l_2} = 1$, and using a simple combinatorial identity we can find

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} = \sum_{i_2=0}^{l_2-1}\lambda_{j_2}^{i_2}\sum_{i_1=0}^{l_1-1}\lambda_{j_1}^{i_1} = \frac{\lambda_{j_2}^{l_2}-1}{\lambda_{j_2}-1}\sum_{i_1=0}^{l_1-1}\lambda_{j_1}^{i_1} = 0. \qquad (5.2.23)$$

If $(j_2, j_1) = 0$, then $\lambda_{j_2} = 1$, $\lambda_{j_1} = 1$ and, therefore,

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} = l_2 l_1.$$

Thus, the expression on the right side of (5.2.22) is equal to $H(\mathcal{S}_2^0\mathcal{S}_1^0x) = H(x)$. This proves the equality (5.2.21).

Now let us prove (5.2.20). It is easy to see that with the help of (5.2.19) and (5.2.21) we can write

$$
\begin{aligned}
F_{(k_2,k_1)}\big[F_{(i_2,i_1)}[H]\big](x) &= \frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{k_2}^{j_2}\lambda_{k_1}^{j_1} F_{(i_2,i_1)}[H]\big(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1}x\big)\\
&= \frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{k_2}^{j_2}\lambda_{k_1}^{j_1}\bar\lambda_{i_2}^{j_2}\bar\lambda_{i_1}^{j_1} F_{(i_2,i_1)}[H](x)\\
&= F_{(i_2,i_1)}[H](x)\frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} (\lambda_{k_2}\bar\lambda_{i_2})^{j_2}(\lambda_{k_1}\bar\lambda_{i_1})^{j_1}\\
&= F_{(i_2,i_1)}[H](x)\frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{k_2-i_2}^{j_2}\lambda_{k_1-j_1}^{j_1}.
\end{aligned}
$$

Since due to (5.2.23) the equality

$$
\frac{1}{l_2 l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{k_2-i_2}^{j_2}\lambda_{k_1-j_1}^{j_1} = \begin{cases} 1 & (k_2,k_1)=(i_2,i_1),\\ 0 & (k_2,k_1)\neq(i_2,i_1). \end{cases},
$$

holds true, then (5.2.20) follows from it. Here it is taken into account that $\lambda_{k_2}\bar\lambda_{i_2} = \lambda_{k_2}\lambda_{-i_2} = \lambda_{k_2-i_2}$. The lemma is proved. □

Example 5.11. Let $l_2 = l_1 = 2$, $\mathcal{S}_1 x = (-x_1, x_2)$, $\mathcal{S}_2 x = (x_1, -x_2)$. Taking into account that $\lambda_{i_2} = (-1)^{i_2}$, $\lambda_{i_1} = (-1)^{i_1}$ we get from (5.2.18) that

$$
\begin{aligned}
F_{(0,0)}[H](x) &= \frac14\big(H(x_1,x_2)+H(-x_1,x_2)+H(x_1,-x_2)+H(-x_1,-x_2)\big),\\
F_{(0,1)}[H](x) &= \frac14\big(H(x_1,x_2)-H(-x_1,x_2)+H(x_1,-x_2)-H(-x_1,-x_2)\big),\\
F_{(1,0)}[H](x) &= \frac14\big(H(x_1,x_2)+H(-x_1,x_2)-H(x_1,-x_2)-H(-x_1,-x_2)\big),\\
F_{(1,1)}[H](x) &= \frac14\big(H(x_1,x_2)-H(-x_1,x_2)-H(x_1,-x_2)+H(-x_1,-x_2)\big).
\end{aligned}
$$

If, for example, the function $H(x)$ is even in x_1, then its components $F_{(i_2,i_1)}[H]$ of generalized parity $(0,1)$ and $(1,1)$ are equal to zero.

Now, for clarity, let $H_m(x_1,x_2)$ be a homogeneous harmonic polynomial of degree m and (r,φ) be the polar coordinates of the point $x=(x_1,x_2)$. Then there exist $\alpha,\beta\in\mathbb{R}$ such that

$$
H_m(x) = \alpha\,\mathrm{Re}(x_1+ix_2)^m + \beta\,\mathrm{Im}(x_1+ix_2)^m = r^m(\alpha\cos m\varphi + \beta\sin m\varphi),
$$

whence follows

$$\begin{aligned}
H_m(-x_1, x_2) &= \alpha\,\mathrm{Re}(-x_1 + ix_2)^m + \beta\,\mathrm{Im}(-x_1 + ix_2)^m \\
&= (-r)^m(\alpha\cos m\varphi - \beta\sin m\varphi), \\
H_m(x_1, -x_2) &= \alpha\,\mathrm{Re}(x_1 - ix_2)^m + \beta\,\mathrm{Im}(x_1 - ix_2)^m \\
&= r^m(\alpha\cos m\varphi - \beta\sin m\varphi), \\
H_m(-x_1, -x_2) &= (-r)^m(\alpha\cos m\varphi + \beta\sin m\varphi).
\end{aligned}$$

From the equalities above it follows that the operator $F_{(i_2,i_1)}[\cdot]$ extracts the following components of the harmonic polynomial $H_m(x)$:

$$\begin{aligned}
F_{(0,0)}[H_m](x) &= \alpha r^m \frac{1 + (-1)^m}{2}\cos m\varphi, \\
F_{(0,1)}[H_m](x) &= \alpha r^m \frac{1 - (-1)^m}{2}\cos m\varphi, \\
F_{(1,0)}[H_m](x) &= \beta r^m \frac{1 - (-1)^m}{2}\sin m\varphi, \\
F_{(1,1)}[H_m](x) &= \beta r^m \frac{1 + (-1)^m}{2}\sin m\varphi.
\end{aligned}$$

Therefore, for $m \in \mathbb{N}_0$ we find

$$\begin{gathered}
F_{(0,0)}[H_{2m}](x) = \alpha r^{2m}\cos 2m\varphi,\ F_{(1,1)}[H_{2m}](x) = \beta r^{2m}\sin 2m\varphi, \\
F_{(0,1)}[H_{2m+1}](x) = \alpha r^{2m+1}\cos(2m+1)\varphi, \\
F_{(1,0)}[H_{2m+1}](x) = \beta r^{2m+1}\sin(2m+1)\varphi,
\end{gathered} \tag{5.2.24}$$

and the remaining components disappear

$$F_{(0,1)}[H_{2m}](x) = F_{(1,0)}[H_{2m}](x) = F_{(0,0)}[H_{2m+1}](x) = F_{(1,1)}[H_{2m+1}](x) = 0.$$

5.2.4 Eigenfunctions and eigenvalues of problem S_2

Let us transform the result of theorem 5.13 to a simpler form.

Theorem 5.14. *The eigenfunctions and eigenvalues of the problem* (5.2.1)-(5.2.2) *can be represented as*

$$\hat{u}_{(i_2,i_1)}(x) = F_{(i_2,i_1)}[w_\mu](x), \quad \lambda_{\mu,(i_2,i_1)} = \mu \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}\bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}, \tag{5.2.25}$$

where the operator $F_{(i_2,i_1)}[\cdot]$ *is defined in* (5.2.18)*, the function* $w_\mu(x) \neq 0$ *is the solution to the problem* (5.2.15)-(5.2.16)

$$\Delta w(x) + \mu w(x) = 0,\ x \in S; \quad w(x) = 0,\ x \in \partial S$$

for some $\mu \in \mathbb{R}_+$*. The eigenfunctions* $\hat{u}_{(i_2,i_1)}(x)$ *for* $(i_2, i_1) = 0, \ldots, (l_2 - 1, l_1 - 1)$ *and fixed* μ *are orthogonal in* $L_2(S)$*.*

The functions $\hat{u}_{(i_2,i_1)}(x)$ *are parts of the eigenfunction* $w_\mu(x)$ *in the sense that*

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} \hat{u}_{(i_2,i_1)}(x) = w_\mu(x). \tag{5.2.26}$$

Proof. Let us denote $\hat{u}_{(i_2,i_1)}(x) = \frac{1}{l_2 l_1} u_{(i_2,i_1)}(x)$. It is clear that $\hat{u}_{(i_2,i_1)}(x)$ is also an eigenfunction of the problem (5.2.1)-(5.2.2). It is easy to see that from (5.2.17) it follows

$$\begin{aligned}\hat{u}_{(i_2,i_1)}(x) &= \frac{1}{l_2 l_1} u_{(i_2,i_1)}(x) = \frac{1}{l_2 l_1} W_\mu(x) \cdot \mathbf{a}_{(i_2,i_1)} \\ &= \frac{1}{l_2 l_1} \big(w_\mu(\mathcal{S}_2^{j_2} \mathcal{S}_1^{j_1} x)\big)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \cdot \big(\lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1}\big)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\ &= \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} w_\mu(\mathcal{S}_2^{j_2} \mathcal{S}_1^{j_1} x) \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} = F_{(i_2,i_1)}[w_\mu](x),\end{aligned}$$

which proves the first equality from (5.2.25).

