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Research Article

On Some One Parameter Lifetime Distributions and their Applications

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Abstract

The analysis and modeling of lifetime data are crucial in almost all applied sciences including medicine, insurance, engineering, and finance, amongst others. In the present paper an attempt has been made to have a comparative study on statistical properties and applications of some well-known one parameter lifetime distributions available in statistics literature to model lifetime data from biomedical science and engineering. This comparative study would be of great help to scientist working in biomedical science and engineering for modeling waiting time and survival time data in their respective fields of knowledg.

Keywords: Lifetime distributions; Index of dispersion; Statistical properties; Estimation of parameter; Applications

Introduction

The time to the occurrence of event of interest is known as lifetime or survival time or failure time in reliability analysis. The event may be failure of a piece of equipment, death of a person, development (or remission) of symptoms of disease, health code violation (or compliance). The modeling and statistical analysis of lifetime data are crucial for statisticians and research workers in almost all applied sciences including engineering, medical science/biological science, insurance and finance, amongst others. The classical lifetime distribution namely exponential distribution and Lindley distribution introduced by Lindley (1958) distribution are popular in statistics for modeling lifetime data. But these two classical lifetime distributions are not suitable from theoretical and applied point of view. Shanker et al (2015) have done a critical and comparative study regarding the modeling of lifetime data using both exponential and Lindley distributions and found that there are several lifetime data where these classical lifetime distributions are not suitable due to their shapes, hazard rate functions and mean residual life functions, amongst others. Recently, a number of one parameter lifetime distributions have been introduced by Shanker, namely Shanker, Akash, Rama, Suja, Sujatha, Amarendra, Devya, Shambhu, Aradhana, and Akshya, respectively. The probability density function (pdf), cumulative distribution function (cdf) one parameter lifetime distributions along with their introducer and years are presented in the following Table 1.

The statistical properties and some applications of these distributions have been given in the respective papers.

Table 1: pdf, cdf of one parameter lifetime distributions and their introducer (year).

Distributions	pdf and cdf	Introducer (Year)
Exponential	$f(x;\theta)=\theta e^{-\theta x} \quad ;x>0,\ \theta>0$	Epstein (1958)
	$F_1(x;\theta)=1-e^{-\theta x} \quad ;x>0,\theta>0$	
Lindley	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ;x>0,\ \theta>0$	Lindley (1958)
	$F(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x} \quad ;x>0,\theta$	

Shanker	$f(x;\theta)=\dfrac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x} \quad ;x>0,\ \theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta x}{\theta^2+1}\right]e^{-\theta x} \quad ;x>0,\theta>0$	Shanker (2015a)
Akash	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x} \quad ;x>0,\ \theta>0$ $F(x;\theta)=1-\left[1+\dfrac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x} \quad ;x>0,\theta>0$	Shanker (2015 b)
Rama	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x} \quad ;x>0,\theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x} \quad ;x>0,\theta>0$	Shanker (2017 a)
Suja	$f(x;\theta)=\dfrac{\theta^5}{\theta^4+24}(1+x^4)e^{-\theta x} \quad ;x>0,\ \theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0$	Shanker (2017 b)
Sujatha	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+\theta+2}(1+x+x^2)e^{-\theta x} \quad ;x>0,\ \theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x};x>0,\theta>0$	Shanker (2016 a)
Amarendra	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+\theta^2+2\theta+6}(1+x+x^2+x^3)e^{-\theta x};x>0,\theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta^3x^3+\theta^2(\theta+3)x^2+\theta(\theta^2+2\theta+6)x}{\theta^3+\theta^2+2\theta+6}\right]e^{-\theta x};x>0,\theta>0$	Shanker (2016 b)
Devya	$f(x;\theta)=\dfrac{\theta^5}{\theta^4+\theta^3+2\theta^2+6\theta+24}(1+x+x^2+x^3+x^4)e^{-\theta x};x>0,\theta>0$ $F(x,\theta)=1-\left[1+\dfrac{\theta^4(x^4+x^3+x^2+x)+\theta^3(4x^3+3x^2+2x)+6\theta^2(2x^2+x)+24\theta x}{\theta^4+\theta^3+2\theta^2+6\theta+24}\right]e^{-\theta x}$	Shanker (2016 c)
Shambhu	$f(x,\theta)=\dfrac{\theta^6}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}(1+x+x^2+x^3+x^4+x^5)e^{-\theta x}$ $F(x,\theta)=1-\left[1+\dfrac{\theta^5(x^5+x^4+x^3+x^2+x)+\theta^4(5x^4+4x^3+3x^2+2x)+2\theta^3(10x^3+6x^2+3x)+12\theta^2(5x^2+2x)+120\theta x}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}\right]e^{-\theta x}$	Shanker (2016 d)
Aradhana	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x} \quad ;x>0,\ \theta>0$ $F(x;\theta)=1-\left[1+\dfrac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x} \quad ;x>0,\theta>0$	Shanker (2016 e)

