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Review Article

The Ampadu-G Family of Distributions with Application to the $T-X(W)$ Class of Distributions

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Abstract

The $T-X(W)$ family of distributions appeared in [1]. In this paper, inspired by the structure of the CDF in the Zubair-G class of distributions [2], we introduce a new family of distributions called the Ampadu-G class of distributions, and use it to obtain a new class of distributions which we will call the $A_T-X(W)$ class of distributions, as a further application of the $T-X(W)$ framework. Sub-models of the Ampadu-G class of distributions and the $A_T-X(W)$ class of distributions are shown to be practically significant in modeling real-life data. The Ampadu-G class of distributions is seen to be strikingly similar in structure to the Exponentiated EP (EEP) model contained in [3], and the Zubair-G class of distributions is seen to be strikingly similar in structure to the Complementary exponentiated Weibull-Poisson (CEWP) model contained in [4].

Keywords: Zubair-G; $T-X(W)$ family of distributions; Ampadu-G

Introduction

Background on the $T-X(W)$ family of distributions

Definition 3.1: [1] Let $r(t)$ be the PDF of a continuous random variable $T\in[a,\ b]$ for $-\infty\le a\le b\le\infty$ and let $R(t)$ be its CDF. Also let the random variable X have CDF $F(x)$ and PDF $f(x)$, respectively. A random variable B is said to be $T-X(W)$ distributed if the CDF can be written as the following integral

$$\int_a^{W(F(x))} r(t)\,dt = R\left(W\left(F(x)\right)\right)$$

where $W\left(F(x)\right)$ satisfies the following conditions

a) $W\left(F(x)\right)\in[a,\ b]$

b) $W\left(F(x)\right)$ is differentiable and monotonically non-decreasing

c) $\lim_{x\to-\infty} W\left(F(x)\right)=a$ and $\lim_{x\to\infty} W\left(F(x)\right)=b$

Remark 3.2: By differentiating the RHS of the above equation with respect to x, the PDF

of the $T-X(W)$ family of distributions can be obtained.

Remark 3.3: If the continuous random variable T has support [0, 1], we can take

$$W(x)=x^{\alpha}$$

where $\alpha>0$. In particular, we will say a random variable B is $T-X(W)$ distributed of type I, if the CDF can be written as the following integral

$$\int_0^{F(x)^{\alpha}} r(t)\,dt = R\left(F(x)^{\alpha}\right)$$

Remark 3.4: If the continuous random variable T has support $[a,\infty)$ with $a\ge 0$ we can take $W(x)=-\log\left(1-x^{\alpha}\right)$ or $W(x)=\frac{x^{\alpha}}{1-x^{\alpha}}$ where $\alpha>0$. In particular, we will say arandom variable B $T-X(W)$ distributed of type II, if the CDF can be written as either one of the following integrals

$$\int_0^{-\log\left(1-F(x)^{\alpha}\right)} r(t)\,dt = R\left(-\log\left(1-F(x)^{\alpha}\right)\right)$$

Or

$$\int_0^{\frac{F(x)^{\alpha}}{1-F(x)^{\alpha}}} r(t)\,dt = R\left(\frac{F(x)^{\alpha}}{1-F(x)^{\alpha}}\right)$$

Remark 3.5: If the continuous random variable T has support $(-\infty,\infty)$ we can take $W(x)=\log\left(-\log\left(1-x^{\alpha}\right)\right)$ or $W(x)=\log\left(\frac{x^{\alpha}}{1-x^{\alpha}}\right)$, where $\alpha>0$. In particular, we will say a random variable B is $T-X(W)$ distributed of type III, if the CDF can be written as either one of the following integrals

$$\int_{-\infty}^{\log(-\log(1-x^{\alpha}))} r(t)\,dt = R\left(\log\left(-\log\left(1-x^{\alpha}\right)\right)\right)$$

or

$$\int_{-\infty}^{\log\left(\frac{F(x)^{\alpha}}{1-F(x)^{\alpha}}\right)} r(t)\,dt = R\left(\log\left(\frac{F(x)^{\alpha}}{1-F(x)^{\alpha}}\right)\right)$$

Remark 3.6: By differentiating the RHS of the equations in Remark 3.3, Remark 3.4, and Remark 3.5, respectively, we obtain the PDF's of the class of $T-X(W)$ distributions of type I, II and III, respectively.

