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Review Article

The Ampadu $APT-qT-X$ Family of Distributions Induced by V with an Illustration to Data in the Health Sciences

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Abstract

The alpha power transform family of distributions ($APT-G$ for short) appeared in [1], and since then variants of it has been proposed, and for example, see [2]. In this paper we propose another variant of the $APT-G$ family of distributions called the Ampadu $APT-G$ family of distributions. Assuming G is given by the CDF of $qT-X$ family of distributions induced by V , and for example, see [3] and [4], we show sub-models of the Ampadu $APT-qT-X(V)$ family of distributions is significant in modeling data in the health sciences, in particular, the breast cancer patients data contained in [5]. As a further development of the $APT-G$ family of distributions, we propose a so-called $\left(\frac{1}{e}\right)^{\alpha}$ power transform family of distributions($\left(\frac{1}{e}\right)^{\alpha}$ PT-G for short).

Keywords: Alpha power transform; Ampadu-G; Quantile generated family of distributions

Background on qT –X Family of Distributions Induced by V

This section is inspired by [3] and [4]

Definition 3.1. Let V be any function such that the following holds:

a) $F(x)\in\left[V(a),V(b)\right]$

b) $F(x)$ is differentiable and strictly increasing

c) $\lim_{x\to-\infty} F(x)=V(a)$ and $\lim_{x\to-\infty} F(x)=V(b)$

then the CDF of the qT – X family induced by V is given by

$$K(x)=\int_a^{V(F(x))}\frac{1}{r(Q(t))}dt$$

where $\frac{1}{r(Q(t))}$ is the quantile density function of random variable $T\in\left[a,b\right]$, for $-\infty\le a<b\le\infty$, and $F(x)$ is the CDF of any random variable X.

Theorem 3.2. The CDF of the qT – X family induced by V is given by

$$K(x)=Q\left[V\left(F(x)\right)\right]$$

Proof. Follows from the previous definition and noting that $Q'=\frac{1}{r^{o}Q}$

Theorem 3.3. The PDF of the qT – X family induced by V is given by

$$k(x)=\frac{f(x)}{r\left[Q\left(V\left(F(x)\right)\right)\right]}V'\left[F(x)\right]$$

Proof. $k=K',Q'=\frac{1}{r^{o}Q},F'=f$, and K is given by Theorem 3.5.2

Remark 3.4. When the support of T is $\left[a,\infty\right)$, where $a\ge0$, we can take V as follows

a) $V(x)=1-e^{-x}$

b) $V(x)=\frac{x}{1+x}$

c) $V(x)=\left[1-e^{-x}\right]^{\frac{1}{\alpha}}, where\alpha>0$

d) $V(x)=\left[\frac{x}{1+x}\right]^{\frac{1}{\alpha}}, where \alpha>0$

Remark 3.5. When the support of T is $(-\infty,\infty)$, we can take V as follows

1. $V(x)=1-e^{-e^{-x}}$
2. $V(x)=\frac{e^{-x}}{1+e^{-x}}$
3. $V(x)=\left[1-e^{-e^{-x}}\right]^{\frac{1}{\alpha}}, where \alpha>0$
4. $V(x)=\left[\frac{e^{-x}}{1+e^{-x}}\right]^{\frac{1}{\alpha}}, where \alpha>0$

The New Family of Distributions

The Ampadu APT-G aamily of distributions

Definition 4.1. A random variable X will be called Ampadu APT-G distributed if the

CDF is given by

$$F(x;\alpha,\xi)=\frac{e^{-G(x;\xi)}-\alpha^{G(x;\xi)}}{e^{-1}-\alpha}$$

and the PDF is given by

$$f(x;\alpha,\xi)=\frac{g(x;\xi)\alpha^{G(x;\xi)}-(-\log(\alpha))-e^{-G(x;\xi)}}{\frac{1}{e}-\alpha}$$

Where $\xi>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e},x\in R$,and G is some baseline distribution.