The eigenvalue of the Dirichlet problem (5.2.1)-(5.2.2) corresponding to the eigenfunction $\hat{u}_{(i_2,i_1)}(x)$, by Theorem 5.13 and the equality (5.2.13) from Corollary 5.7 can be represented as

$$\lambda_{\mu,(i_2,i_1)} = \mu \bar{\lambda}_{(i_2,i_1)} = \mu \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \bar{\lambda}_{i_2}^{j_2} \bar{\lambda}_{i_1}^{j_1}.$$

Now we prove that the functions $\hat{u}_{(i_2,i_1)}(x)$ and $\hat{u}_{(j_2,j_1)}(x)$ for $(i_2, i_1) \neq (j_2, j_1)$ are orthogonal in $L_2(S)$. Indeed, if $(i_2 - j_2, i_1 - j_1) \neq 0$, then either $i_2 - j_2 \neq 0 \bmod l_2$ or $i_1 - j_1 \neq 0 \bmod l_1$. Let, for example, $i_2 - j_2 \neq 0$ and, therefore, $\lambda_{i_2 - j_2} \neq 1$. Then, according to Lemma 5.5 for $g \in C(S)$ the following equality holds true:

$$\int_S g(\mathcal{S}_2 \xi)\, d\xi = \int_S g(\xi)\, d\xi.$$

Therefore, using the equality (5.2.19) from Lemma 5.9, we obtain

$$\int_S \hat{u}_{(i_2,i_1)}(x)\bar{\hat{u}}_{(j_2,j_1)}(x)\,dx = \int_S F_{(i_2,i_1)}[w_\mu](x)\bar{F}_{(j_2,j_1)}[w_\mu](x)\,dx$$
$$= \int_S F_{(i_2,i_1)}[w_\mu](\mathcal{S}_2 x)\bar{F}_{(j_2,j_1)}[w_\mu](\mathcal{S}_2 x)\,dx = \bar{\lambda}_{i_2}\lambda_{j_2}\int_S F_{(i_2,i_1)}[w_\mu](x)$$
$$\times\bar{F}_{(j_2,j_1)}[w_\mu](x)\,dx = \bar{\lambda}_{i_2-j_2}\int_S \hat{u}_{(i_2,i_1)}(x)\bar{\hat{u}}_{(j_2,j_1)}(x)\,dx. \quad (5.2.27)$$

Since $\lambda_{i_2-j_2}\neq 1$, this immediately implies orthogonality

$$\int_S \hat{u}_{(i_2,i_1)}(x)\bar{\hat{u}}_{(j_2,j_1)}(x)\,dx = 0.$$

Finally, the equality (5.2.26) is a consequence of the equality (5.2.21) from Lemma 5.9 for $H(x)=w_\mu(x)$. The theorem is proved. □

Corollary 5.8. If $H(x)$ is a harmonic polynomial with real coefficients, then the harmonic polynomials $F_{(i_2,i_1)}[H](x)$ for $(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)$ are orthogonal and linearly independent.

Proof. Indeed, let $(i_2,i_1)\neq(j_2,j_1)$, which is possible, for example, for $i_1\neq j_1$ whence $\lambda_{i_1-j_1}\neq 1$. By Lemma 5.5, similarly to (5.2.27), we obtain

$$\int_{\partial S} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds = \int_{\partial S} F_{(i_2,i_1)}[H](\mathcal{S}_2 x)\bar{F}_{(j_2,j_1)}[H](\mathcal{S}_2 x)\,ds$$
$$= \bar{\lambda}_{i_1}\lambda_{j_1}\int_{\partial S} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds$$
$$= \bar{\lambda}_{i_1-j_1}\int_{\partial S} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds.$$

Since $\lambda_{i_1-j_1}\neq 1$, it follows that polynomials $F_{(i_2,i_1)}[H](x)$ and $F_{(j_2,j_1)}[H](x)$ are orthogonal on ∂S and therefore they are linearly independent. The case $i_2\neq j_2$ is considered in a similar way. The corollary is proved. □

Corollary 5.9. Matrix $E(\mathbf{a})=\left(\mathbf{a}_{(j_2,j_1)}\right)_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$, consisting of eigenvectors of the matrix $A_{(2)}(\mathbf{a})$, is orthogonal and symmetric.

Proof. Let $\mathbf{a}_{(j_2,j_1)}$ and $\mathbf{a}_{(i_2,i_1)}$ be two different columns of the matrix $E(\mathbf{a})$ from (5.2.12). Then, using the equality $\lambda_{j_2}\bar{\lambda}_{i_2}=\lambda_{j_2-i_2}$, we write

$$
\begin{aligned}
\mathbf{a}_{(j_2,j_1)}\cdot\bar{\mathbf{a}}_{(i_2,i_1)} &= \left(\lambda_{j_2}^{k_2}\lambda_{j_1}^{k_1}\right)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)}\cdot\left(\bar{\lambda}_{i_2}^{k_2}\bar{\lambda}_{i_1}^{k_1}\right)^T_{(k_2,k_1)=0,\dots,(l_2-1,l_1-1)}\\
&= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)}\lambda_{j_2}^{k_2}\lambda_{j_1}^{k_1}\bar{\lambda}_{i_2}^{k_2}\bar{\lambda}_{i_1}^{k_1} = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)}(\lambda_{j_2}\bar{\lambda}_{i_2})^{k_2}(\lambda_{j_1}\bar{\lambda}_{i_1})^{k_1}\\
&= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)}\lambda_{j_2-i_2}^{k_2}\lambda_{j_1-i_1}^{k_1} = \sum_{k_2=0}^{l_2-1}\lambda_{j_2-i_2}^{k_2}\sum_{k_1=0}^{l_1-1}\lambda_{j_1-i_1}^{k_1}.
\end{aligned}
$$

If $(j_2,j_1)\neq(i_2,i_1)$, then $(j_2-i_2,j_1-i_1)\neq 0$, which means that one of the equalities $j_2-i_2\neq 0$ or $j_1-i_1\neq 0$ is satisfied. Therefore, either $\lambda_{j_2-i_2}\neq 1$ or $\lambda_{j_1-i_1}\neq 1$, and therefore, similarly to (5.2.23) we get $\mathbf{a}_{(j_2,j_1)}\cdot\bar{\mathbf{a}}_{(i_2,i_1)}=0$.

The symmetry of the matrix $E(\mathbf{a})$ follows from the equalities

$$
\begin{aligned}
E^T(\mathbf{a}) &= \left(\lambda_{j_2}^{i_2}\lambda_{j_1}^{i_1}\right)^T_{\substack{(i_2,i_1)=0,\dots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}}\\
&= \left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)_{\substack{(i_2,i_1)=0,\dots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}} = \left(\lambda_{j_2}^{i_2}\lambda_{j_1}^{i_1}\right)_{\substack{(i_2,i_1)=0,\dots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}} = E(\mathbf{a}).
\end{aligned}
$$

The corollary is proved. □

Note that the matrices of eigenvectors in the case of multiple involution and for $l_1=\cdots=l_n=2$ have a similar property [177].

Now we examine the completeness of the eigenfunctions of the problem $\mathcal{S}_2$. Let $\mathcal{H}_m$ be the space of homogeneous harmonic polynomials of degree m. By Lemma 5.9 it can be divided into the sum of l_2l_1 orthogonal on ∂S subspaces $F_{(i_2,i_1)}[\mathcal{H}_m]$, where $(i_2,i_1)=0,\dots,(l_2-1,l_1-1)$, consisting of homogeneous harmonic polynomials of parity (i_2,i_1). Let $\mathcal{H}_m^{(i_2,i_1)}=\{H_m^{(i_2,i_1),k}(x):k=1,\dots,m_{(i_2,i_1)}\}$ be a complete system in $F_{(i_2,i_1)}[\mathcal{H}_m]$ of harmonic polynomials orthogonal on ∂S.