Background on Zubair-G family of distributions

Definition 3.7: [2] A random variable B* is said to be Zubair-G distributed if the CDF is given by

$$F(x;\alpha,\xi) = \frac{e^{\alpha G(x;\xi)^2} - 1}{e^{\alpha} - 1}$$

Where $\alpha,\xi > 0, x \in R$ and G is the CDF of the baseline distribution by differentiating the CDF in the above definition we obtain the PDF of the Zubair-G class of distributions as

$$f(x;\alpha,\xi) = \frac{2\alpha g(\alpha;\xi) G(\alpha;\xi) e^{\alpha g(\alpha;\xi)^2}}{e^{\alpha} - 1}$$

Where $\alpha,\xi > 0, x \in R$, G is the CDF of the baseline distribution, and g is the PDF of the baseline distribution

The New Family of Distributions

The Ampadu-G family of distributions

Definition 4.1: Let $\lambda > 0, \xi > 0$ be a parameter vector all of whose entries are positive, and $x \in R$. A random variable X will be said to follow the Ampadu-G family of distributions if the CDF is given by

$$F(x;\lambda,\xi) = \frac{1-e^{-\lambda G(\alpha;\xi)^2}}{1-e^{-\lambda}}$$

and the PDF is given by

$$F(x;\lambda,\xi) = \frac{2\lambda g(x;\xi) G(x;\xi) e^{-\lambda G(\alpha;\xi)^2}}{1-e^{-\lambda}}$$

where the baseline distribution has CDF $G(x,\xi)$ *and PDF* $g(x,\xi)$

Generalized $A_T - X(W)$ Family of Distributions of type I

Definition 4.2: Assume the random variable T with support [0, 1] has CDF $G(t;\xi)$ and

PDF $g(t;\xi)$. We say a random variable S is generalized $A_T - X(W)$ distributed of type I if the CDF can be expressed as the following integral

$$\int_0^{F(x,w)^{\beta}} \frac{2\lambda g(t;\xi) G(t;\xi) e^{-\lambda G(t;\xi)^2}}{1-e^{-\lambda}}\,dt = \frac{1-e^{-\lambda G\left(F(x,w)^{\beta};\xi\right)^2}}{1-e^{-\lambda}}$$

Where $\lambda,\xi,\beta > 0$ and the random variable X with parameter vector ! has CDF $F(x,w)$ and PDF $f(x,w)$

Remark 4.3: If $\beta = 1$ in the above definition we say S is $A_T - X(W)$ distributed of type I

Generalized $A_T - X(W)$ Family of Distributions of type II

Definition 4.4: Assume the random variable T with support $[a,\infty)$ has CDF $G(t,\xi)$ and

PDF $g(x,\xi)$. We say a random variable S is generalized A_T-X(W) distributed of type II if the CDF can be expressed as either one of the following integrals

$$\int_0^{-\log\left(1-F(x,w)^{\beta}\right)} \frac{2\lambda g(t;\xi) G(t;\xi) e^{-\lambda G(t;\xi)^2}}{1-e^{-\lambda}}\,dt = \frac{1-e^{-\lambda G\left(-\log\left(1-F(x,w)^{\beta}\right);\xi\right)^2}}{1-e^{-\lambda}}$$

Or

$$\int_0^{\frac{F(x,w)^{\beta}}{1-F(x,w)^{\beta}}} \frac{2\lambda g(t;\xi) G(t;\xi) e^{-\lambda G(t;\xi)^2}}{1-e^{-\lambda}}\,dt = \frac{1-e^{-\lambda G\left(\frac{F(x,w)^{\beta}}{1-F(x,w)^{\beta}}\right)^2}}{1-e^{-\lambda}}$$

where $\lambda,\xi,\beta > 0$ and the random variable X with parameter vector ! has CDF $F(x,w)$ and PDF $f(x,w)$