Application to the qT – X family of distributions induced by V

Definition 4.2. A random variable J_1 will be called Ampadu APT – qT – X distributed of type I if the CDF is given by

$$F(x;\alpha,\beta,\xi)=\frac{e^{-Q_T\left(G(x;\xi)^{\frac{1}{\beta}}\right)}-\alpha^{Q_T\left(G(x;\xi)^{\frac{1}{\beta}}\right)}}{e^{-1}-\alpha}$$

Where $\xi>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e},x\in R$, G is some baseline distribution associated with the

random variable X, $\beta>0$, and the random variable T with support [0, 1] has quantile Q_T

Remark 4.3. The PDF can be obtained by differentiating the CDF above

Definition 4.4. A random variable J_2 will be called Ampadu APT – qT – X distributed of type II if the CDF is given either by

$$F(x;\alpha,\beta,\xi)=\frac{e^{-Q_T\left(1-e^{-G(x;\xi)^{\frac{1}{\beta}}}\right)}-\alpha^{Q_T\left(1-e^{G(x;\xi)^{\frac{1}{\beta}}}\right)}}{e^{-1}-\alpha}$$

Or

$$F(x;\alpha,\beta,\xi)=\frac{e^{-Q_T\left(\left(\frac{G(x;\xi)}{1+G(x;\xi)}\right)^{\frac{1}{\beta}}\right)}-\alpha^{Q_T\left(\left(\frac{G(x;\xi)}{1+G(x;\xi)}\right)^{\frac{1}{\beta}}\right)}}{e^{-1}-\alpha}$$

where $\xi>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e},x\in\mathbb{R}$, G is some baseline distribution associated with the random variable X, $\beta>0$, and the random variable T with support [0,1] has quantile QT

Remark 4.5. The PDF can be obtained by differentiating the CDF immediately above

Definition 4.6. A random variable J_3 will be called Ampadu APT – qT – X distributed

of type III if the CDF is given either by

$$F(x;\alpha,\beta,\xi)=\frac{e^{-Q_T\left(\left[1-e^{-e^{G(x;\xi)}}\right]^{\frac{1}{\beta}}\right)}-\alpha^{Q_T\left(\left[1-e^{-e^{G(x;\xi)}}\right]^{\frac{1}{\beta}}\right)}}{e^{-1}-\alpha}$$

Or

$$F(x;\alpha,\beta,\xi)=\frac{e^{-Q_T\left(\left[\frac{e^{G(x;\xi)}}{1+e^{G(x;\xi)}}\right]^{\frac{1}{\beta}}\right)}-\alpha^{Q_T\left(\left[\frac{e^{G(x;\xi)}}{1+e^{G(x;\xi)}}\right]^{\frac{1}{\beta}}\right)}}{e^{-1}-\alpha}$$

where $\xi>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e},x\in R$, G is some baseline distribution associated with the

random varaible X, $\beta>0$, and the random variable T with support $(-\infty,\infty)$ has quantileQT

Remark 4.7. The PDF can be obtained by differentiating the CDF immediately above

An Illustration to Breast Cancer Patients Data

Application of Definition 4.2

We assume $\beta=1$, T is uniform on [0, 1], and the random variable X follows the Weibull distribution, so that the CDF of X is given by

$$G(x;a,b)=1-e^{-\left(\frac{x}{b}\right)^a}$$

and the PDF of X is given by

$$g(x;a,b)=\frac{ae^{-\left(\frac{x}{b}\right)^a}-\left(\frac{x}{b}\right)^{a-1}}{b}$$

Now from Definition 4.2, we have the following

Theorem 5.1. The CDF of the Ampadu APT-{Quantile Uniform-Weibull} distribution of type I is given by

$$F(x;a,b,\alpha)=\frac{e^{e^{-\left(\frac{x}{b}\right)^{a}}-1}-\alpha^{1-e^{-\left(\frac{x}{b}\right)^{a}}}}{\frac{1}{e}-\alpha}$$

Where $x,a.b>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e}$

Remark 5.2. If a random variable B (say) has CDF given by the previous theorem we Write $B\sim \text{AAPTQCW}(a,b,\alpha)$ (Figure 1)

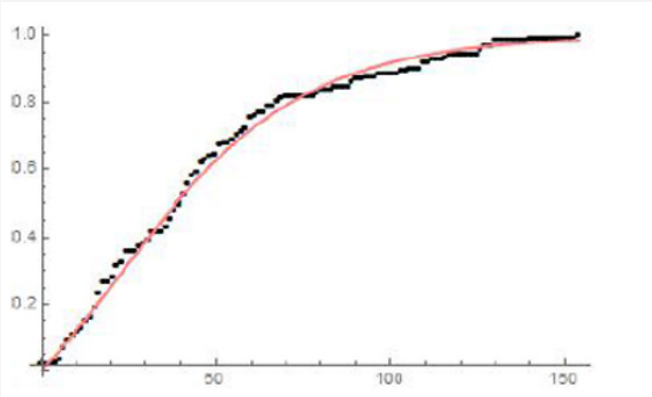

Figure 1: The CDF of AAPTQUW (1.00786, 30.0313, 10.0178) fitted to the empirical distribution of the data on patients with breast cancer, Section 5.2 [5].