Akshya	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3 e^{-\theta x}\quad ;x>0,\ \theta>0$ $F(x;\theta)=1-\left[1+\dfrac{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta(\theta^2+2\theta+2)x}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x}$	Shanker (2017 c)

Dispersion of One Parameter Lifetime Distributions

The conditions of dispersion of one parameter lifetime distributions for values of the parameter θ have been presented in Table 2.

Table 2: Over-dispersion, equi-dispersion and under-dispersion of one parameter lifetime distributions for values of their parameter θ.

Distributions	Over-dispersion $(\mu<\sigma^2)$	Equi-dispersion $(\mu=\sigma^2)$	Under-dispersion $(\mu>\sigma^2)$
Exponential	$\theta<1$	$\theta=1$	$\theta>1$
Lindley	$\theta<1.170086487$	$\theta=1.170086487$	$\theta>1.170086487$
Shanker	$\theta<1.171535555$	$\theta=1.171535555$	$\theta>1.171535555$
Akash	$\theta<1.515400063$	$\theta=1.515400063$	$\theta>1.515400063$
Rama	$\theta<1.950164618$	$\theta=1.950164618$	$\theta>1.950164618$
Suja	$\theta<2.493120984$	$\theta=2.493120984$	$\theta>2.493120984$
Sujatha	$\theta<1.364271174$	$\theta=1.364271174$	$\theta>1.364271174$
Amarendra	$\theta<1.525763580$	$\theta=1.525763580$	$\theta>1.525763580$
Devya	$\theta<1.451669994$	$\theta=1.451669994$	$\theta>1.451669994$
Shambhu	$\theta<1.149049973$	$\theta=1.149049973$	$\theta>1.149049973$
Aradhana	$\theta<1.283826505$	$\theta=1.283826505$	$\theta>1.283826505$
Akshya	$\theta<1.327527885$	$\theta=1.327527885$	$\theta>1.327527885$

Maximum likelihood estimation of parameter of one parameter lifetime distributions

Assuming $(x_1,x_2,...,x_n)$ as a random sample of size n from the respective distribution, the log likelihood function to find maximum likelihood estimate (MLE) of parameter of one parameter lifetime distributions is presented in the following table 3. Note that these log likelihood function have been maximized using R software for the MLE of parameter of respective distribution (Table 3).

Table 3: Log likelihood functions for one parameter lifetime distributions.

Distribution	Log Likelihood function
Exponential	$\ln L=n\ln\theta-n\theta\bar{x}$
Lindley	$\ln L=n\ln\left(\dfrac{\theta^2}{\theta+1}\right)+\sum_{i=1}^{n}\ln(1+x_i)-n\theta\bar{x}$
Shanker	$\log L=n\ln\left(\dfrac{\theta^2}{\theta^2+1}\right)+\sum_{i=1}^{n}\ln(\theta+x_i)-n\theta\bar{x}$
Akash	$\ln L=n\ln\left(\dfrac{\theta^3}{\theta^2+2}\right)+\sum_{i=1}^{n}\ln\left(1+x_i^{\,2}\right)-n\theta x$
Rama	$\ln L=n\ln\left(\dfrac{\theta^4}{\theta^3+6}\right)+\sum_{i=1}^{n}\ln\left(1+x_i^{\,3}\right)-n\theta\bar{x}$
Suja	$\ln L=n\ln\left(\dfrac{\theta^5}{\theta^4+24}\right)+\sum_{i=1}^{n}\ln\left(1+x_i^{\,4}\right)-n\theta\bar{x}$

