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Non-Newtonian Generating Functions

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Abstract

In this study, generating functions will be introduced in Non-Newtonian calculus. Their properties will be shown. In different calculus, generating functions of same sequence and sequence generated by same function will be examined. Important remarks and non-Newtonian Fibonacci numbers will be obtained.

Keywords: Non-Newtonian generating function; Non-Newtonian formal power series; Non-Newtonian Fibonacci numbers

Introduction and Preliminaries

Generating functions are one of most suprising topics in mathematics. They are used many fields of cryptograhy, mathematics, computational science and else. Non-Newonian calculus was firstly introduced by Grossman and Katz as an alternative to classical calculus[1]. It provides wide diversity of mathematical instruments which are useful in engineering, mathematics and science. Non-Newtonian calculus was used by Meginniss to create a theory of probability that is adapted to human behavior and decision making[2]. Rybaczuk and Stopel used the bigeometric calculus on fractals and material science[3]. In a study which is made by Aniszewska and Rybaczuk, the bigeometric calculus was used on a multiplicative Lorenz system[4]. In a study which is made by Bashirov and Riza, differentiation analysed as complex multiplicative[5]. The non-Newtonian calculus was used in the study of biomedical image analysis by Florack and van Assen[6]. Çakmak and Başar obtained some results and sequence spaces with respect to non-Newtonian calculus[7]. In a study which is made by Tekin and Başar, sequence spaces were examined on non-Newtonian complex field[8]. Duyar, Sagir and Oğur got some basic topologic properties on non-Newtonian real line[9]. Duyar and Erdoğan investigated non-Newtonian real number series and non-Newtonian improper integrals, and they obtained convergence tests for them[10, 11]. Sağir and Erdoğan studied non-Newtonian power series[12]. Güngör examined geometric properties of non-Newtonian sequence space $l_p(N)$[13].

A generator is one-to-one function whose domain is $\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$. The range of generator α is called non-Newtonian real line and we denote it by $\mathbb{R}(N)_\alpha$. By $\alpha-$arithmetic we mean the arithmetic whose realm $\mathbb{R}(N)_\alpha$ and whose operations and ordering relation are defined as follows:

$\alpha-$ addition	$y\dot{+}z=\alpha\left\{\alpha^{-1}(y)+\alpha^{-1}(z)\right\}$

$\alpha-$subtraction	$y\dot{-}z=\alpha\left\{\alpha^{-1}(y)-\alpha^{-1}(z)\right\}$
$\alpha-$multiplication	$y\dot{\times}z=\alpha\left\{\alpha^{-1}(y)\times\alpha^{-1}(z)\right\}$
$\alpha-$division	$y\dot{/}z=\alpha\left\{\alpha^{-1}(y)/\alpha^{-1}(z)\right\}$
$\alpha-$order	$y\dot{<}z\left(y\dot{\leq}z\right)\Leftrightarrow\alpha^{-1}(y)<\alpha^{-1}(z)\ \left(\alpha^{-1}(y)\leq\alpha^{-1}(z)\right)$

In this case, it is said that α generates $\alpha-$arithmetic. For example, the identity function I generates the classical arithmetic and the exponential function exp generates geometric arithmetic. Each generator generates exactly one arithmetic and, conversely, each arithmetic is generated by exactly one generator[1].

The $\alpha-$positive numbers are the numbers in $\mathbb{R}(N)_\alpha$ such that $x\dot{>}\dot{0}$, similarly the $\alpha-$negative numbers are the numbers in $\mathbb{R}(N)_\alpha$ such that $x\dot{<}\dot{0}$. $\alpha-$zero and $\alpha-$one numbers are denoted by $\dot{0}=\alpha(0)$ and $\dot{1}=\alpha(1)$ respectively. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. Hence $\alpha-$integers are as follows:

$$\ldots,\alpha(-2),\alpha(-1),\alpha(0),\alpha(1),\alpha(2),\ldots.$$

For each integer n, we set $\dot{n}=\alpha(n)$. If $\dot{n}$ is an $\alpha-$positive integer, then it is n times $\alpha-$sum of $\dot{1}$ [1].

$\alpha-$absolute value of a number $x\in\mathbb{R}(N)_\alpha$ is defined by

$$|x|_\alpha=\alpha\left(\left|\alpha^{-1}(x)\right|\right)=\begin{cases}x & if \quad x\dot{>}\dot{0}\\ \dot{0} & if \quad x=\dot{0}\\ \dot{0}\dot{-}x & if \quad x\dot{<}\dot{0}\end{cases}.$$

For $x\in\mathbb{R}(N)_\alpha$, $\sqrt[p]{x}^{\alpha}=\alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ and

$x^{p_\alpha}=\alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$ [1,7].

A closed $\alpha-$interval on $\mathbb{R}(N)$ is represented by

$$[a,b]^{\cdot}=\left\{x\in\mathbb{R}(N)_\alpha : a\dot{\leq}x\dot{\leq}b\right\}$$

$$=\left\{x\in\mathbb{R}(N)_\alpha : \alpha^{-1}(a)\leq\alpha^{-1}(x)\leq\alpha^{-1}(b)\right\}=\alpha\left(\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]\right),$$

similarly an open $\alpha-$interval $(a,b)^{\cdot}$ can be represented. It is said that an $\alpha-$interval has $\alpha-$extent $b\dot{-}a$ [9].