Theorem 5.15. *Let all numbers $\lambda_{(i_2,i_1)}$ defined in* (5.2.13) *not be equal to zero. Then the system of eigenfunctions of the Dirichlet problem* (5.2.1)-(5.2.2) *is complete in $L_2(S)$ and has the form*

$$
u_{\mu,m,(i_2,i_1),k}(x)=\frac{1}{|x|^{m+n/2-1}}J_{m+n/2-1}\left(\sqrt{\mu}|x|\right)H_m^{(i_2,i_1),k}(x), \tag{5.2.28}
$$

where $m\in\mathbb{N}_0$, $(i_2,i_1)=0,\dots,(l_2-1,l_1-1)$, $k=1,\dots,m_{(i_2,i_1)}$, $J_\nu(t)$ is the Bessel function of the first kind, $\sqrt{\mu}$ is the root of the Bessel function $J_{m+n/2-1}(t)$. The *eigenvalues of problem $\mathcal{S}_2$ are the numbers $\lambda_{\mu,(i_2,i_1)}$, defined in* (5.2.25)

$$\lambda_{\mu,(i_2,i_1)} = \mu \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \bar{\lambda}_{i_2}^{j_2} \bar{\lambda}_{i_1}^{j_1}.$$

Proof. According to Theorem 5.14, to find the eigenfunctions of the problem (5.2.1)-(5.2.2) it is necessary to find the function $w_\mu(x)$, which is the solution to the problem (5.2.15)-(5.2.16), and then write down the function $\hat{u}_{(i_2,i_1)}(x) = F_{(i_2,i_1)}[w_\mu](x)$. The maximum system of eigenfunctions of the problem (5.2.15)-(5.2.16) has the form (see Remark 1.4 on page 39)

$$w_{\mu,m,j}(x) = \frac{1}{|x|^{m+n/2-1}} J_{m+n/2-1}\left(\sqrt{\mu}|x|\right) H_m^j(x), \tag{5.2.29}$$

where $\left\{H_m^j(x) : j = 1, \ldots, h_m\right\}$, $h_m = \frac{2m+n-2}{n-2}\binom{m+n-3}{n-3}$ $(n > 2)$ is the maximal system of linearly independent homogeneous harmonic polynomials of degree m from Definition 1.5, $\sqrt{\mu}$ is the root of the Bessel function $J_{m+n/2-1}(t)$. Then, since $|\mathcal{S}_2\mathcal{S}_1 x| = |x|$, then

$$\begin{aligned}
\hat{u}_{(i_2,i_1)}(x) &= F_{(i_2,i_1)}[w_{\mu,m,j}](x) = \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} w_{\mu,m,j}\left(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1} x\right) \\
&= \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} \frac{1}{|\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1} x|^{m+n/2-1}} J_{m+n/2-1}\left(\sqrt{\mu}|\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1} x|\right) \\
&\quad \times H_m^j\left(\mathcal{S}_2^{j_2}\mathcal{S}_1^{j_1} x\right) = \frac{1}{|x|^{m+n/2-1}} J_{m+n/2-1}\left(\sqrt{\mu}|x|\right) F_{(i_2,i_1)}[H_m^j](x).
\end{aligned} \tag{5.2.30}$$

Since $F_{(i_2,i_1)}[H_m^j] \subset F_{(i_2,i_1)}[\mathcal{H}_m]$, then we choose in the subspace of polynomials $F_{(i_2,i_1)}[\mathcal{H}_m]$ a complete system of polynomials $\mathcal{H}_m^{(i_2,i_1)} = \{H_m^{(i_2,i_1),k}(x) : k = 1, \ldots, m_{(i_2,i_1)}\}$ orthogonal on ∂S. Note that for some values of m it is possible $m_{(i_2,i_1)} = 0$, that is, for such m the component $H_m^{(i_2,i_1)}(x)$ is absent (see Example 5.11) and therefore $\hat{u}_{(i_2,i_1)} = 0$. By choosing in the resulting expression for $\hat{u}_{(i_2,i_1)}(x)$ instead of $F_{(i_2,i_1)}[H_m^j](x)$ the harmonic polynomials $H_m^{(i_2,i_1),k}(x)$ and adding to $\hat{u}_{(i_2,i_1)}$ indices indicating the dependence of this function $\hat{u}_{(i_2,i_1)}(x)$ on k, m and μ, we get (5.2.28).

Since in the equality (5.2.28) $H_m^{(i_2,i_1),k}(x)$ is a homogeneous harmonic polynomial of degree m, then the function $u_{\mu,m,(i_2,i_1),k}(x)$ has the form (5.2.29) and, therefore, is an eigenfunction of the problem (5.2.15)-(5.2.16). The reverse is also true. Each homogeneous harmonic polynomial $H_m^j(x)$ according to the formula (5.2.21) can be represented as a linear combination of harmonic polynomials of the form $H_m^{(i_2,i_1)}(x)$, and those are linear combinations of polynomials $H_m^{(i_2,i_1),k}(x)$ and, therefore, any function from (5.2.29) is a linear combination of functions of the form (5.2.28). The eigenvalues of the problem (5.2.1)-(5.2.2), in accordance with Theorem (5.14), are found from the equality (5.2.25).

Let us study the orthogonality of the eigenfunctions $u_{\mu,m,(i_2,i_1),k}(x)$. It is easy to see that the equality

$$\int_S u_{\mu_1,m_1,(i_2,i_1),k_1}(x)\bar{u}_{\mu_2,m_2,(j_2,j_1),k_2}(x)\,dx = \int_0^1 \rho J_{m_1+n/2-1}(\sqrt{\mu_1}\rho) \\ \times J_{m_2+n/2-1}(\sqrt{\mu_2}\rho)\,d\rho \cdot \int_{\partial S} H_{m_1}^{(i_2,i_1),k_1}(\xi)\bar{H}_{m_2}^{(j_2,j_1),k_2}(\xi)\,ds_\xi = 0$$

holds true. Let us consider the right side of the resulting equality. For $\mu_1 \neq \mu_2$ and $m_1 = m_2$, due to the properties of the Bessel functions (orthogonality in $L_2((0,1);t)$), the first factor is equal to zero. If $m_1 \neq m_2$, then by the property of harmonic polynomials the second factor on the right is equal to zero. If $\mu_1 = \mu_2$, $m_1 = m_2$, then for $(i_2,i_1) \neq (j_2,j_1)$ the second factor on the right is equal to zero by Corollary 5.8. Finally, if $(i_2,i_1) = (j_2,j_1)$ and $k_1 \neq k_2$, then again the second factor is equal to zero by the construction of the polynomials $H_m^{(i_2,i_1),k}(x)$.

The constructed system of eigenfunctions (5.2.28) is complete in $L_2(S) = L_2((0,1)\times \partial S)$ by Lemma 2 from [183, p.33], since the system of functions

$$\left\{J_{m+n/2-1}(\sqrt{\mu_k}\rho), k : J_{m+n/2-1}(\sqrt{\mu_k}) = 0\right\}$$

is orthogonal and complete in $L_2((0,1);t)$ for each m, and the system of polynomials $\{H_m^{(i_2,i_1),k)}(\xi)\}$ is orthogonal and complete in $L_2(\partial S)$ for different m, (i_2,i_1) and k. The theorem is proved. □

Example 5.12. 1^0. Let $n = 2$, $l_2 = l_1 = 2$, $\mathcal{S}_1 x = (-x_1,x_2)$, $\mathcal{S}_2 x = (x_1,-x_2)$, then problem $\mathcal{S}_2$ has the form

$$\begin{gathered} a_0\Delta u(x_1,x_2) + a_1\Delta u(-x_1,x_2) + a_2\Delta u(x_1,-x_2) \\ +a_3\Delta u(-x_1,-x_2) + \lambda u(x) = 0,\ x\in S; \quad u(x) = 0,\ x\in\partial S. \end{gathered} \tag{5.2.31}$$