Remark 4.5: If $\beta = 1$ in the above definition we say S is $A_T - X(W)$ distributed of type II

Generalized $A_T - X(W)$ Family of Distributions of type III

Definition 4.6: Assume the random variable T with support $(-\infty,\infty)$ has CDF $G(t;\xi)$ and PDF $g(t;\xi)$. We say a random variable S is generalized $A_T - X(W)$ distributed of type III if the CDF can be expressed as either one of the following integrals

$$\int_{-\infty}^{\log\left(-\log\left(1-F(x,w)^{\beta}\right)\right)} \frac{2\lambda g(t;\xi) G(t;\xi) e^{-\lambda G(t;\xi)^2}}{1-e^{-\lambda}}\,dt = \frac{1-e^{-\lambda G\log\left(\left(-\log\left(1-F(x,w)^{\beta}\right);\xi\right)^2\right)}}{1-e^{-\lambda}}$$

or

$$\int_{-\infty}^{\log\left(\frac{F(x,w)^{\beta}}{1-F(x,w)^{\beta}}\right)} \frac{2\lambda g(t;\xi) G(t;\xi) e^{-\lambda G(t;\xi)^2}}{1-e^{-\lambda}}\,dt = \frac{1-e^{-\lambda G\log\left(\left(\frac{F(x,w)^{\beta}}{1-F(x,w)^{\beta}}\right)^2\right)}}{1-e^{-\lambda}}$$

where $\lambda,\xi,\beta > 0$ and the random variable X with parameter vector w has CDF $F(x,w)$ and PDF $f(x,w)$

Remark 4.7. If $\beta = 1$ in the above definition, we say S is $A_T - X(W)$ distributed of type III

Practical Application to Real-life Data

Illustration of Ampadu-G family of distributions

We consider the data set [5] which is on the breaking stress of carbon fibers of 50 mm in length. We assume the baseline distribution is Weibull distributed, so that for $x,a,b > 0$, the CDF is given by

$$G(x;a,b) = 1 - e^{-\left(\frac{x}{a}\right)^{a}}$$

and the PDF is given by

$$g(x;a,b)=\frac{ae^{-\left(\frac{x}{a}\right)^{a}}\left(\frac{x}{a}\right)^{a-1}}{b}$$

Theorem 5.1. The CDF of the Ampadu-Weibull distribution is given by

$$F(x,a,b,\lambda)=\frac{1-e^{-\lambda\left(1-e^{-\left(\frac{x}{a}\right)^{a}}\right)^{2}}}{1-e^{-\lambda}}$$

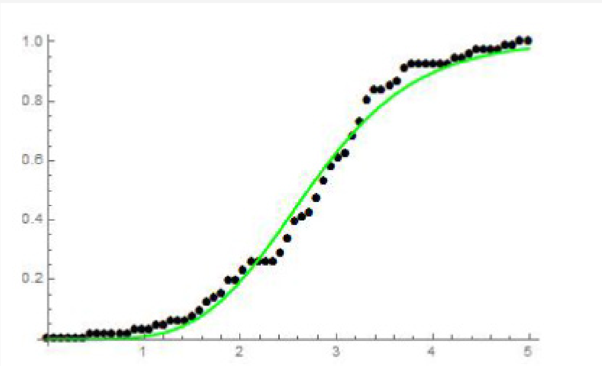
Where $x,a,b,\lambda>0$

Proof. Since the baseline distribution is Weibull distributed, then for $x,a,b>0$, the CDF is given by

$$G(x;a,b)=1-e^{-\left(\frac{x}{a}\right)^{a}}$$

So the result follows from Definition 4.1

Remark 5.2: If a random variable R is Ampadu-Weibull distributed, we write $R\sim AW(a,b,\lambda)$ (Figure 1).



The parameter estimates in the AW distribution wereobtained using the software MATHEMATICA

Figure 1: The CDF of AW (2.43975, 3.13163, 2.42517) fitted to the empirical distribution [5].