Let $\{u_n\}$ be an infinite sequence of the numbers in $\mathbb{R}(N)_\alpha$. If each open $\alpha-$interval containing an element u includes all elements except for a finite numbers of elements of the sequence $\{u_n\}$, then it is said that the sequence $\{u_n\}$ $\alpha-$converges to u and the element u is called as $\alpha-$limit of the sequence $\{u_n\}$. This is denoted by ${}^{\alpha}\lim_{n\to\infty}u_n=u$. This convergence becomes the classic convergence if $\alpha=I$. Classic and geometric convergence are equivalent in the sense that a positive number sequence $\{p_n\}$ converges as geometric to a positive number p iff $\{p_n\}$ converges as classic to p [1].

An infinite $\alpha-$sum

$$a_0\dot{+}a_1\dot{+}a_2\dot{+}\ldots\dot{+}a_n\dot{+}\ldots={}_\alpha\sum_{n=0}^{\infty}a_n$$

is called the non-Newtonian real number series or $\alpha-$series. The sequence $\{S_m\}$ with the general term $S_m={}_\alpha\sum_{n=0}^{m}a_n$ is called as the non-Newtonian partial sums sequence of the series ${}_\alpha\sum_{n=0}^{\infty}a_n$. If the sequence $\{S_m\}$ is $\alpha-$convergent, then it is said that the series ${}_\alpha\sum_{n=0}^{\infty}a_n$ is $\alpha-$convergent. If ${}^{\alpha}\lim_{m\to\infty}S_m=S$, then it is written ${}_\alpha\sum_{n=0}^{\infty}a_n=S$. If the limit ${}^{\alpha}\lim_{m\to\infty}S_m$ is not exist or equal to $\dot{-}\infty$ or $\dot{+}\infty$, then it is said that the series ${}_\alpha\sum_{n=0}^{\infty}a_n$ is $\alpha-$divergent [10].

Let α and β be arbitrary chosen generators which image the set $\mathbb{R}$ to A and B respectively. *-Calculus is defined as an ordered pair of the arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic) and the following notations are used:

	$\alpha-$arithmetic	$\beta-$arithmetic
Universe(Realm)	$A\left(=\mathbb{R}(N)_\alpha\right)$	$B\left(=\mathbb{R}(N)_\beta\right)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}\left(or\ -\alpha\right)$	$\ddot{/}\left(or\ -\beta\right)$
Ordering	$\dot{<}$	$\ddot{<}$

$\alpha-$arithmetic is used on inputs and $\beta-$arithmetic is used on outputs. In particular, the changes of inputs and outputs are measured by $\alpha-$arithmetic and $\beta-$arithmetic, respectively. The operators in *-calculus are applied to functions whose inputs and outputs belong to A and B, respectively [1].

If the generators α and β are chosen as one of I and exp, the following special calculuses are obtained.

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function ι (iota) that possesses the following three properties:

- (i) ι is one to one,
- (ii) ι is on A and onto B,
- (iii) For any number u and v in A,

$$\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v),$$

$$\iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v),$$

$$\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v),$$

$$\iota(u \dot{/} v) = \iota(u) \ddot{/} \iota(v), \quad v \neq \dot{0}$$

$$u \dot{<} v \Leftrightarrow \iota(u) \ddot{<} \iota(v).$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every number x in A, and that

$\iota(\dot{n}) = \ddot{n}$ for every integer n. Any statement in $\alpha-$arithmetic can easily transformed into a statement in $\beta-$arithmetic thanks to the isomorphism ι [1].

The general statement of *-power series of $\iota(x \dot{-} x_0) = \iota(x) \ddot{-} \iota(x_0)$ is formed

$$_{\beta}\sum_{k=0}^{\infty} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta} = a_0 \ddot{+} [a_1 \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))] \ddot{+} [a_2 \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{2_\beta}] \ddot{+} \ldots$$

where $a_0, a_1, \ldots, a_k, \ldots \in \mathbb{R}(N)_\beta$ are constants and

$x, x_0 \in \mathbb{R}(N)_\alpha$ [12]. If $x_0 = \dot{0}$, then the *-power series is

$$_{\beta}\sum_{k=0}^{\infty} a_k \ddot{\times} \iota(x)^{k_\beta} = a_0 \ddot{+} [a_1 \ddot{\times} \iota(x)] \ddot{+} [a_2 \ddot{\times} \iota(x)^{2_\beta}] \ddot{+} \ldots .$$

Non-Newtonian Generating Functions

Definition 4.1: Let (a_n) be a sequence in $\mathbb{R}(N)_\alpha$, namely, be an $\alpha-$sequence. Then the *-power series $f(t) = {_\beta}\sum_{n=0}^{\infty} \iota(a_n) \ddot{\times} \iota(t)^{n_\beta}$ which has sequence coefficients $(\iota(a_n))$ is called *-generating function or non-Newtonian generating function for the sequence (a_n). Similarly, when (b_n) is a sequence in $\mathbb{R}(N)_\beta$, namely a $\beta-$sequence, *-power series

$g(t) = {_\beta}\sum_{n=0}^{\infty} b_n \ddot{\times} \iota(t)^{n_\beta}$ which has sequence coefficients (b_n) is called *-generating function or non-Newtonian generating function for the sequence (b_n).