Let us find the eigenfunctions of problem (5.2.31) using the example 5.11. The eigenfunctions of the problem (5.2.15)-(5.2.16) in the polar coordinate system are determined according to the equality (5.2.29) in the form

$$w_{\mu,m,0}(x) = J_m(\sqrt{\mu}r)\cos m\varphi, \quad w_{\mu,m,1}(x) = J_m(\sqrt{\mu}r)\sin m\varphi, \quad m \in \mathbb{N}_0,$$

where $\sqrt{\mu}$ is the positive root of the Bessel function $J_m(t)$

$$J_m(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+m+1)\Gamma(j+1)}\Big(\frac{t}{2}\Big)^{2j+m}.$$

Using (5.2.30), we find

$$u_{\mu,m,(i_2,i_1)}(x) = J_m(\sqrt{\mu}|x|)F_{(i_2,i_1)}[H_m](x/|x|).$$

The written equality does not indicate the dependence of the eigenfunction $u_{\mu,m,(i_2,i_1)}(x)$ on the index k, since in accordance with Example 5.11 the dimension of the space $F_{(i_2,i_1)}[\mathcal{H}_m]$ is equal to 1. According to (5.2.24) and taking into account (5.2.25) we write

$$\begin{aligned}
u_{\mu,2m,(0,0)}(x) &= J_{2m}(\sqrt{\mu}r)\cos(2m\varphi),\\
u_{\mu,2m+1,(0,1)}(x) &= J_{2m+1}(\sqrt{\mu}r)\cos((2m+1)\varphi),\\
u_{\mu,2m+1,(1,0)}(x) &= J_{2m+1}(\sqrt{\mu}r)\sin((2m+1)\varphi),\\
u_{\mu,2m,(1,1)}(x) &= J_{2m}(\sqrt{\mu}r)\sin(2m\varphi),
\end{aligned}$$

and also $\lambda_{\mu,(0,0)} = \mu(a_0+a_1+a_2+a_3)$, $\lambda_{\mu,(0,1)} = \mu(a_0-a_1+a_2-a_3)$, $\lambda_{\mu,(1,0)} = \mu(a_0+a_1-a_2-a_3)$, $\lambda_{\mu,(1,1)} = \mu(a_0-a_1-a_2+a_3)$, where $m \in \mathbb{N}_0$, $\sqrt{\mu}$ is the root of the Bessel function $J_m(t)$ and $\lambda_{\mu,(i_2,i_1)} \neq 0$. It is clear that the resulting system of eigenfunctions is complete in $L_2(S)$.

2^0. Let $n = 3$ and $\mathcal{S}_1 x = (-x_1, x_2, x_3)$, $\mathcal{S}_2 x = (x_1, -x_2, x_3)$. By Theorem 1.5 on page 44 the maximal system of homogeneous harmonic polynomials of degree m from (5.2.29) in 3 variables has the form $\{H_m^{j,0}(x), H_m^{j,1}(x) : j = 0,\dots,m\}$, where

$$\begin{aligned}
H_m^{j,0}(x_1,x_2,x_3) &= G_{m-j}^j(x)H_j^0(x_1,x_2),\\
H_m^{j,1}(x_1,x_2,x_3) &= G_{m-j}^j(x)H_j^1(x_1,x_2),
\end{aligned} \tag{5.2.32}$$

and $H_j^0(x_1,x_2) = \mathrm{Re}(x_1+ix_2)^j$, $H_j^1(x_1,x_2) = \mathrm{Im}(x_1+ix_2)^j$, and $G_m^j(x)$ is the G-polynomial from Definition 1.4

$$G_m^j(x) \equiv G_m^j(x_1,x_2,x_3) = \sum_{k=0}^{[m/2]}(-1)^k \frac{(x_1^2+x_2^2)^k x_3^{m-2k}}{(2k)!!(2j+2k)!!(m-2k)!}.$$

The system (5.2.32) has $2m+1$ elements for each m, since $H_0^{j,1}(x)=0$. In the spherical coordinate system (r,φ,θ) these polynomials can be written more compactly

$$H_m^{j,0}(x_1,x_2,x_3)=r^m G_{m-j}^j(r,\varphi,\theta)\cos j\varphi,$$
$$H_m^{j,1}(x_1,x_2,x_3)=r^m G_{m-j}^j(r,\varphi,\theta)\sin j\varphi,$$

where

$$G_m^j(r,\varphi,\theta)=\sum_{k=0}^{[m/2]}(-1)^k\frac{\sin^{2k+j}\theta\cos^{m-2k}\theta}{(2k)!!(2j+2k)!!(m-2k)!},$$

since $\cos\theta=x_3/|x|$, $\sin\theta=\sqrt{x_1^2+x_2^2}/|x|$ and $\cos\varphi=x_1/\sqrt{x_1^2+x_2^2}$. In this case, the dimension of the space $F_{(i_2,i_1)}[\mathcal{H}_m]$ is greater than 1. It is easy to see that

$$F_{(i_2,i_1)}[H_m^{k,0}(x_1,x_2,x_3)]=G_{m-k}^k(x)F_{(i_2,i_1)}[H_k^0(x_1,x_2)],$$

and the values $F_{(i_2,i_1)}[H_k^0(x_1,x_2)]$ and $F_{(i_2,i_1)}[H_k^1(x_1,x_2)]$ are already calculated in (5.2.24) from Example 5.11. Since

$$F_{(0,0)}[\alpha H_{2k}^0+\beta H_{2k}^1]=\alpha H_{2k}^0,\ F_{(0,0)}[\alpha H_{2k+1}^0+\beta H_{2k+1}^1]=0,$$

then in the subspace $F_{(0,0)}[\mathcal{H}_m]$ we can choose the following maximal system of polynomials $\{H_m^{2k,0}(r,\varphi,\theta):k=0,\ldots,[m/2]\}$, which follows

$$u_{\mu,m,(0,0),k}(x)=r^{-1/2}J_{m+1/2}(\sqrt{\mu}r)G_{m-2k}^{2k}(r,\varphi,\theta)\cos(2k\varphi),$$

for $k=0,\ldots,[m/2]$. Similarly we find

$$F_{(0,1)}[\alpha H_{2k}^0+\beta H_{2k}^1]=0,\ F_{(0,1)}[\alpha H_{2k+1}^0+\beta H_{2k+1}^1]=\alpha H_{2k+1}^0,$$

whence

$$u_{\mu,m,(0,1),k}(x)=r^{-1/2}J_{m+1/2}(\sqrt{\mu}r)G_{m-2k-1}^{2k+1}(r,\varphi,\theta)\cos((2k+1)\varphi),$$

for $k=0,\ldots,[(m-1)/2]$. The remaining eigenfunctions of problem (5.2.31) for $n=3$ have the form

$$u_{\mu,m,(1,0),k}(x)=r^{-1/2}J_{m+1/2}(\sqrt{\mu}r)G_{m-2k-1}^{2k+1}(r,\varphi,\theta)\sin((2k+1)\varphi),$$
$$u_{\mu,m,(1,1),k}(x)=r^{-1/2}J_{m+1/2}(\sqrt{\mu}r)G_{m-2k}^{2k}(r,\varphi,\theta)\sin(2k\varphi).$$

5.2.5 Conclusion

The results obtained allow one to find explicitly, using the equality (5.2.28), the eigenfunctions and eigenvalues of the boundary value problem (5.2.1)-(5.2.2) for the nonlocal differential equation with double involution. The completeness of the system of eigenfunctions make it possible to use the Fourier method to construct solutions of initial-boundary value problems for nonlocal parabolic and hyperbolic equations.

Possible applications of the obtained results can be found in the modeling of optical systems, since differential equations with involution are an important part of the general theory of functional differential equations, which has numerous applications in optics. Applications of equations with involution in modeling optical systems are given, for example, in [130, 151]. In particular, in [130], mathematical models important for applications in nonlinear optics are considered in the form of nonlinear functional-differential equations of a parabolic type with feedback and transformation of spatial variables, which is specified by the involution operator. The following parabolic functional differential equation is considered

$$\frac{\partial u}{\partial t} + u = \mu\Delta u + K\left(1 + \gamma\cos Qu\right), \; r \in [r_1, r_2], \; t \geq 0, \; \mu > 0,$$

which describes the dynamics of phase modulation of a light wave in an optical system with an involution operator Q such that $Q^l = I$.