By differentiating the CDF given by Theorem 5.1, we get the following

Theorem 5.3. The PDF of the Ampadu APT-{Quantile Uniform-Weibull} distribution

of type I is given by

$$F(x;a,b,\alpha)=\frac{-\frac{a\log(\alpha)e^{-\left(\frac{x}{a}\right)^{a}}\left(\frac{x}{a}\right)^{a-1}\alpha^{1-e^{-\left(\frac{x}{a}\right)^{a}}}}{b}-\frac{ae^{-\left(\frac{x}{a}\right)^{a}+e^{-\left(\frac{x}{a}\right)^{a}}-1}+\left(\frac{x}{a}\right)^{a-1}}{b}}{\frac{1}{e}-\alpha}$$

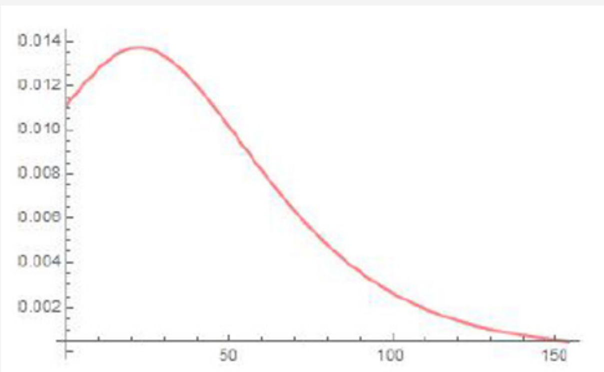

Figure 2: The PDF of AAPTQUW(1.00786, 30.0313, 10.0178) over [0.3, 154].

Where $x,a.b>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e}$ (Figure 2)

Application of definition 4.4

We assume T is a fully specified exponential distribution. In particular, we assume T has CDF

$$1-e^{-0.714286t}$$

so that the quantile of T in this case is given by

$$Q_T(t)=-1.4\log(1-t)$$

We assume $\beta=1$ and consider X to be Weibull distributed with CDF and PDF as defined

in the previous section. The second CDF in Definition 4.4 implies the following

Theorem 5.4. The CDF of the Ampadu APT-{Quantile Exponential-Weibull} distribution

of type II is given by

$$F(x;a,b,\alpha)=\frac{\left(1-\frac{1-e^{-\left(\frac{x}{a}\right)^{a}}}{2-e^{-\left(\frac{x}{a}\right)^{a}}}\right)^{1.4}-\alpha^{-1.1\log\left(1-\frac{1-e^{-\left(\frac{x}{a}\right)^{a}}}{2-e^{-\left(\frac{x}{a}\right)^{a}}}\right)}}{\frac{1}{e}-\alpha}$$

$$x,a.b>0,\alpha>\frac{1}{e},\alpha\neq\frac{1}{e}$$

Remark 5.5. If a random variable L has CDF given by the theorem immediately above, we write

$L\sim \text{AAPTQEW}(a,b,\alpha)$ (Figure 3)

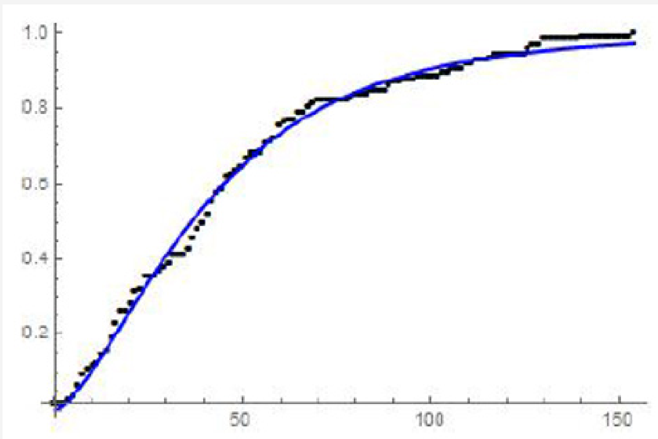

Figure 3: The CDF of AAPTQEW(1.53908, 80.6998, 0.68369) fitted to the empirical distribution of the data on patients with breast cancer, Section 5.2 [5].