Illustration of $A_T-X(W)$ Family of Distributions of type I

The data set refers to the remission times (in months) of a random sample of 128 bladder cancer patients studied in [6]. We assume the random variable T follows the Burr X (BX) family of distributions so that for t, a, b > 0, the CDF is given by

$$G(t;a,b)=\left(1-e^{-a^2t^2}\right)^{b}$$

and the PDF is given by

$$g(t;a,b)=2a^2bte^{-a^2t^2}\left(1-e^{-a^2t^2}\right)^{b-1}$$

We assume the random variable X is Lomax distributed so the for x, c, d > 0, the CDF is given by

$$F(x,c,d)=1-\left(\frac{x}{c}+1\right)^{-d}$$

and the PDF is given by

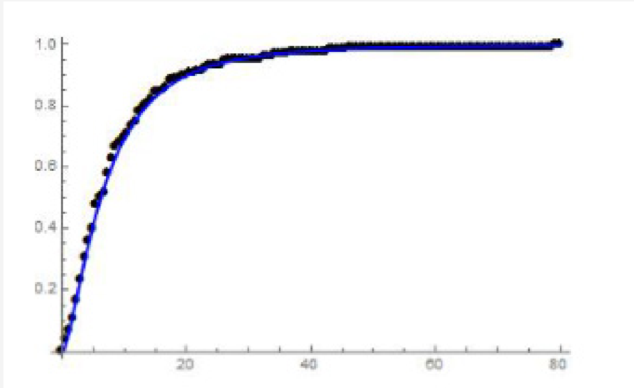
$$f(x,c,d)=\frac{d\left(\frac{x}{c}+1\right)^{-d-1}}{c}$$

Now we consider Remark 4.3 in Definition 4.2, then we get the following

Theorem 5.3: The CDF of the A_{Burr}X–Lomax family of distributions, for $x,a,b,c,d,\lambda>0$ is given by

$$V(x;a,b,c,d,\lambda)=\frac{1-\exp\left(\lambda\left(-\left(1-e^{-a^2\left(1-\left(\frac{x}{c}+1\right)^{-d}\right)^{2}}\right)^{2b}\right)\right)}{1-e^{-\lambda}}$$

Remark 5.4: If a random variable W has CDF given by the ABurrX – Lomax family of distributions, we write $W\sim ABXL(a,b,c,d,\lambda)$ (Figure 2).



The parameter estimates in the ABXL distribution were obtained using the software MATHEMATICA

Figure 2: The CDF of ABXL(1.62958, 0.387959, 10.1454, 0.128371, 25.8032) fitted to the empirical distribution of the bladder cancer patients data [6].

Illustration of $A_T-X(W)$ Family of Distributions of type II

The second data set is on 30 successive March precipitation (in inches) observations obtained from [7] and recorded in Section 7 of [8]. We assume the random variable T with support $[0,\infty)$ follows the Weibull distribution, so that for t > 0, and b, c > 0, the CDF is given by

$$G(t;b,c)=1-e^{-\left(\frac{t}{c}\right)^{b}}$$

and the PDF is given by

$$g(t;b,c)=\frac{be^{-\left(\frac{x}{c}\right)^{b}}\left(\frac{x}{c}\right)^{b-1}}{c}$$

We also assume the random variable X follows the Rayleigh distribution, so that for x, a > 0, the PDF is given by

$$f(x;a)=\frac{xe^{-\frac{x^2}{2a^2}}}{a^2}$$

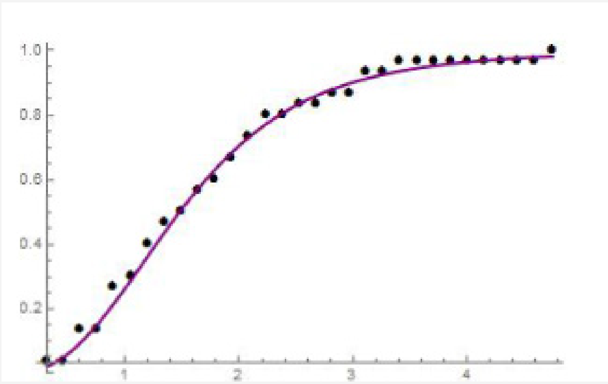
and the CDF is given by

$$F(x;a) = 1 - e^{-\frac{x^2}{2a^2}}$$

Considering Remark 4.5 in the first integral of Definition 4.4, we get the following Theorem 5.5. The CDF of the AWeibull – Rayleigh family of distributions of type II is given by

