Review Article

Maximum Likelihood Estimation in the Ampadu-G Family of Distributions

Clement Boateng Ampadu*

Department of Biostatistics, USA

*Corresponding author: Clement Boateng Ampadu, Department of Biostatistics, USA.

Abstract

In this paper the maximum likelihood estimation for parameters in the Ampadu-G family of distributions [1] is discussed. When G is given by the normal distribution, we assess the performance of the estimation procedure via simulation study carried out in the software package R [2]. As a further development we conjecture the methods of maximum product spacings and least squares are adequate for estimating the parameters in the Ampadu-G family of distributions.

Keywords: Ampadu-G; Normal distribution; Maximum likelihood estimation; Ordinary least squares estimation; Maximum product spacings estimation; Monte carlo simulation

Introduction and Preliminaries

At first, we recall the following

Definition 3.1: [1] Let $\lambda > 0, \xi > 0$ be a parameter vector all of whose entries are positive, and $x \in \mathbb{R}$. A random variable X will be said to follow the Ampadu-G family of distributions if the CDF is given by

$$F(x; \lambda, \xi) = \frac{1 - e^{-\lambda G(x;\xi)^2}}{1 - e^{-\lambda}}$$

and the PDF is given by

$$f(x; \lambda, \xi) = \frac{2\lambda g(x;\xi) G(x;\xi) e^{-\lambda G(x;\xi)^3}}{1 - e^{-\lambda}}$$

where the baseline distribution has CDF $G(x;\xi)$ and PDF $g(x;\xi)$

The Quantile

Theorem 4.1. Let $Q_G(\bullet) = G^{-1}(\bullet)$ denote the quantile of some base distribution with CDF G and PDF g, $\lambda > 0$, and $0 < u < 1$. The quantile function of the Ampadu-G family of distributions is given by

$$Q_G\left[\left\{\frac{1}{\lambda}\ln\left(\frac{1}{1 - u + ue^{-\lambda}}\right)\right\}^{\frac{1}{2}}\right]$$

Proof. Let $0 < u < 1$. Since $Q_G(\bullet) = G^{-1}(\bullet)$ the quantile function can be obtained by solving the following equation for y

$$u = \frac{1 - e^{-\lambda G(y)^2}}{1 - e^{-\lambda}}$$

Random Number Generation

If $u \sim U(0,1)$, that is, u is a uniform random variable, then a random sample from the Ampadu-G family of distributions can be obtained via

$$X = Q_G\left[\left\{\frac{1}{\lambda}\ln\left(\frac{1}{1 - u + ue^{-\lambda}}\right)\right\}^{\frac{1}{2}}\right]$$

Where $Q_G(\bullet) = G^{-1}(\bullet)$ denote the quantile of some base distribution with CDF G and PDF g, and $\lambda > 0$

Parameter Estimation

In this section, we obtain the maximum likelihood estimators (MLEs) for the parameters of the Ampadu-G family of distributions. For this, let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the Ampadu-G family of distributions. The likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{2\lambda g(x_i;\xi) G(x_i;\xi) e^{-\lambda G(x_i;\xi)^2}}{1 - e^{-\lambda}}$$

$$= (2\lambda)^n \prod_{i=1}^{n} \{g(x_i;\xi) G(x_i;\xi)\} e^{-\lambda \sum_{i=1}^{n} G(x_i;\xi)^2} (1 - e^{-\lambda})^{-n}$$

From the above the log-likelihood function is given by

$$\ln L = n\ln(2\lambda) + \sum_{i=1}^{n} \ln\{g(x_i;\xi) G(x_i;\xi)\} - \lambda \sum_{i=1}^{n} G(x_i;\xi)^2 - n\ln(1 - e^{-\lambda})$$

The MLE's of ξ and λ can be obtained by maximizing the equation immediately above. The derivatives of the equation immediately above with respect to the unknown parameters are given as follows

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} G(x_i;\xi)^2 - \frac{ne^{-\lambda}}{1 - e^{-\lambda}}$$

$$\frac{\partial \ln L}{\partial \xi} = \sum_{i=1}^{n} \frac{\frac{\partial g(x_i;\xi)}{\partial \xi} G(x_i;\xi) + \frac{\partial G(x_i;\xi)}{\partial \xi} g(x_i;\xi)}{g(x_i;\xi) G(x_i;\xi)} - 2\lambda \sum_{i=1}^{n} G(x_i;\xi) \frac{\partial G(x_i;\xi)}{\partial \xi}$$