Remark 5.6. By differentiating the CDF of the AAPTQEW distribution, the PDF can be obtained (Figure 4).

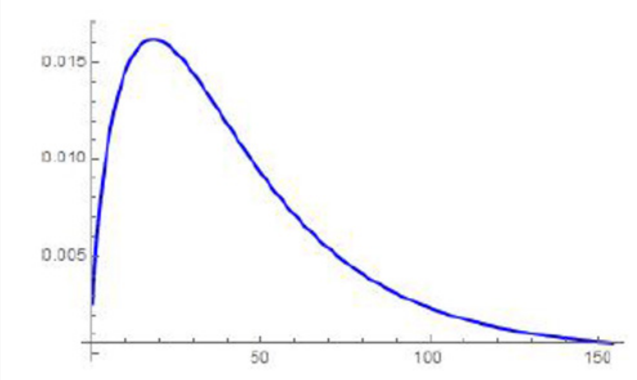

Figure 4: The PDF of AAPTQEW(1.53908, 80.6998, 0.68369) over [0.3, 154].

Application of Definition 1.6

We assume $\beta = 1, \alpha = 1.3$, X is Weibull distributed with CDF and PDF as defined in Section 5.1, and T is a fully specified Cauchy distribution. In particular, we assume T has CDF

$$\frac{\tan^{-1}(0.869565t)}{\pi} + \frac{1}{2}$$

so that the quantile of T is given by

$$Q_T(t) = 1.15 \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$$

The second CDF in Definition 3.6 implies the following

Theorem 5.7. The CDF of the Ampadu APT-{Quantile Cauchy-Weibull} distribution of type III is given by

$$F(x:a,b) = -1.07282\left(\exp\left(-1.15\tan\left(\pi\left(\frac{e^{1-e^{-\left(\frac{x}{a}\right)^{a}}}}{e^{1-e^{-\left(\frac{x}{a}\right)^{a}}}+1} - \frac{1}{2}\right)\right)\right) - 1.3^{1.15\tan\left(\pi\left(\frac{e^{1-e^{-\left(\frac{x}{a}\right)^{a}}}}{e^{1-e^{-\left(\frac{x}{a}\right)^{a}}}+1} - \frac{1}{2}\right)\right)}\right)$$

where x, a, b > 0

Remark 5.8. If a random variable V has CDF given by the theorem immediately above,

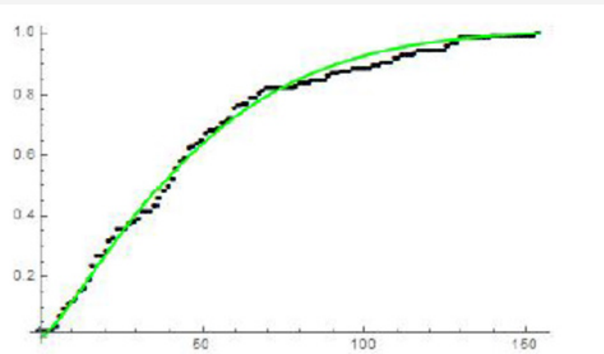

Figure 5: The CDF of AAPTQCW(1.36338, 52.8276) fitted to the empirical distribution of the data on patients with breast cancer, Section 5.2 [5]

we write $V \sim \text{AAPTQCW}(a,b)$ (Figure 5)

Remark 5.9. By differentiating the CDF of the AAPTQCW distribution, the PDF can be obtained (Figure 6)

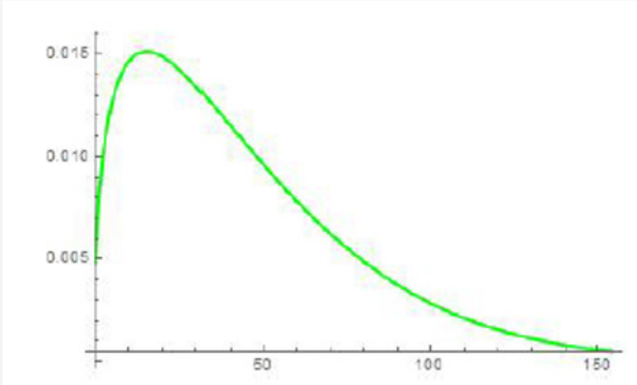

Figure 6: The PDF of AAPTQCW(1.36338, 52.8276) over [0.3, 154].