Example 4.2: Let *-generating function of constant $\alpha-$sequence $(\dot{1})$ and constant $\beta-$sequence $(\ddot{1})$ be found. Additionally let these *-generating functions be examined in geometric, anageometric and bigeometric calculus.

Solution 4.3: The *-generating function of constant $\alpha-$sequence $(\dot{1})$ is the *-power series

$$\iota(\dot{1}) \ddot{+} \iota(\dot{1}) \ddot{\times} \iota(t) \ddot{+} \iota(\dot{1}) \ddot{\times} \iota(t)^{2_\beta} \ddot{+} \iota(\dot{1}) \ddot{\times} \iota(t)^{3_\beta} \ddot{+} \ldots = {_\beta}\sum_{n=0}^{\infty} \iota(\dot{1}) \ddot{\times} \iota(t)^{n_\beta} .$$

By virtue of example 4 in [10], we get

$$_{\beta}\sum_{n=0}^{\infty} \iota(\dot{1}) \ddot{\times} \iota(t)^{n_\beta} = \frac{\iota(\dot{1})}{\ddot{1} \ddot{-} \iota(t)}\beta \qquad \textbf{(4.1)}$$

Since $\iota(\dot{1}) = \ddot{1}$ here, *-generating functions of constant $\alpha-$sequence $(\dot{1})$ and constant $\beta-$sequence $(\ddot{1})$ are same for $|\iota(t)|_\beta \ddot{<} \ddot{1}$ and it is *-function

$$f(t) = \frac{\iota(\dot{1})}{\ddot{1} \ddot{-} \iota(t)}\beta = \frac{\ddot{1}}{\ddot{1} \ddot{-} \iota(t)}\beta$$

As in classical generating functions, we don't consider *-convergence intervals of *-generating functions and we consider that they are *-convergent. Now, let these sequence and functions be examined for special calculus. Since $\alpha = I$, $\beta = \exp$ for geometric calculus, constant $\alpha-$sequence $(\dot{1})$ is constant sequence (1) and *-generating function is

$$\frac{\ddot{1}}{\ddot{1} \ddot{-} \iota(t)}\beta = \frac{e}{e \ddot{-} e^t}\beta = \frac{e}{e^{\ln e - \ln e^t}}\beta = e^{\left(\frac{\ln e}{\ln e^{\ln e - \ln e^t}}\right)} = e^{\left(\frac{1}{1-t}\right)}.$$

From here, it is seen that *-generating functions of constant sequence (1) in geometric calculus is $e^{\left(\frac{1}{1-t}\right)}$. It is well known that generating function of constant sequence (1) in classical calculus is the function $\frac{1}{1-t}$. This shows us that generating functions of same sequence can be different in different calculus.

Since $\alpha = \exp$, $\beta = I$ for anageometric calculus, constant $\alpha-$sequence $(\dot{1})$ is constant sequence (e) and *-generating function is

$$\frac{\ddot{1}}{\ddot{1} \ddot{-} \iota(t)}\beta = \frac{1}{1 - \ln t}.$$

It is easily seen that generating function of constant sequence (e) in classical calculus is $\frac{e}{1-t}$. This shows us that generating functions of same sequence can be different in different calculus.

Since $\alpha=\beta=\exp$ for bigeometric calculus, constant $\alpha-$ sequence $(\dot{1})$ is the constant sequence (e) and *-generating function is

$$\frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(t)}\beta=\frac{e}{e\ddot{-}t}\beta=e^{\left(\frac{\ln e}{\ln(e\ddot{-}t)}\right)}=e^{\left(\frac{1}{\ln e^{(\ln e-\ln t)}}\right)}=e^{\left(\frac{1}{1-\ln t}\right)}.$$

Again, this shows us that generating functions of same sequence can be different in different calculus.

Example 4.4: Find that *-generating function and general term of sequence (a_n) which holds recurrence $a_{n+1}=\left(\dot{3}\dot{\times}a_n\right)\dot{+}\dot{2}$ for $n\geq 0$ and $a_0=\dot{0}$.