Sujatha	$\ln L = n\ln\left(\frac{\theta^3}{\theta^2+\theta+2}\right)+\sum_{i=1}^{n}\ln\left(1+x_i+x_i^2\right)-n\theta\bar{x}$
Amarendra	$\ln L = n\ln\left(\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\right)+\sum_{i=1}^{n}\ln\left(1+x_i+x_i^2+x_i^3\right)-n\theta\bar{x}$
Devya	$\ln L = n\ln\left(\frac{\theta^5}{\theta^4+\theta^3+2\theta^2+6\theta+24}\right)+\sum_{i=1}^{n}\ln\left(1+x_i+x_i^2+x_i^3+x_i^4\right)-n\theta\bar{x}$
Shambhu	$\ln L = n\ln\left(\frac{\theta^6}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}\right)+\sum_{i=1}^{n}\ln\left(1+x_i+x_i^2+x_i^3+x_i^4+x_i^5\right)-n\theta\bar{x}$
Aradhana	$\ln L = n\ln\left(\frac{\theta^3}{\theta^2+2\theta+2}\right)+2\sum_{i=1}^{n}\ln\left(1+x_i\right)-n\theta\bar{x}$
Akshya	$\ln L = n\ln\left(\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\right)+3\sum_{i=1}^{n}\ln\left(1+x_i\right)-n\theta\bar{x}$

Applications

The one parameter lifetime distributions have been fitted to a number of real lifetime data to test their goodness of fit. The goodness of fit of these one parameter life time distributions based on maximum likelihood estimates are presented in this section.

In order to compare one parameter life time distributions values of $-2\ln L$, AIC (Akaike Information Criterion), K-S Statistics (Kolmogorov-Smirnov Statistics) for all five real lifetime datasets have been computed. The formulae for computing AIC and K-S Statistics are as follows:

$AIC = -2\ln L + 2k$, and $D = \operatorname{Sup}_x \left|F_n(x) - F_0(x)\right|$, where k the number of parameters, n is the sample size and $F_n(x)$ is the empirical distribution function. The best distribution corresponds to lower values of $-2\ln L$, AIC and K-S statistics. The following real lifetime datasets have been considered.

Data set 1: This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105).

1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7 4.1 1.8 1.5 1.2 1.4 3 1.7 2.3 1.6 2

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller et al (1994):

18.83 20.8 21.657 23.03 23.23 24.05 24.321 25.5 25.52 25.8 26.69 26.77 26.78 27.05 27.67 29.9 31.11 33.2 33.73 33.76 33.89 34.76 35.75 35.91 36.98 37.08 37.09 39.58 44.045 45.29 45.381

Data Set 3: The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm (Bader and Priest, 1982):

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997 2.006 2.021 2.027 2.055 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490 2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.858

The ML estimates, $-2\ln L$, AIC , K-S and p-value of the respective distribution for data sets 1, 2, and 3 have been presented in tables 4, 5, and 6. The best distribution has been marked with bold (Tables 4-6).

Table 4: MLE's, -2ln L, AIC K-S Statistics and p value of the fitted distributions of data sets 1.