Bibliography

[1] Ahmad A., Ali M., Malik S.A. Inverse problems for diffusion equation with fractional Dzherbashian-Nersesian operator. Fractional Calculus and Applied Analysis. 2021. 24, 1899–1918.

[2] Ahmad B., Alsaedi A., Kirane M., Tapdigoglu R.G. An inverse problem for space and time fractional evolution equations with an involution perturbation. Quaestiones Mathematicae. 2017. 40, 151–160.

[3] Akel M., Begehr H. Neumann function for a hyperbolic strip and a class of related plane domains. Mathematische Nachrichten. 2017. 290:4, 490–506.

[4] Alimov Sh.A. On a problem with an oblique derivative. Differ. Uravn. 1981. 17:10, 1738–1751.

[5] Almansi E. Sull' integrazione dell' equazione differenziale $\Delta^n u = 0$. Ann. Mat. Pura. Appl. Ser. 3. 1899. 11, 5–19.

[6] Al-Salti N., Kerbal S., Kirane M. Initial-boundary value problems for a time-fractional differential equation with involution perturbation. Mathematical Modelling of Natural Phenomena. 2019. 14, 315.

[7] Ali A.H.A., Raslan K.R. Variational iteration method for solving biharmonic equations. Physics Letters A. 2007. 370:5/6, 441–448.

[8] Andersson L-E., Elfving T., Golub G.H. Solution of biharmonic equations with application to radar imaging. J. Comput. Appl. Math. 1998. 94:2, 153–180.

[9] Andreev A.A. Analogs of Classical Boundary Value Problems for a Second-Order Differential Equation with Deviating Argument. Differ. Equ. 2004. 40, 1192–1194.

[10] Ashyralyev A., Ibrahim S., Hincal E. On stability of the third order partial delay differential equation with involution and Dirichlet condition. Bulletin of the Karaganda University. Mathematics series. 2021. 2, 25–34.

[11] Ashyralyev A., Al-Hazaimeh H. Stability of the time-dependent identification problem for the telegraph equation with involution. International Journal of Applied Mathematics. 2022. 35, 447–459.

[12] Ashyralyev A., Sarsenbi A. Well-Posedness of a Parabolic Equation with Involution. Numer. Funct. Anal. Optim. 2017. 38, 1295–1304.

[13] Ashyralyev A., Sarsenbi A.M. Well-posedness of an elliptic equation with involution. Electron. J. Differ. Equ. 2015. 2015, 1–8.

[14] Babbage C.H. Essays towards the Calculus of the Functions; W. Bulmer and Co.: London, UK, 1815.

[15] Bateman H., Erdélyi A. Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953.

[16] Baskakov A.G., Krishtal I.A., Romanova E.Y. Spectral analysis of a differential operator with an involution. Journal of Evolution Equations. 2017. 17, 669–684.

[17] Baskakov A.G., Krishtal I.A., Uskova N.B. Similarity techniques in the spectral analysis of perturbed operator matrices. Journal of Mathematical Analysis and Applications. 2019. 477, 669–684.

[18] Baskakov A.G., Krishtal I.A., Uskova N.B. On the spectral analysis of a differential operator with an involution and general boundary conditions. Eurasian Mathematical Journal. 2020. 11, 30–39.

[19] Bavrin I.I. Operators for harmonic functions and their applications. Differ. Uravn. 1985. 21:1, 9–15.

[20] Begehr H. Biharmonic Green functions. Matematiche. 2006. 61, 395–405.

[21] Begehr H., Du J., Wang Y. A Dirichlet problem for polyharmonic functions. Annali di Matematica Pura ed Applicata. 2008. 187:3, 435–457.

[22] Begehr H., Vaitekhovich T. Modified harmonic Robin function. Complex Variables and Elliptic Equations. 2013. 58:4, P. 483–496.

[23] Begehr H., Burgumbayeva S., Shupeyeva B. Remark on Robin problem for Poisson equation. Complex Variables and Elliptic Equations. 2017. 62:10, 1589–1599.

[24] Begehr H., Burgumbayeva S., Dauletkulova A., Lin H. Harmonic Green functions for the Almaty apple. Complex Variables and Elliptic Equations. 2020. 65:11, 1814–1825.

[25] Begerh H., Vu T.N.H., Zhang Z.-X. Polyharmonic Dirichlet Problems. Proceedings of the Steklov Institute of Math. 2006. 255, 13–34.

[26] Bekaeva A., Karachik V.V., Turmetov B. Kh. Solvability of some boundary-value problems for polyharmonic equation with Hadamard-Marchaud boundary operator. Russian Mathematics, 2014. 58:7, 11–24

[27] Berdyshev A.S., Cabada A., Turmetov B.Kh. On solvability of a boundary value problem for a nonhomogeneous biharmonic equation with a boundary operator of a fractional order. Acta Mathematica Scientia. 2014. 34:6, Ser. B, 1695–1706.

[28] Bitsadze A.V. Uravneniya Matematicheskoi Fiziki (Equations of Mathematical Physics), 5th ed.; Nauka: Moscow, Russia, 1982.

[29] Bitsadze A.V. Some properties of polyharmonic functions. 1988. Differ. Equations. 24:5, 543–548.

[30] Bitsadze A.V. On the Neumann problem for harmonic functions. Dokl. Akad. Nauk SSSR 1990. 311(1), 11–13.

[31] Bjorstad P. Fast numerical solution of the biharmonic Dirichlet problem on rectangles. SIAM J. Numer. Anal. 1983. 20, 59–71.

[32] Boggio T. Sulle funzioni di Green d'ordine m. Palermo Rend. 1905. 20, 97–135.

[33] Bondarenko B.A. Polyharmonic polynomials. FAN: Tashkent, 1968.

[34] Bondarenko B.A. Operator algorithms in differential equations. FAN: Tashkent, 1984.

[35] Bondarenko B.A. Generalized Pascal Triangles and Pyramids:Their Fractals, Graphs and Applications. Santa Clara, Calif: The Fibonacci Association, 1993.

[36] Buoso D., Provenzano L. A few shape optimization results for a biharmonic Steklov problem. Journal of Differential Equations. 2015. 259, 1778–1818.

[37] Burlutskaya, M.S.H., Khromov A.P. Fourier method in an initial-boundary value problem for a first-order partial differential equation with involution. Comput. Math. Math. Phys. 2011. 51, 2102–2114.

[38] Burlutskaya M.Sh. Some properties of functional-differential operators with involution $v(x) = 1 - x$ and their applications. Russian Math. 2021. 65, 69–76.

[39] Cabada A., Tojo F.A.F. Differential Equations with Involutions, 1rd ed.; Atlantis Press: Paris, France, 2015; ISBN 978-94-6239-120-8.

[40] Cabada A., Tojo F.A.F. On linear differential equations and systems with reflection. Appl. Math. Comp. 2017. 305, 84–102.

[41] Constantin E., Pavel N.H. Green function of the Laplacian for the Neumann problem in $\mathbb{R}^n_+$. Libertas Math. 2010. XXX, 57–69.

[42] Dalmasso R. On the mean-value property of polyharmonic functions. Studia Sci. Math. Hungar. 2010. 47:1, 113–117.

[43] Dang Q.A. Iterative method for solving the Neumann boundary problem for biharmonic type equation. J. Comput. Appl. Math. 2006. 196:2, 634–643.

[44] Dang Quang A., Mai Xuan Thao. Iterative method for solving a problem with mixed boundary conditions for biharmonic equation arising in fracture mechanics. Boletim da Sociedade Paranaense de Matematica. 2013. 31:1, 65–78.

[45] Ehrlich L.N., Gupta M.M. Some difference schemes for the biharmonic equation. SIAM J. Numer. Anal. 1975. 12:5, 773–790.

[46] Garkavenko G.V., Uskova N.B. Decomposition of linear operators and asymptotic behavior of eigenvalues of differential operators with growing potencial. Journal of Mathematical Sciences. 2020. 246, 812–827.

[47] Gazzola F., Grunau H.C., Sweers G. Polyharmonic Boundary Value Problems. Springer, 2010.

[48] Gazzola F., Sweers G. On positivity for the biharmonic operator under Steklov boundary conditions. Arch. Rat. Mech. Anal. 2008. 188, 399–427.

[49] Gomez-Polanco A., Guevara-Jordan J. M., Molina B. A mimetic iterative scheme for solving biharmonic equations. Mathematical and Computer Modelling. 2013. 57:9-10, 2132–2139.