Solution 4.5: Primarily, let's take image of each side of recurrence under ι.

$$\iota(a_{n+1})=\iota\left(\left(\dot{3}\dot{\times}a_n\right)\dot{+}\dot{2}\right)$$
$$\iota(a_{n+1})=\iota\left(\dot{3}\dot{\times}a_n\right)\ddot{+}\iota\left(\dot{2}\right)$$
$$\iota(a_{n+1})=\ddot{3}\ddot{\times}\iota(a_n)\ddot{+}\ddot{2}$$

If each side of equality above be $\beta-$multiplicated by $\iota(t)^{n_\beta}$ and be $\beta-$summed for $n\geq 0$, then

$$\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta}=\left(\ddot{3}\ddot{\times}\iota(a_n)\ddot{+}\ddot{2}\right)\ddot{\times}\iota(t)^{n_\beta}$$
$${}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta}={}_\beta\sum_{n\geq 0}\left(\ddot{3}\ddot{\times}\iota(a_n)\ddot{+}\ddot{2}\right)\ddot{\times}\iota(t)^{n_\beta}$$
$${}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta}=\ddot{3}\ddot{\times}{}_\beta\sum_{n\geq 0}\iota(a_n)\ddot{\times}\iota(t)^{n_\beta}\ddot{+}{}_\beta\sum_{n\geq 0}\ddot{2}\ddot{\times}\iota(t)^{n_\beta}$$

is found. Now, let *-generating function for (a_n) be defined as $A(t)={}_\beta\sum_{n\geq 0}\iota(a_n)\ddot{\times}\iota(t)^{n_\beta}$. In the right hand of last equality, it is seen that $\beta-$sum is $A(t)$ and, by virtue of example 4 in [10], second $\beta-$sum is $\frac{\ddot{2}}{\ddot{1}\ddot{-}\iota(t)}\beta$. If we substitute them, we find

$${}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta}=\ddot{3}\ddot{\times}A(t)\ddot{+}\frac{\ddot{2}}{\ddot{1}\ddot{-}\iota(t)}\beta.$$

If equality be $\beta-$multiplicated by $\iota(t)$ in order to make left hand sum similar to the function $A(t)$,

$${}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{(n+1)_\beta}=\ddot{3}\ddot{\times}\iota(t)\ddot{\times}A(t)\ddot{+}\frac{\ddot{2}\ddot{\times}\iota(t)}{\ddot{1}\ddot{-}\iota(t)}\beta \qquad \textbf{(4.2)}$$

is obtained. Here, only difference of $\beta-$series at left side of 2.2 from $A(t)$ is that there is no term $\iota(a_0)$ in $\beta-$sum. Hence left side is $A(t)\ddot{-}\iota(a_0)$. $\iota(a_0)=\ddot{0}$ since $a_0=\dot{0}$, thus left side is found as $A(t)\ddot{-}\ddot{0}=A(t)$. If we substitute this in 2.2, then we find *-generating function of (a_n) as

$$A(t)=\ddot{3}\ddot{\times}\iota(t)\ddot{\times}A(t)\ddot{+}\frac{\ddot{2}\ddot{\times}\iota(t)}{\ddot{1}\ddot{-}\iota(t)}\beta$$
$$A(t)\ddot{-}\left(\ddot{3}\ddot{\times}\iota(t)\ddot{\times}A(t)\right)=\frac{\ddot{2}\ddot{\times}\iota(t)}{\ddot{1}\ddot{-}\iota(t)}\beta$$
$$A(t)\ddot{\times}\left[\ddot{1}\ddot{-}\left(\ddot{3}\ddot{\times}\iota(t)\right)\right]=\frac{\ddot{2}\ddot{\times}\iota(t)}{\ddot{1}\ddot{-}\iota(t)}\beta$$
$$A(t)=\frac{\ddot{2}\ddot{\times}\iota(t)}{\left[\ddot{1}\ddot{-}\left(\ddot{3}\ddot{\times}\iota(t)\right)\right]\ddot{\times}\left[\ddot{1}\ddot{-}\iota(t)\right]}\beta.$$

Now, we must expand the *-generating function $A(t)$ in partial fractions to find general term of the sequence (a_n). Therefore, let .

$$A(t)=\frac{\ddot{2}\ddot{\times}\iota(t)}{\left[\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))\right]\ddot{\times}\left[\ddot{1}\ddot{-}\iota(t)\right]}\beta=\frac{m}{\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))}\beta\ddot{+}\frac{k}{\ddot{1}\ddot{-}\iota(t)}\beta$$