Distributions	ML estimates	$-2\ln L$	AIC	K-S	p-value
Exponential	$\hat{\theta}=0.526314$	65.67	67.67	0.471	0.0002
Lindley	$\hat{\theta}=0.81611$	60.49	62.49	0.450	0.0012
Shanker	$\hat{\theta}=0.80387$	59.78	61.78	0.442	0.0008
Akash	$\hat{\theta}=1.15692$	59.52	61.52	0.442	0.0007
Rama	$\hat{\theta}=1.52133$	59.70	61.70	0.468	0.0003
Suja	$\hat{\theta}=1.89537$	60.40	62.40	0.493	0.0001
Sujatha	$\hat{\theta}=1.13674$	57.49	59.49	0.442	0.0007

Amarendra	$\hat{\theta}=1.48076$	55.63	57.63	0.465	0.0003
Devya	$\hat{\theta}=1.84192$	54.50	56.50	0.548	0.0000
Shambhu	$\hat{\theta}=2.21539$	53.89	55.89	0.504	0.0000
Aradhana	$\hat{\theta}=1.12319$	56.37	58.37	0.442	0.0008
Akshya	$\hat{\theta}=1.44168$	**53.01**	**55.01**	**0.464**	**0.0000**

Table 5: MLE's, -2ln L, AIC K-S Statistics and p value of the fitted distributions of data sets 2.

Distributions	ML estimates	$-2\ln L$	AIC	K-S	p-value
Exponential	$\hat{\theta}=0.03244$	274.52	276.52	0.458	0.0000
Lindley	$\hat{\theta}=0.06299$	253.98	255.98	0.365	0.0000
Shanker	$\hat{\theta}=0.64716$	252.35	254.35	0.358	0.0004
Akash	$\hat{\theta}=0.09706$	240.68	242.68	0.298	0.0059
Rama	$\hat{\theta}=0.12978$	232.79	234.79	0.253	0.0301
Suja	$\hat{\theta}=0.16227$	**227.25**	**229.25**	**0.223**	**0.0774**
Sujatha	$\hat{\theta}=0.09561$	241.50	243.50	0.303	0.0051
Amarendra	$\hat{\theta}=0.12829$	233.41	235.41	0.257	0.0269
Devya	$\hat{\theta}=0.16087$	227.68	229.68	0.422	0.0000
Shambhu	$\hat{\theta}=0.19339$	223.39	225.39	0.199	0.1477
Aradhana	$\hat{\theta}=0.09432$	242.22	244.22	0.306	0.0044
Akshya	$\hat{\theta}=0.12574$	234.44	236.44	0.262	0.0223

Table 6: MLE's, -2ln L, AIC K-S Statistics and p value of the fitted distributions of data sets 3.

Distributions	ML estimates	$-2\ln L$	AIC	K-S	p-value
Exponential	$\hat{\theta}=0.40794$	261.73	263.73	0.448	0.0000
Lindley	$\hat{\theta}=0.65450$	238.38	240.38	0.401	0.0000
Shanker	$\hat{\theta}=0.65803$	233.00	235.00	0.369	0.0000
Akash	$\hat{\theta}=0.96472$	224.27	226.27	0.362	0.0000
Rama	$\hat{\theta}=1.30241$	211.21	213.21	0.324	0.0000
Suja	$\hat{\theta}=1.65273$	199.21	201.21	0.291	0.0000
Sujatha	$\hat{\theta}=0.93612$	221.60	223.60	0.364	0.0000
Amarendra	$\hat{\theta}=1.24425$	207.95	209.95	0.332	0.0000
Devya	$\hat{\theta}=1.57229$	**196.15**	**198.15**	**0.519**	**0.0000**
Shambhu	$\hat{\theta}=1.91529$	185.74	187.74	0.278	0.0000
Aradhana	$\hat{\theta}=0.91702$	219.90	221.90	0.364	0.0000
Akshya	$\hat{\theta}=1.18939$	204.89	206.89	0.336	0.0000

Concluding Remarks

The analysis and modeling of lifetime data are crucial in almost all applied sciences including medicine, insurance, engineering, and finance, amongst others. In the present paper three examples of observed real lifetime datasets have been considered for testing the goodness of fit of various one parameter lifetime distributions. The index of dispersion of these one parameter lifetime distributions has been presented which will give idea about the distribution to be used as per the dispersion of the data. This study would be of great help to scientist working in biomedical science and engineering for modeling waiting time and survival time data in their respective fields of knowledge.

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Conflict of Interest

No conflict of interest.

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