Now solving the system below for ξ and λ gives the maximum likelihood estimators, $\hat{\lambda}$ and $\hat{\xi}$ of the unknown parameters:

$$\frac{\partial \ln L}{\partial \lambda} = 0 \text{ and } \frac{\partial \ln L}{\partial \xi} = 0$$

Simulation Study

In this section, a Monte Carlo simulation study is carried out to assess the performance of the estimation method, when G is given by the normal distribution. First, we present the CDF and PDF of the Ampadu-Normal distribution.

Theorem 7.1. The CDF of the Ampadu-Normal family of distributions is given by

$$\frac{1 - e^{-\frac{1}{4}\lambda erfc\left(\frac{c-x}{\sqrt{2d}}\right)^2}}{1 - e^{-\lambda}}$$

where $erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$

By differentiating the CDF in the previous theorem, we get the following

Theorem 7.2. The PDF of the Ampadu-Norma family of distribution is given by

$$\frac{\lambda erfc\left(\frac{c-x}{\sqrt{2d}}\right) \exp\left(-\frac{1}{4}\lambda erfc\left(\frac{c-x}{\sqrt{2d}}\right)^2 - \left(\frac{c-x}{\sqrt{2d}}\right)^2\right)}{\sqrt{2\pi}(1 - e^{-\lambda})d}$$

Where $erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$

Remark 7.3. If a random variable W follows the Ampadu-Normal family of distributions, we write $W \sim AN(\lambda, c, d)$ (Figure 1 &2).
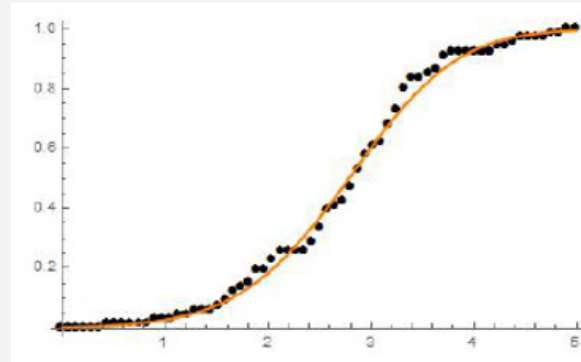

Figure 1: The CDF of AN (9.326129, 3.864083, 1.789715) fitted to empirical distribution of Table 1 [3].
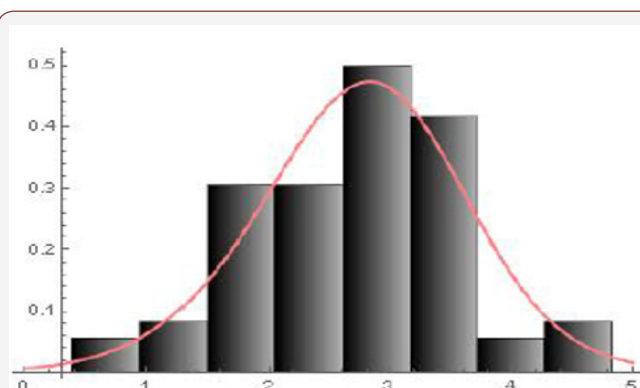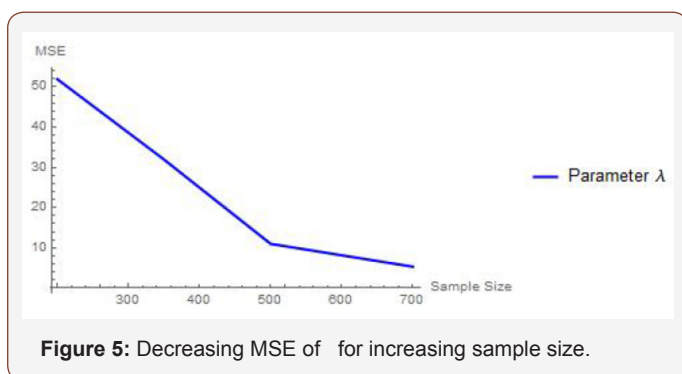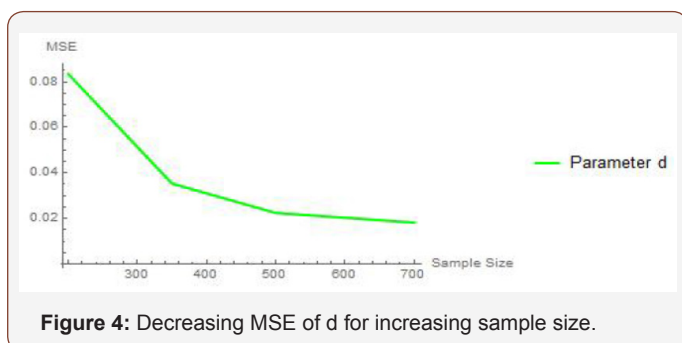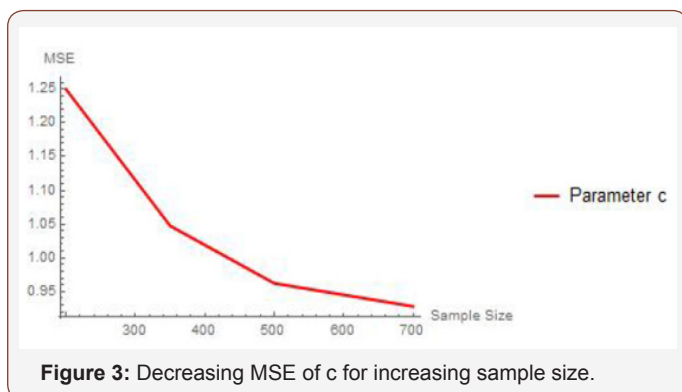