From here, it is seen that

$$\frac{\ddot{2}\ddot{\times}\iota(t)}{\left[\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))\right]\ddot{\times}\left[\ddot{1}\ddot{-}\iota(t)\right]}\beta=\frac{m}{\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))}\beta\ddot{+}\frac{k}{\ddot{1}\ddot{-}\iota(t)}\beta$$
$$\beta\left(\frac{\beta^{-1}\left(\ddot{2}\ddot{\times}\iota(t)\right)}{\beta^{-1}\left(\left[\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))\right]\ddot{\times}\left[\ddot{1}\ddot{-}\iota(t)\right]\right)}\right)=\beta\left(\frac{\beta^{-1}(m)}{\beta^{-1}\left[\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))\right]}\right)\ddot{+}\beta\left(\frac{\beta^{-1}(k)}{\beta^{-1}\left(\ddot{1}\ddot{-}\iota(t)\right)}\right)$$
$$\beta\left(\frac{2.\beta^{-1}(\iota(t))}{\beta^{-1}\left[\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))\right].\beta^{-1}\left[\ddot{1}\ddot{-}\iota(t)\right]}\right)=\beta\left(\frac{\beta^{-1}(m)}{1-\beta^{-1}\left(\ddot{3}\ddot{\times}\iota(t)\right)}\right)\ddot{+}\beta\left(\frac{\beta^{-1}(k)}{1-\beta^{-1}(\iota(t))}\right)$$
$$\beta\left(\frac{2.\beta^{-1}(\iota(t))}{\left[1-\beta^{-1}\left(\ddot{3}\ddot{\times}\iota(t)\right)\right].\left[1-\beta^{-1}(\iota(t))\right]}\right)=\beta\left(\frac{\beta^{-1}(m)}{1-\beta^{-1}\left(\ddot{3}\ddot{\times}\iota(t)\right)}+\frac{\beta^{-1}(k)}{1-\beta^{-1}(\iota(t))}\right)$$
$$\beta\left(\frac{2.\alpha^{-1}(t)}{\left[1-3.\beta^{-1}(\iota(t))\right].\left[1-\alpha^{-1}(t)\right]}\right)=\beta\left(\frac{\beta^{-1}(m)}{1-3.\beta^{-1}(\iota(t))}+\frac{\beta^{-1}(k)}{1-\alpha^{-1}(t)}\right)$$
$$\beta\left(\frac{2.\alpha^{-1}(t)}{\left[1-3.\alpha^{-1}(t)\right].\left[1-\alpha^{-1}(t)\right]}\right)=\beta\left(\frac{\beta^{-1}(m)}{1-3.\alpha^{-1}(t)}+\frac{\beta^{-1}(k)}{1-\alpha^{-1}(t)}\right).$$

Since β is one-to-one,

$$\frac{2.\alpha^{-1}(t)}{\left(1-3.\alpha^{-1}(t)\right).\left(1-\alpha^{-1}(t)\right)}=\frac{\beta^{-1}(m)}{1-3.\alpha^{-1}(t)}+\frac{\beta^{-1}(k)}{1-\alpha^{-1}(t)}$$

is written. When this well known operation of expanding in partial fractions is completed, it is found that $\beta^{-1}(m)=1$ and $\beta^{-1}(k)=-1$. Hence $m=\ddot{1}$ and $k=\ddot{0}\ddot{-}\ddot{1}$. From here,

$$A(t)=\frac{\ddot{1}}{\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))}\beta\ddot{+}\frac{\ddot{0}\ddot{-}\ddot{1}}{\ddot{1}\ddot{-}\iota(t)}\beta$$
$$=\frac{\ddot{1}}{\ddot{1}\ddot{-}(\ddot{3}\ddot{\times}\iota(t))}\beta\ddot{-}\frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(t)}\beta$$

is obtained. In view of *-generating function 2.1, if we expand the *-generating function $A(t)$ in *-power series, we get

$$A(t) = {}_{\beta}\sum_{n=0}^{\infty} \ddot{1} \ddot{\times} \left(\ddot{3} \ddot{\times} \iota(t)\right)^{n_\beta} \ddot{-} {}_{\beta}\sum_{n=0}^{\infty} \ddot{1} \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \ddot{1} \ddot{\times} \ddot{3}^{n_\beta} \ddot{\times} \iota(t)^{n_\beta} \ddot{-} {}_{\beta}\sum_{n=0}^{\infty} \ddot{1} \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \ddot{3}^{n_\beta} \ddot{\times} \iota(t)^{n_\beta} \ddot{-} {}_{\beta}\sum_{n=0}^{\infty} \ddot{1} \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\ddot{3}^{n_\beta} \ddot{-} \ddot{1}\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\beta\left(3^n\right) \ddot{-} \iota\left(\dot{1}\right)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\beta\left(\alpha^{-1}\left(\alpha\left(3^n\right)\right)\right) \ddot{-} \iota\left(\dot{1}\right)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\iota\left(\alpha\left(3^n\right)\right) \ddot{-} \iota\left(\dot{1}\right)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\iota\left(\alpha\left(\alpha^{-1}(\dot{3})^{n_\alpha}\right)\right) \ddot{-} \iota\left(\dot{1}\right)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \left(\iota\left(\dot{3}^{n_\alpha}\right) \ddot{-} \iota\left(\dot{1}\right)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$$= {}_{\beta}\sum_{n=0}^{\infty} \iota\left(\dot{3}^{n_\alpha} \dot{-} \dot{1}\right) \ddot{\times} \iota(t)^{n_\beta}$$

Therefore $a_n = \dot{3}^{n_\alpha} \dot{-} \dot{1}$ is obtained.

Example 4.6: Find that *-generating function for sequence (a_n) with $n \geq 1$ and $a_n = \frac{1}{n}\alpha$ in geometric calculus.