[50] Granilshchikova Ya.A., Shkalikov A.A. Spectral properties of a differential operator with involution. Vestnik Moskovskogo Universiteta.Seriya 1. Matematika.Mekhanika. 2022. 4, 67–71 .

[51] Grebenkov D.S., Traytak S.D. Semi-analytical computation of Laplacian Green functions in three-dimensional domains with disconnected spherical boundaries. Journal of Computational Physics. 2019. 379, 91–117.

[52] Grunau H.C., Sweers G. Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions. Math. Ann. 1997. 307:4, 589–626.

[53] Hsu C.-W., Hwu C. Green's functions for unsymmetric composite laminates with inclusions. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 2020. 476:2233, 20190437.

[54] Jill Pipher and Gregory Verchota. Area integral estimates for the biharmonic operator in Lipschitz domains. Trans. Amer. Math. Soc. 1991. 327:2, 903–917.

[55] Joseph D. D., Sturgers L. The convergence of biorthogonal series for biharmonic and Stokes flow edge problems: Part II. SIAM J. Appl. Math. 1978. 34, 7–26.

[56] Kal'menov T.Sh., Koshanov B.D., Nemchenko M.Yu. Green function representation in the Dirichlet problem for polyharmonic equations in a ball. Dokl. Math. 2008. 421:3, 528–530.

[57] Kal'menov T.Sh. and Suragan D. On a new method for constructing the Green function of the Dirichlet problem for the polyharmonic equation. Differ. Equations. 2012. 48:3, 441–445.

[58] Kanguzhin B.E., Koshanov B.D. Necessary and sufficient conditions for the solvability of boundary value problems for nonhomogeneous polyharmonic equation in a ball. Ufim. Mat. Zh.2010. 2:2, 41–52.

[59] Karachik V.V. A certain class of problems for the polyharmonic equation in a ball. Dokl. Akad. Nauk UzSSR. 1989. 9, 7–8.

[60] Karachik V.V. A problem for the polyharmonic equation in the sphere. Siberian Mathematical Journal. 1991. 32:5, 767–774.

[61] Karachik V.V. Solvability of a boundary value problem for the Helmholtz equation with higher-order normal derivatives on the boundary. Differ. Uravn. 1992. 28:5, 907–909.

[62] Karachik V.V. A problem with higher-order normal derivatives on the boundary for the Poisson equation. Differential Equations. 1996. 32:3, 421–424.

[63] Karachik V.V. P-Latin matrices and Pascal's triangle modulo a prime. The Fibonacci Quarterly. 1996. 34:4, 362–372.

[64] Karachik V.V. Harmonic polynomials and the divisibility problem. Proceedings of AMS. 1997. 125:11, 3257–3258.

[65] Karachik V.V. On one set of orthogonal harmonic polynomials. Proceedings of AMS. 1998. 126:12, 3513–3519.

[66] Karachik V.V. A generalized Neumann problem for harmonic functions in the half-space. Differential Equations. 1999. 35:7, 949–955.

[67] Karachik V.V. Polynomial solutions to the systems of partial differential equations with constant coefficients. Yokohama Mathematical Journal. 2000. 47(2), 121–142.

[68] Karachik V.V. Normalized system of functions with respect to the Laplace operator and its applications. Journal of Mathematical Analysis and Applications. 2003. 287:2, 577–592.

[69] Karachik V.V. On some special polynomials. Proceedings of AMS. 2004. 132:4, 1049–1058.

[70] Karachik V.V. On One Representation of Analytic Functions by Harmonic Functions, Siberian Adv. Math. 2008. 18:2, 103–117.

[71] Karachik V.V. On an Expansion of Almansi Type. Math. Notes. 2008. 83:3, 335–344.

[72] Karachik V.V., Antropova N.A. On the solution of a nonhomogeneous polyharmonic equation and the nonhomogeneous Helmholtz equation. Differential Equations. 2010. 46:3, 387–399.

[73] Karachik V.V. Construction of polynomial solutions to some boundary value problems for Poisson's equation. Computational Mathematics and Mathematical Physics. 2011. 51:9, 1567–1587.

[74] Karachik V.V., Antropova N.A. On polynomial solutions of the Dirichlet problem for the biharmonic equation in a ball. Sib. Zh. Ind. Mat. 2012. 15:2, 86–98.

[75] Karachik V.V. Method for constructing solutions of linear ordinary differential equations with constant coefficients. Comput. Math. Math. Phys. 2012. 52:2, 219–234.

[76] Karachik V.V. Application of the Almansi formula for constructing polynomial solutions to the Dirichlet problem for a second-order equation. Russian Math. 2012. 56:6, 20–29.

[77] Karachik V.V. Polynomial solutions to Dirichlet boundary value problem for the 3-harmonic equation in a ball. J. Sib. Fed. Univ.. Math. Phys. 2012. 5:4, 527–546.

[78] Karachik V.V., Turmetov B.Kh., Torebek B.T. On some integro-differential operators in the class of harmonic functions and their applications. Siberian Adv. Math. 2012. 22:2, 115–134.

[79] Karachik V.V. On some special polynomials and functions. Sib. Elektron. Mat. Izv. 2013. 10, 205–226.

[80] Karachik V.V., Antropova N. A. Polynomial solutions of the Dirichlet problem for the biharmonic equation in the ball. Differential Equations. 2013. 49:2, 251–256.

[81] Karachik V.V. On the mean-value property for polyharmonic functions. Bulletin of the South Ural State University. Ser. Mathematical Modelling & Programming. 2013. 6:3, 59–66.

[82] Karachik V.V. Rows of Pascal's triangle modulo a prime. Diskretn. Anal. Issled. Oper. 2013. 20:2, 26–46.

[83] Karachik V.V., Antropova N.A. Construction of Polynomial Solutions to the Dirichlet Problem by Almansi Representation. Journal of Mathematical Sciences, Springer: New York, January 2013. 188:3, 256–267.

[84] Karachik V.V. On solvability conditions for the Neumann problem for a polyharmonic equation in the unit ball. Journal of Applied and Industrial Mathematics. 2014. 8:1, 63–75.

[85] Karachik V.V. On the mean value property for polyharmonic functions in the ball. Siberian Advances in Mathematics. 2014. 24:3, 169–182.

[86] Karachik V.V. Construction of Polynomial Solutions to the Dirichlet Problem for the Polyharmonic Equation in a Ball. Computational Mathematics and Mathematical Physics. 2014. 54:7, 1122–1143.

[87] Karachik V.V. On the Arithmetic Triangle Arising from the Solvability Conditions for the Neumann Problem. Math. Notes. 2014. 96:2, 217–227.

[88] Karachik V.V. Solvability Conditions for the Neumann Problem for the Homogeneous Polyharmonic Equation. Differential Equations. 2014. 50:11, 1449–1456.

[89] Karachik V.V. Method of normalized systems of functions. Chelyabinsk: Izd. Tsentr Yuzhno-Ural'skii Gosudarstvennyi Universitet. 2014. 452p. (ISBN 978-5-696-04562-7/hbk).

[90] Karachik V.V., Yuan HongFen Dunkl-Poisson Equation and Related Equations in Superspace, Mathematical Modelling and Analysis. 2015. 20:6, 768–781.

[91] Karachik V.V. Solution of the Dirichlet Problem with Polynomial Data for the Polyharmonic Equation in a Ball. Differential Equations. 2015. 51:8, 1033–1042.

[92] Karachik V.V., Sadybekov M.A., Torebek B.T. Uniqueness of solutions to boundary-value problems for the biharmonic equation in a ball. Electronic Journal of Differential Equations. 2015. 244, 1–9.

[93] Karachik V.V., Torebek T. On one mathematical model described by boundary value problem for the biharmonic equation. Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software (Bulletin SUSU MMCS). 2016. 9:4, 40–52.

[94] Karachik V.V., Turmetov B.Kh. Solvability of some Neumann-type boundary value problems for biharmonic equations. Electronic Journal of Differential Equations. 2017. 2017:218, 1–17.

[95] Karachik V.V. A Neumann-type problem for the biharmonic equation. Siberian Adv. Math. 2017. 27:2, 103–118.

[96] Karachik V.V. Generalized Third Boundary Value Problem for the Biharmonic Equation. Differential Equations. 2017. 53:6, 756–765.