Further Developments and Concluding Remarks

Inspired by the APT-G distribution in [1], we ask the reader to investigate properties and applications of a so-called $\left(\frac{1}{e}\right)^{\alpha}$ power transform distribution $\left(\frac{1}{e}\right)^{\alpha}$ PT-G for short)

Definition 6.1. A random variable Z will be called $\left(\frac{1}{e}\right)^{\alpha}$ PT-G distributed if the CDF is given by

$$F(x;\alpha,\xi) = \frac{1 - e^{-\alpha G(x;\xi)}}{1 - e^{-\alpha}}$$

For $\alpha \neq \frac{1}{e}, \alpha > \frac{1}{e}, x \in \mathbb{R} and \xi > 0$

Remark 6.2. The PDF can be obtained by differentiating the CDF

The new distribution appears practically significant in modeling real-life data as shown below. We assume G is given by the CDF of the Weibull distribution, as in Section 5.1. Thus, from the above definition the following is immediate

Theorem 6.3. The CDF of the $\left(\frac{1}{e}\right)^{\alpha}$ PT-Weibull distribution is given by

$$F(x;a,b,\alpha) = \frac{1 - e^{-\alpha\left(1 - e^{-\left(\frac{x}{b}\right)^{a}}\right)}}{1 - e^{-\alpha}}$$

For $x, a, b > 0, \alpha \neq \frac{1}{e} and \alpha > \frac{1}{e}$

Remark 6.4. If a random variable M has CDF given by the theorem immediately above,

we write $M \sim \left(\frac{1}{e}\right)^{\alpha} PTW(a,b,\alpha)$ (Figure 7)

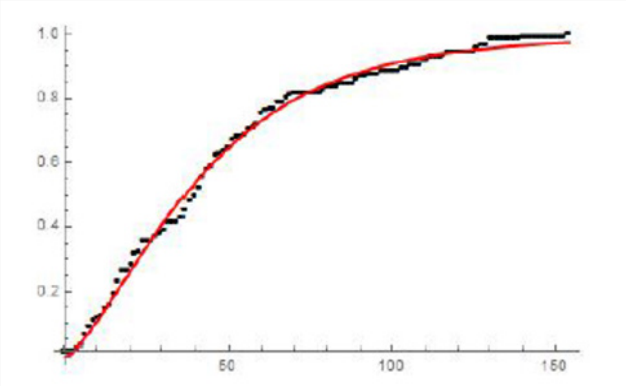

Figure 7: The CDF of $\left(\frac{1}{e}\right)^{\alpha}$ PTW (1.47236, 117.014, 4.06969) fitted to the empirical distribution of the data on patients with breast cancer, Section 5.2 [5].

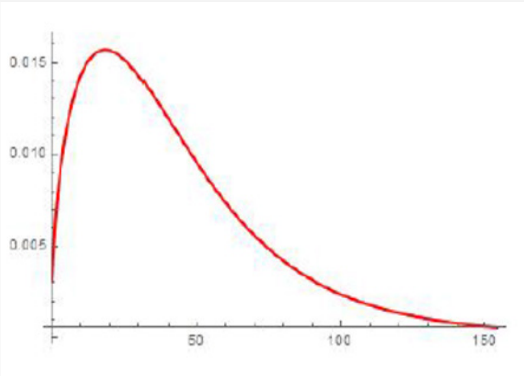

Figure 8: The PDF of $\left(\frac{1}{e}\right)^{\alpha}$ PTW (1.47236, 117.014, 4.06969) over [0.3, 154].

Remark 6.5. By differentiating the CDF of the $\left(\frac{1}{e}\right)^{\alpha}$ PTW distribution, the PDF can be obtained (Figure 8).

Whilst we have shown the distributions are practically significant in the health sciences, we hope they find applications in other disciplines. Also we hope the researchers will further develop the mathematical and statistical properties of these distributions.

Acknowledgement

None.

Conflict of Interest

No conflict of interest.

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