Solution 4.7: Since $\alpha = I$, $\beta = \exp$ in geometric calculus, sequence (a_n) is the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$. Hence $\iota(a_n) = \iota\left(\frac{1}{n}\right) = e^{\frac{1}{n}}$. Then desired *-generating function is obtained as

$${}_{\beta}\sum_{n\geq 1} e^{\frac{1}{n}} \ddot{\times} \iota(t)^{n_\beta} = {}_{\beta}\sum_{n\geq 1} e^{\frac{1}{n}} \ddot{\times} e^{(\ln \iota(t))^n}$$

$$= {}_{\beta}\sum_{n\geq 1} e^{\frac{1}{n}} \ddot{\times} e^{(t^n)}$$

$$= {}_{\beta}\sum_{n\geq 1} e^{\left(\frac{1}{n}.t^n\right)}$$

$$= e^{\left[\sum_{n\geq 1} \ln e^{\left(\frac{t^n}{n}\right)}\right]}$$

$$= e^{\left[\sum_{n\geq 1} \frac{t^n}{n}\right]}$$

$$= e^{\ln \frac{1}{1-t}}$$

$$= \frac{1}{1-t}$$

Remark 4.8: At example 2.6 above, it was seen that the function $\frac{1}{1-t}$ generates the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$ in geometric calculus. It is known that same function generates the constant sequence $(1)_{n\geq 0}$ in classical calculus. This shows us that sequences which are generated by same function in different calculus can not be same.

Definition 4.9: Let $\alpha-$series ${}_{\alpha}\sum_{n=0}^{\infty} a_n$ and $\alpha-$series ${}_{\alpha}\sum_{n=0}^{\infty} b_n$ are given. Then $\alpha-$series ${}_{\alpha}\sum_{n=0}^{\infty} c$ with general term $c_n = {}_{\alpha}\sum_{i=0}^{n} a_i \dot{\times} b_{n-i}$ is called $\alpha-$Cauchy product of ${}_{\alpha}\sum_{n=0}^{\infty} a_n$ and ${}_{\alpha}\sum_{n=0}^{\infty} b_n$. This situation is denoted by

$${}_{\alpha}\sum_{n=0}^{\infty} a_n \dot{\times} {}_{\alpha}\sum_{n=0}^{\infty} b_n = {}_{\alpha}\sum_{n=0}^{\infty}\left({}_{\alpha}\sum_{i=0}^{n} a_i \dot{\times} b_{n-i}\right) = {}_{\alpha}\sum_{n=0}^{\infty} c_n. \qquad \textbf{(4.3)}$$

Definition 4.10: A *-formal power series is an expression of the form

$$\iota(a_0) \ddot{+} \iota(a_1) \ddot{\times} \iota(t) \ddot{+} \iota(a_2) \ddot{\times} \iota(t)^{2_\beta} \ddot{+} \iota(a_3) \ddot{\times} \iota(t)^{3_\beta} \ddot{+} \ldots$$

or

$$b_0 \ddot{+} b_1 \ddot{\times} \iota(t) \ddot{+} b_2 \ddot{\times} \iota(t)^{2_\beta} \ddot{+} b_3 \ddot{\times} \iota(t)^{3_\beta} \ddot{+} \ldots$$

where the $\alpha-$sequence (a_n) and the $\beta-$sequence (b_n) are called sequence of coefficients.

To say that two *-formal power series are equal is to say that coefficient sequences are the same. Since the function ı is an isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic, $(\iota(a_n))$ is a $\beta-$sequence for given $\alpha-$sequence (a_n) and $(\iota^{-1}(b_n))$ is an $\alpha-$sequence for given $\beta-$sequence (b_n). Due to this situation, features for *-formal power series will be shown over only $\alpha-$sequences or only $\beta-$sequences. By virtue of the function ı, having a feature in one will require it to have in the other.

*-formal power series can be $\beta-$summed or $\beta-$subtracted as

$${}_{\beta}\sum_{n\geq 0} \iota(a_n) \ddot{\times} \iota(t)^{n_\beta} \ddot{+} {}_{\beta}\sum_{n\geq 0} \iota(b_n) \ddot{\times} \iota(t)^{n_\beta} = {}_{\beta}\sum_{n\geq 0} \left(\iota(a_n) \ddot{+} \iota(b_n)\right) \ddot{\times} \iota(t)^{n_\beta}$$

$${}_{\beta}\sum_{n\geq 0} \iota(a_n) \ddot{\times} \iota(t)^{n_\beta} \ddot{-} {}_{\beta}\sum_{n\geq 0} \iota(b_n) \ddot{\times} \iota(t)^{n_\beta} = {}_{\beta}\sum_{n\geq 0} \left(\iota(a_n) \ddot{-} \iota(b_n)\right) \ddot{\times} \iota(t)^{n_\beta}$$

and they can be $\beta-$multiplied by $\beta-$Cauchy product in 2.3, namely

$$\left({}_{\beta}\sum_{n\geq 0} \iota(a_n) \ddot{\times} \iota(t)^{n_\beta}\right) \ddot{\times} \left({}_{\beta}\sum_{n\geq 0} \iota(b_n) \ddot{\times} \iota(t)^{n_\beta}\right) = {}_{\beta}\sum_{n\geq 0}\left({}_{\beta}\sum_{k} \iota(a_k) \ddot{\times} \iota(b_{n-k})\right) \ddot{\times} \iota(t)^{n_\beta}.$$

One can easily see that *-formal power series forms a ring with operations mentioned above.

Proposition 4.11: A *-formal power series $f = {}_{\beta}\sum_{n\geq 0} \iota(a_n) \ddot{\times} \iota(t)^{n_\beta}$ has a $\beta-$reciprocal if and only if $a_0 \neq \dot{0}$. In that case $\beta-$reciprocal is unique.