[97] Karachik V.V., Torebek B.T. On the Dirichlet-Riquier Problem for Biharmonic Equations. Mathematical Notes. 2017. 102:1, 31–42.

[98] Karachik V.V. Neumann type problem for polyharmonic equation in a ball. Chelyab. Fiz.-Mat. Zh. 2017. 2:4, 420–429.

[99] Karachik V.V. Integral identities on a sphere for normal derivatives of polyharmonic functions, Siberian electronic mathematical reports. 2017. 14, 533–551.

[100] Karachik V.V., Turmetov B.Kh. On solvability of some boundary value problems for a biharmonic equation with periodic conditions. Filomat. 2018. 32:3, 947–953.

[101] Karachik V.V. Riquier-Neumann Problem for the Polyharmonic Equation in a Ball. Differential Equations. 2018. 54:5, 648–657.

[102] Karachik V.V. Solving a Problem of Robin Type for Biharmonic Equation. Russian Mathematics. 2018. 62:2, 34–48.

[103] Karachik V.V., Turmetov B.Kh. On solvability of some nonlocal boundary value problems for polyharmonic equation. Kazakh Mathematical Journal. 2019. 19:1, 39–49.

[104] Karachik V.V., Turmetov B.Kh. On the Green's Function for the Third Boundary Value Problem. Siberian Advances in Mathematics. 2019. 29:1, 32–43.

[105] Karachik V.V. The Green Function of the Dirichlet Problem for the Biharmonic Equation in a Ball. Computational Mathematics and Mathematical Physics. 2019. 59:1, 66–81.

[106] Karachik V.V. Greens function of Dirichlet problem for biharmonic equation in the ball. Complex Variables and Elliptic Equations. 2019. 64:9, 1500–1521.

[107] Karachik V.V., Turmetov B.Kh. On Green's function of the Robin problem for the Poisson equation. Advances in Pure and Applied Mathematics. 2019. 10:3, 203–214.

[108] Karachik, V.V., Sarsenbi, A.M., Turmetov, B.K. On the solvability of the main boundary value problems for a nonlocal Poisson equation. Turkish Journal of Mathematics. 2019. 43, 1604–1625.

[109] Karachik V., Turmetov B. On solvability of some nonlocal boundary value problems for biharmonic equation. Mathematica Slovaca. 2020. 70:2, 329–342.

[110] Karachik V.V. The Green Function of the Dirichlet Problem for the Triharmonic Equation in the Ball. Mathematical Notes. 2020. 107:1, 105–120.

[111] Karachik V.V. Solvability conditions of the $\mathcal{N}_2$ Neumann-type problem for polyharmonic equation in a ball. Bulletin of the South Ural State University Series “Mathematics. Mechanics. Physics”. 2020. 12:2, 13–20.

[112] Karachik V.V. Presentation of solution of the Dirichlet problem for biharmonic equation in the unit ball through the Green function. Chelyabinsk Physical and Mathematical Journal. 2020. 5:4-1, 391–399.

[113] Karachik V.V. Class of Neumann-Type Problems for the Polyharmonic Equation in a Ball. Computational Mathematics and Mathematical Physics. 2020. 60:1, 144–162.

[114] Karachik V. Neumann type problems for the polyharmonic equation in ball. Journal of Mathematical Sciences. 2020. 249:6, 974–988.

[115] Karachik V., Turmetov B. Solvability of one nonlocal Dirichlet problem for the Poisson equation. Novi Sad Journal of Mathematics. 2020, 50:1, 67–88.

[116] Karachik V.V. Dirichlet and Neumann boundary value problems for the polyharmonic equation in the unit ball. Mathematics. 2021. 9:16, 1907.

[117] Karachik V.V. Green’s Functions of the Navier and Riquier-Neumann Problems for the Biharmonic Equation in the Ball. Differential Equations. 2021. 57:5, 654–668.

[118] Karachik V.V. Sufficient Conditions for Solvability of One Class of Neumann-Type Problems for the Polyharmonic Equation. Computational Mathematics and Mathematical Physics. 2021. 61:8, 1276–1288.

[119] Karachik V. Green’s functions of some boundary value problems for the biharmonic equation. Complex Variables and Elliptic Equations. 2022. 67:7, 1712–1736.

[120] Karachik V.V. Solution to the Dirichlet Problem for the Polyharmonic Equation in the Ball. Siberian Advances in Mathematics. 2022. 32:3, 197–210.

[121] Karachik V., Turmetov B., Yuan H. Four Boundary Value Problems for a Nonlocal Biharmonic Equation in the Unit Ball. Mathematics. 2022. 10:7, 1158.

[122] Karachik V.V. The Green function of the Navier problem for the polyharmonic equation in a ball. Journal of Mathematical Sciences. 2023. 269:2, 189–204.

[123] Karachik V.V. On Green function of the Dirichlet problem for polyharmonic equation in the ball. Axioms. 2023. 12:6, 543.

[124] Karachik V.V. Riquier-Neumann problem for the polyharmonic equation in a ball. Mathematics. 2023. 11:4, 1000.

[125] Karachik V.V. On one integral representation of solutions of polyharmonic equation. Lobachevskii Journal of Mathematics. 2023. 44:7, 2749–2756.

[126] Karachik V.V. A solution to the Riquier-Neymann problem for polyharmonic equations in a ball. Bulletin of the South Ural State University, Ser. Mathematics. Mechanics. Physics. 2023. 15:1, 26–33.

[127] Karachik V.V. Representation of the Green’s Function of the Dirichlet Problem for the Polyharmonic Equation in the Ball. Differential Equations. 2023. 59:8, 1061–1074.

[128] Kirane M., Al-Salti N. Inverse problems for a nonlocal wave equation with an involution perturbation. J. Nonlinear Sci. Appl. 2016. 9, 1243–1251.

[129] Kirane M. Sadybekov M.A. Sarsenbi A.A. On an inverse problem of reconstructing a subdiffusion process from nonlocal data. Mathematical Methods in the Applied Sciences. 2019. 42, 2043–2052.

[130] Kornuta A.A., Lukianenko V.A. Dynamics of solutions of nonlinear functional differential equation of parabolic type. Izvestiya Vysshikh Uchebnykh Zavedeniy. Prikladnaya Nelineynaya Dinamika. 2022. 30:2, 132–151.

[131] Koshanov B.D. About solvability of boundary value problems for the nonhomogeneous polyharmonic equation in a ball. Advances in Pure and Applied Mathematics. 2013. 4:4, 351–373.

[132] Koshanov B.D., Soldatov A.P. Boundary value problem with normal derivatives for a higher order elliptic eguation on the plane. Differential Equations. 2016. 52:12, 1594–1609.

[133] Koshlyakov N.S., Gliner E.B., Smirnov M.M. Differential Equations of Mathematical Physics; North-Holland: Amsterdam, The Netherlands, 1964.

[134] Kritskov L.V., Sarsenbi A.M. Spectral properties of a nonlocal problem for a second-order differential equation with an involution. Differ. Equ. 2015. 51, 984–990.

[135] Kritskov L.V., Sarsenbi A.M. Basicity in L_p of root functions for differential equations with involution. Electron. J. Differ. Equ. 2015. 2015, 1–9.

[136] Kritskov L.V., Sadybekov M.A., Sarsenbi A.M. Properties in L_p of root functions for a nonlocal problem with involution. Turkish Journal of Mathematics. 2019. 43, 393–401.

[137] Kritskov L.V., Sarsenbi A.M. Riesz basis property of system of root functions of second-order differential operator with involution. Differ. Equ. 2017. 53, 33–46.

[138] Lai M-C., Liu H-C. Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows. Appl. Math. Comput. 2005. 164:3, 679–695.

[139] Lin H. Harmonic Green and Neumann functions for domains bounded by two intersecting circular arcs. Complex Variables and Elliptic Equations. 2020. 67, 79–95.

[140] Linkov A. Substantiation of a method the fourier for boundary value problems with an involute deviation. Vestnik Samarskogo Universiteta. Estestvenno-Nauchnaya Seriya. 1999. 2, 60–66.

[141] Love A. E. H. Biharmonic analysis, especially in a rectangle, and its application to the theory of elasticity. J. London Math. Soc. 1928. 3, 144–156.