Figure 2: The PDF of AN (9.326129, 3.864083, 1.789715) fitted to histogram of Table 1 [3].

Samples of sizes 200, 350, 500, and 700, are drawn from the Ampadu-Normal family of distributions. The samples have been drawn for $(\lambda, c, d) = (9.3, 3.8, 1.7)$, and the maximum likelihood estimators for the parameters λ, c, and d are obtained. The procedure has been repeated 200 times and the mean and mean square error for the estimates are computed, and the results are summarized in Table 1 below.

Table 1: Result of Simulation Study.

Parameter λ		
Sample Size	Average Estimate	MSE
200	11.02128	51.92315
350	11.10743	31.96127
500	10.06477	10.99207
700	9.476436	5.315401
Parameter c		
Sample Size	Average Estimate	MSE
200	3.834995	1.250143
350	3.834198	1.047542
500	3.83159	0.962348
700	3.829701	0.927999
Parameter d		
Sample Size	Average Estimate	MSE
200	1.67182	0.083467
350	1.727176	0.035323
500	1.724912	0.022401
700	1.720403	0.018161

Figure 3: Decreasing MSE of c for increasing sample size.

Figure 4: Decreasing MSE of d for increasing sample size.

Figure 5: Decreasing MSE of   for increasing sample size.

From Table 1 above, we find that the simulated estimates are close to the true values of the parameters and hence the estimation method is adequate. We have also observed that the estimated mean square errors (MSEs) consistently decrease with increasing sample size as seen in the Figures below (Figure 3,4,5).

Concluding Remarks and Further Developments

In this paper we have shown that estimating the parameters in the Ampadu-Normal family of distributions by the method of maximum likelihood is adequate via a brief simulation study. As a further development we propose the following

Exercise 8.1. Conduct a simulation study to determine if the methods of maximum product spacings, and ordinary least squares are adequate in estimating the parameters in the Ampadu-Normal family of distributions.

Remark 8.2. The methods of maximum product spacings and ordinary least squares are discussed in [4].

Acknowledgement

None.

Conflict of Interest

No conflict of interest.

References

1. Clement Boateng Ampadu (2019) The Ampadu-G Family of Distributions with Application to the T-X(W) Class of Distributions. Annal Biostat and Biomed Appli 1(4).

2. R version 3.5.2 (2018). Copyright (C) 2018 The R Foundation for Statistical Computing.

3. Ayman Alzaatreh, Carl Lee and Felix Famoye (2014) T-normal family of distributions: a new approach to generalize the normal distribution. Journal of Statistical Distributions and Applications 1:16

4. Zubair Ahmad (2018) The Zubair-G Family of Distributions: Properties and Applications. Annals of Data Science.