Proof. Let *f* have a β–reciprocal, namely $\frac{\ddot{1}}{f}\beta = {}_{\beta}\sum_{n\geq 0} \iota(b_n) \ddot{\times} \iota(t)^{n_\beta}$. Then $f \ddot{\times} \frac{\ddot{1}}{f}\beta = \ddot{1}$ and, by virtue of 2.3, $c_0 = \ddot{1} = \iota(a_0) \ddot{\times} \iota(b_0)$, so $\iota(a_0) \neq \ddot{0}$, hence $a_0 \neq \dot{0}$. Additionally, in this case according to 2.3,

$$c_n = \ddot{0} = {}_{\beta}\sum_{k} \iota(a_k) \ddot{\times} \iota(b_{n-k}) \quad \text{for } n \geq 1.$$

Hence

$$c_n = \ddot{0} = \iota(a_0)\ddot{\times}\iota(b_n)\ddot{+}\iota(a_1)\ddot{\times}\iota(b_{n-1})\ddot{+}\ldots\ddot{+}\iota(a_n)\ddot{\times}\iota(b_0) \qquad (n\geq 1)$$

and therefore

$$\iota(b_n) = \frac{\ddot{0}\ddot{-}\ddot{1}}{\iota(a_0)}\beta\ddot{\times}\sum_{k\geq 1} a_k b_{n-k} \quad (n\geq 1) \qquad \textbf{(4.4)}$$

is obtained. This determines $b_1, b_2, b_3\ldots$ uniquely, as claimed.

Conversely suppose that $a_0 \neq \dot{0}$ Then $\iota(b_0) = \frac{\ddot{1}}{\iota(a_0)}\beta$ and $b_1, b_2, b_3\ldots$ can be determined from 2.4, the resulting series ${}_\beta\sum_{n\geq 0}\iota(b_n)\ddot{\times}\iota(t)^{n_\beta}$ is the $\beta-$reciprocal of *f*. This completes proof.

Definition 4.12: The symbol $f \xleftrightarrow{*,\alpha} (a_n)_0^\infty$ means that the *-formal power series *f* is the *-generating function for the $\alpha-$sequence $(a_n)_0^\infty$. Namely, it means that $f = {}_\beta\sum_{n\geq 0}\iota(a_n)\ddot{\times}\iota(t)^{n_\beta}$. Analogously, the symbol $g \xleftrightarrow{*,\alpha} (b_n)_0^\infty$ means that the *-formal power series *g* is the *-generating function for the $\beta-$ sequence $(b_n)_0^\infty$, namely, it means that $g = {}_\beta\sum_{n\geq 0} b_n \ddot{\times}\iota(t)^{n_\beta}$.

Proposition 4.13: Let $f \xleftrightarrow{*,\alpha} (a_n)_0^\infty$. Then $\frac{f(x)\ddot{-}\iota(a_0)}{\iota(t)}\beta \xleftrightarrow{*,\alpha} (a_{n+1})_0^\infty$.

Proof. $f = {}_\beta\sum_{n\geq 0}\iota(a_n)\ddot{\times}\iota(t)^{n_\beta}$ since $f \xleftrightarrow{*,\alpha} (a_n)_0^\infty$. Then

$$\begin{aligned}
{}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta} &= \frac{\iota(t)}{\iota(t)}\beta\ddot{\times}{}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{n_\beta} \\
&= \frac{\ddot{1}}{\iota(t)}\beta\ddot{\times}{}_\beta\sum_{n\geq 0}\iota(a_{n+1})\ddot{\times}\iota(t)^{(n+1)_\beta} \\
&= \frac{\ddot{1}}{\iota(t)}\beta\ddot{\times}{}_\beta\sum_{m\geq 1}\iota(a_m)\ddot{\times}\iota(t)^{m_\beta} \\
&= \frac{\ddot{1}}{\iota(t)}\beta\ddot{\times}\left(f(t)\ddot{-}\iota(a_0)\right) \\
&= \frac{f(t)\ddot{-}\iota(a_0)}{\iota(t)}\beta .
\end{aligned}$$

Let's consider the proposition above. Which *-generating function generates $\alpha-$sequence $(a_{n+2})_0^\infty$ if $f \xleftrightarrow{*,\alpha} (a_n)_0^\infty$? It is just iterating of proposition 2.13 two times. That is

$$(a_{n+2})_0^\infty \xleftrightarrow{*,\alpha} \frac{\left(\frac{f\ddot{-}\iota(a_0)}{\iota(t)}\beta\right)\ddot{-}\iota(a_1)}{\iota(t)}\beta = \frac{\frac{f\ddot{-}\iota(a_0)\ddot{-}\iota(a_1)\ddot{\times}\iota(t)}{\iota(t)}\beta}{\iota(t)}\beta = \frac{f\ddot{-}\iota(a_0)\ddot{-}\iota(a_1)\ddot{\times}\iota(t)}{\iota(t)^{2_\beta}}\beta .$$

In this way, general case of proposition 2.13 is given below. One can easily prove that by induction.