[142] Mikhlin S.G. Variational methods in mathematical physics. Nauka: Moscow, Russia, 1957.

[143] Mikhlin S.G. Course of mathematical physics. Nauka: Moscow, Russia, 1968.

[144] Miranda C. Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche de due variabili. Giorn. Mat. Battaglini. 1948. 2:78, 97–118.

[145] Nahushev A.M. Equations of Mathematical Biology. Nauka: Moscow, Russia, 1995.

[146] Nicolescu M. Sur le problème de Riquier. Comptes Rendus. t.CXCIV, 1932.

[147] Nicolescu M. Les fonctions polyharmoniques, Paris, Hermann ed., 1936.

[148] Nicolescu N. Probléme de l'analyticité par rapport á un opérateur linéaire // Studia Math. 1958. 16, 353–363.

[149] Otis A.B., and Barnett M.P. Surface Harmonic Expantions of Products of Cartesian Coordinates. Tehnical Notes. 1963. 26, Cooperative Computing Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., USA.

[150] Otis A.B., Barnett M.P. Surface Harmonic of Products of Cartesian Coordinates. Mathematics of Computation. 1964. 18:88, USA.

[151] Razgulin, A.V. Nonlinear Models of Optical Synergetics, MAKS Press: Moscow, 2008.

[152] Przeworska-Rolewicz, D. Equations with Transformed Argument: An Algebraic Approach; Warszawa: Amsterdam, The Netherlands, 1973.

[153] Przeworska-Rolewicz, D. Some boundary value problems with transformed argument. Commentat. Math. 1974. 17, 451–457.

[154] Rainville E.D. Special Functions, The Macmillan Company, New York, 1960.

[155] Roumaissa S., Nadjib B., Faouzia R. A variant of quasi-reversibility method for a class of heat equations with involution perturbation. Mathematical Methods in the Applied Sciences. 2021. 44, 11933–11943.

[156] Sadybekov M. A., Torebek B. T., and Turmetov B. Kh. On an explicit form of the Green function of the Robin problem for the Laplace operator in a circle. Adv. Pure Appl. Math. 2015. 6:3, 163–172.

[157] Sadybekov M.A., Torebek B.T., Turmetov B.Kh. Representation of Green's function of the Neumann problem for a multi-dimensional ball. Complex Variables and Elliptic Equations. 2016. 61:1, 104–123.

[158] Sadybekov M.A., Torebek B.T., Turmetov B.Kh. Representation of the Green's function of the exterior Neumann problem for the Laplace operator. Siberian Math. J. 2017. 58:1, 153–158.

[159] Sadybekov M., Dildabek, G., Ivanova, M. Direct and inverse problems for nonlocal heat equation with boundary conditions of periodic type. Boundary Value Problems. 2022. 53, 1–24.

[160] Sarsenbi, A.M. Unconditional bases related to a nonclassical second-order differential operator. Diff. Equ. 2010. 46, 509–514.

[161] Sarsenbi A.A., Sarsenbi A.M. On eigenfunctions of the boundary value problems for second order differential equations with involution. Symmetry. 2021. 13, 1972.

[162] Sobolev S.L. Cubature Formulas and Modern Analysis: An Introduction; Nauka: Moscow, Russia, 1974; Gordon and Breach: Montreux, Switzerland, 1992.

[163] Sokolovsky V.B. On one generalization of the Neumann problem. Differential equations. 1988. 24:4, 714–716.

[164] Soldatov A.P. On the Fredholm property and index of the generalized Neumann problem. Differ. Equations. 2020. 56:2, 212–220.

[165] Stein E.M., Weiss G. Introduction to Fourier Analysis on Euclidean Spaces, Princeton, New Jersey, Princeton University Press, 1971.

[166] Sweers G. A survey on boundary conditions for the biharmonic. Complex Variables and Elliptic Equations. 2009. 54, 79–93.

[167] Teodorescu P.P. Asupra polinoamelor armonice si biarmonice, Studia Univ. Babes-Bolyai., 1963. 1, 821–832. (Romanian)

[168] Tojo F.A.F. Computation of Green's functions through algebraic decomposition of operators. Bound. Value Probl. 2016. 167, 1–15.

[169] Torebek B.T., Tapdigoglu R. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. Math. Meth. Appl. Sci. 2017. 40, 6468–6479.

[170] Turmetov B.Kh., Ashurov R.R. On solvability of the Neumann boundary value problem for a non-homogeneous biharmonic equation in a ball. British J. Math. Comput. Sci. 2014. 4, 557–571.

[171] Turmetov B.Kh. Solvability of fractional analogues of the Neumann problem for a nonhomogeneous biharmonic equation. Electronic Journal of Differential Equations. 2015. 2015. No. 82. 1–21.

[172] Turmetov B.Kh. On a method for constructing a solution of integro-differential equations of fractional order, Electronic Journal of Qualitative Theory of Differential Equations, 2018, no. 25, 1–14.

[173] Turmetov B., Karachik V. On eigenfunctions and eigenvalues of a nonlocal Laplace operator with multiple involution. Symmetry. 2021. 13, 1781.

[174] Turmetov B., Karachik V., Muratbekova M. On a boundary value problem for the biharmonic equation with multiple involutions. Mathematics. 2021. 9, 2020.

[175] Turmetov B.Kh., Karachik V.V. On solvability of the Dirichlet and Neumann boundary value problems for the Poisson equation with multiple involution, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki. 2021. 32:4, 651–667.

[176] Turmetov B.Kh., Karachik V.V. On a Dirichlet problem for a nonlocal polyharmonic equation. Bulletin of the South Ural State University, Ser. Mathematics. Mechanics. Physics. 2021. 13:2, 37–45.

[177] Turmetov B., Karachik V. On eigenfunctions and eigenvalues of a nonlocal Laplace operator with multiple involution. Symmetry. 2021. 13, 1781.

[178] Turmetov B.Kh., Usmanov K.I., Nazarova K.Zh. On the operator method for solving linear integro-differential equations with fractional conformable derivatives, Fractal and Fractional, 2021, 5:3, 1–21.

[179] Turmetov B., Karachik V. Construction of Eigenfunctions to One Nonlocal Second-Order Differential Operator with Double Involution, Axioms, 2022, 11:10, 543.

[180] Turmetov B.Kh., Karachik V.V. Neumann boundary condition for a nonlocal biharmonic equation. Bulletin of the South Ural State University, Ser. Mathematics. Mechanics. Physics. 2022. 14:2, 51–58.

[181] Turmetov B.Kh., Karachik V.V. Solvability of nonlocal Dirichlet problem for generalized Helmholtz equation in a unit ball. Complex variables and elliptic equations. 2023. 68:7, 1204–1218.

[182] Vladimirov V.S. Collection of problems on the equations of mathematical physics. FIZMATLIT: Moscow, 2001.

[183] Vladimirov V.S. Equations of Mathematical Physics (Marcel Dekker, New York, 1971; Nauka: Moscow, 1981).

[184] Vladykina V.E., Shkalikov, A.A. Spectral Properties of Ordinary Differential Operators with Involution. Dokl. Math. 2019, 99, 5–10.

[185] Wiener J. Generalized Solutions of Functional Differential Equations; World Sci.: Singapore, 1993.

[186] Yarka U., Fedushko S., Veselý P. The Dirichlet problem for the perturbed elliptic equation. Mathematics 8. 43, 2108.

[187] Ying Wang, Liuqing Ye. Biharmonic Green function and biharmonic Neumann function in a sector. Complex Variables Elliptic Equ. 2013. 58: 1, 7–22.

[188] Ying Wang. Tri-harmonic boundary value problems in a sector. Complex Variables Elliptic Equ. 2014. 59:5, 732–749.

[189] Yuan H., Karachik V. Representations of super-holomorphic functions and their applications. Complex Variables and Elliptic Equations, 2016, 61: 11, 1517–1532.

[190] Yuan H., Karachik V. Solutions of the generalized Dirichlet problem for the iterated slice Dirac equation. Czechoslovak Mathematical Journal. 2022. 72:2, 523–539.

[191] Yuan H., Karachik V. Dirichlet and Neumann Boundary Value Problems for Dunkl Polyharmonic Equations. Mathematics. 2023. 11:9, 2185.

[192] Zaremba S. Sur l'integration de l'equation biharmonique. Bull. Acad. Sci. Cracovie. 1908. 1–29.