$$H(x;a,b,c,\lambda) = \frac{1-\exp\left(\lambda\left(-\left(1-e^{-\left(-\frac{\log\left(e^{-\frac{x^2}{2a^2}}\right)}{c}\right)^b}\right)^2\right)\right)}{1-e^{-\lambda}}$$

Remark 5.6: When a random variable J* has CDF given by Theorem 3.5, we write $J^* \sim AWR(a,b,c,\lambda)$ (Figure 3).



The parameter estimates in the AWR distribution were obtained using the software MATHEMATICA

Figure 3: The CDF of AWR(1.90154, 0.638893, 0.887386, 4.37718) fitted to the empirical distribution of the precipitation data [8].

Ilustration of $A_T - X(W)$ Family of Distributions of type III

In this application we consider the data set in [9] from [10], on the breaking stress of carbon fibers of 50 mm in length. We assume the random variable T with support $(-\infty,\infty)$ follows the Cauchy distribution, so that for t, a 2 R, b > 0, the CDF is given by

$$G(t;a,b) = \frac{\tan^{-1}\left(\frac{t-a}{b}\right)}{\pi} + \frac{1}{2}$$

and the PDF is given by

$$g(t;a,b) = \frac{1}{\pi b\left(\frac{(x-b)^2}{b^2}+1\right)}$$

We also assume the random variable X follows the Weibull distribution, so that for x > 0, and c, d > 0, the CDF is given by

$$F(x;c,d) = 1 - e^{-\left(\frac{x}{d}\right)^c}$$

and the PDF is given by now considering Remark 4.7 in the second integral of Definition 4.6, we get the following

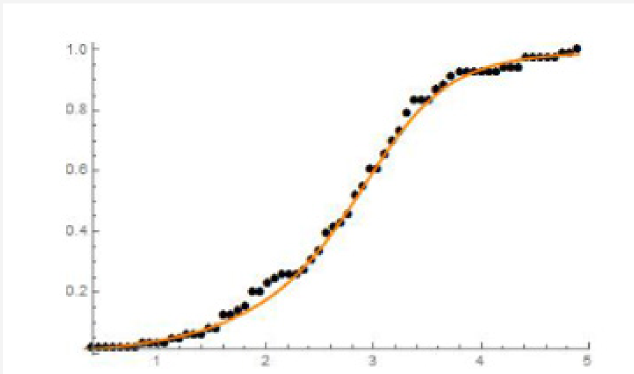
Theorem 5.7: The CDF of the A_{Cauchy} – Weibull family of distributions of type III is given by

$$P(x;a,b,c,d,\lambda) = \frac{1-\exp\left(-\lambda\left(\frac{\tan^{-1}\left(\frac{\log\left(e^{\left(\frac{x}{d}\right)^c}\left(1-e^{\left(\frac{x}{d}\right)^c}\right)\right)-a}{b}\right)}{\pi}+\frac{1}{2}\right)^2\right)}{1-e^{-\lambda}}$$

Where $x;a,b,c,d,\lambda > 0$

Remark 5.8: By differentiating the CDF of the A_{Cauchy}–Weibull family of distributions of type III, the PDF can be obtained

Remark 5.9: When a random variable N* has CDF given by Theorem 5.7, we write (Figure 4) $N^* \sim ACW(a,b,c,d,\lambda)$



The parameter estimates in the ACW distribution were obtained using the software MATHEMATICA

Figure 4: The CDF of ACW (0.0364987, 2.51544, 6.13094, 3.16873, 3.4696) fitted to the empirical distribution of the data on breaking stress of carbon fibers of 50 mm in length [9].

Acknowledgement

None.

Conflict of Interest

No conflict of interest.

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