Proposition 4.14: If $f \xleftrightarrow{*,\alpha} (a_n)_0^\infty$,

$$(a_{n+k})_0^\infty \xleftrightarrow{*,\alpha} \frac{f\ddot{-}\iota(a_0)\ddot{-}\iota(a_1)\ddot{\times}\iota(t)\ddot{-}\ldots\ddot{-}\iota(a_{h-1})\ddot{\times}\iota(t)^{(h-1)_\beta}}{\iota(t)^{h_\beta}}\beta$$

for integer $k > 0$.

Example 4.15: (Non-Newtonian Fibonacci sequence) The $\alpha-$sequence of (F_n) is called $\alpha-$Fibonacci sequence or non-Newtonian Fibonacci sequence which holds recurrence

$$F_{n+2} = F_{n+1}\dot{+}F_n \qquad \textbf{(4.5)}$$

for $n\geq 0$, $F_0 = \dot{0}$ and $F_1 = \dot{1}$. Find that *-generating function and general term of $\alpha-$Fibonacci sequence.

Solution 4.16: $F(t) = \sum_{n\geq 0}\iota(F_n)\ddot{\times}\iota(t)^{n_\beta}$ Proposition 2.14 easily coverts recurrence relation 2.5 to relation of *-generating function

$$\frac{F\ddot{-}\iota(t)}{\iota(t)^{2_\beta}}\beta = \frac{F\ddot{+}F\ddot{\times}\iota(t)}{\iota(t)}\beta$$

$$F\ddot{-}\iota(t) = F\ddot{\times}\iota(t)\ddot{+}F\ddot{\times}\iota(t)^{2_\beta}$$

$$F\ddot{-}F\ddot{\times}\iota(t)\ddot{-}F\ddot{\times}\iota(t)^{2_\beta} = \iota(t)$$

$$F\ddot{\times}\left(\ddot{1}\ddot{-}\iota(t)\ddot{-}\iota(t)^{2_\beta}\right) = \iota(t)$$

$$F = \frac{\iota(t)}{\ddot{1}\ddot{-}\iota(t)\ddot{-}\iota(t)^{2_\beta}}\beta$$

If this is solved for *F*, then

$$\frac{F\ddot{-}\iota(F_0)\ddot{-}\iota(F_1)\ddot{\times}\iota(t)}{\iota(t)^{2_\beta}}\beta = \frac{F\ddot{-}\iota(F_0)}{\iota(t)}\beta\ddot{+}F$$

$$\frac{F-\iota(t)}{\iota(t)^{2_\beta}}\beta = \frac{F}{\iota(t)}\beta\ddot{+}F .$$

is obtained. Hence *-generating function has been founded. When

$$r_+ = \frac{\dot{1}\dot{+}\sqrt{\dot{5}}^\alpha}{\dot{2}}\alpha \text{ and } r_- = \frac{\dot{1}\dot{-}\sqrt{\dot{5}}^\alpha}{\dot{2}}\alpha$$

if *-generating function is expanded in partial fractions, then

$$\begin{aligned}
\frac{\iota(t)}{\ddot{1}\ddot{-}\iota(t)\ddot{-}\iota(t)^{2_\beta}}\beta &= \frac{\iota(t)}{\left(\ddot{1}\ddot{-}\iota(r_+)\ddot{\times}\iota(t)\right)\ddot{\times}\left(\ddot{1}\ddot{-}\iota(r_-)\ddot{\times}\iota(t)\right)}\beta \\
&= \frac{\ddot{1}}{\iota(r_+)\ddot{-}\iota(r_-)}\beta\ddot{\times}\left(\frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(r_+)\ddot{\times}\iota(t)}\beta\ddot{-}\frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(r_-)\ddot{\times}\iota(t)}\beta\right) \\
&= \frac{\ddot{1}}{\sqrt{\ddot{5}}^\beta}\beta\ddot{\times}\left({}_\beta\sum_{n\geq 0}\iota(r_+)^{n_\beta}\ddot{\times}\iota(t)^{n_\beta}\ddot{-}{}_\beta\sum_{n\geq 0}\iota(r_-)^{n_\beta}\ddot{\times}\iota(t)^{n_\beta}\right) \\
&= \frac{\ddot{1}}{\sqrt{\ddot{5}}^\beta}\beta\ddot{\times}{}_\beta\sum_{n\geq 0}\left(\iota(r_+)^{n_\beta}\ddot{-}\iota(r_-)^{n_\beta}\right)\ddot{\times}\iota(t)^{n_\beta} \\
&= \iota\left(\frac{\dot{1}}{\sqrt{\dot{5}}^\alpha}\alpha\right)\ddot{\times}{}_\beta\sum_{n\geq 0}\iota\left(r_+^{n_\alpha}\dot{-}r_-^{n_\alpha}\right)\ddot{\times}\iota(t)^{n_\beta}
\end{aligned}$$

is obtained. In the last equality, if we look at coefficient of $\iota(t)^{n_\beta}$, we get

$$F_n = \frac{\dot{1}}{\sqrt{\dot{5}}^\alpha}\alpha\dot{\times}\left(r_+^{n_\alpha}\dot{-}r_-^{n_\alpha}\right)$$

for $n\geq 0$.

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Conflict of Interest

No conflict of interest.

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