Optimal Designs for M-Allel Crosses in One and Two-Way Setting

MK Sharma* and Mekonnen Tadesse
Addis Ababa University, Addis Ababa, Ethiopia

*Corresponding author: MK Sharma, Addis Ababa University, Addis Ababa, Ethiopia.

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Abstract

Recently authors [1] constructed optimal Block and Row-Column designs for complete diallel cross (CDC) methods by using nested balanced incomplete block designs. In this article we are presenting method of construction of three series of optimal block and row-column designs for partial double cross by using these designs. Our proposed optimal block and row-column designs for partial double cross are new and not available in statistical literature. AMS classification: 62K 05

Keywords: Nested balanced incomplete block design; Complete diallel cross; Partial double cross; Row and Column design; Block design

Introduction

Mating design involving multi–allele cross m (≥2) lines play very important role to study the genetic properties of a set of inbred lines in plant breeding experiments. A most common mating design in genetics is the diallel cross which consist of p(p-1)/2 crosses of p inbred lines such that the crosses of the type (i × j) = (j × i) for i, j = 1, 2, ..., p. This type of mating design is called complete diallel cross (CDC) method 4 of Griffing [2]. The concept of CDC can be easily extended to double cross designs. A double cross design is obtained by crossing two unrelated f₁ hybrids symbolized as (i × j) and (k × l), where i ≠ j ≠ k ≠ l ≠ i, are 4 parents and (i × j) and (k × l) are two f₁’s [3].

Let n denote the total number of crosses (experimental units) involved in a m-allele cross experiment, where m=2 or 4. Generally double cross experiments are conducted using a completely randomized design (CRD) or a randomized complete block (RCB) design involving some or all n crosses as treatments. The number of crosses in such a mating design increases rapidly with increase in the number of lines. It leads to an overall inefficient experiment. It is for this reason that the use of incomplete block design as environment design is needed for double cross experiments [3].

Parsad et al. [4] constructed optimal block designs for double cross experiments by using balanced incomplete block designs and nested balanced incomplete block designs of Morgan et al. [5]. Sharma & Tadesse [6] constructed double cross designs for even and odd value of p by using initial block of unreduced balanced incomplete block designs given by Bose et al. [7] and initial block of row-column designs given by Gupta & Choi [8], respectively.

In this paper we have used nested balanced incomplete block designs of Dey et al. [9] and Morgan et al. [5] for the construction of three series of optimal block and row-column designs for partial double cross experiment. The parameters of our proposed optimal block and row-column designs for partial double cross experiments are different from Parsad et al. [4] designs. We have used those designs of Morgan et al [5] for the construction of partial double cross experiment which Parsad et al. [4] did not use. In our proposed designs every cross is replicated equal number of times in a design. We have considered the model that includes the gca effects, apart from block effects, but no specific combining ability effects.

Some definitions

a. Definition 2.1: The double cross has been defined by Rawlings and Cockerham (1962 b) as a cross between two unrelated F₁ hybrids, say denoted by (i × j) and (k × l), where i ≠ j ≠ k ≠ l ≠ i, are denoting the grandparents and no two of them are same. Ignoring reciprocal crosses, with p grandparents, there will be \( \binom{p}{2} \) double crosses.
b. Definition 2.2 An arrangement of treatments each replicated $r^*$ times in two systems of blocks is said to be a nested balanced incomplete design (NBIBD) with parameters $(v, b, k, r^*, \lambda)$ if

- The second system is nested within the first, with each block from the first system, containing exactly $m$ blocks from the second system (sub blocks);
- ignoring the second system leaves a balanced incomplete block design with parameters $v, b, k, r^*, \lambda$;
- ignoring the first system leaves a balanced incomplete block design with parameters $v, b, k, r^*, \lambda$.

The following parametric relations hold for a nested balanced incomplete block design:

(i) $v r^* = b_1 k_1 = m b_2 k_2$

(ii) $(v-1) \lambda_1 = (k_1 -1) r^*$

(iii) $(v-1) \lambda_2 = (k_2 -1) r^*$

Universal optimality of designs for 1-way heterogeneity

Following Parsad et al. [4], let $db$ be a block design for an $m$-allel cross experiment involving $p$ inbred lines, $b$ blocks each of size $k$. This means that there are $k$ crosses in each of the blocks of $d$. Further, $r_{ag}$ and $s_{ai}$ denote the number of replication of the $a$th cross and the number of replications of the $i$th line in different crosses, respectively, in $d$ ($t = 1, 2, \ldots, n; \ j = 1, 2, \ldots, p$). Evidently, $\Sigma r_{ai} = bk, \Sigma s_{ai} = mbk = mn$ and $n = bk$, the total number of observations. We also considered this and took the following additive model for the observations obtained from design $d$.

$$y = m 1_a + \Delta_1^t g + \Delta_2^t b + e \quad \ldots (3.1)$$

where $y$ is an $n \times 1$ vector of observations, $1_a$ is an $n \times 1$ vector of ones, $\Delta_1$ is an $n \times p$ design matrix for lines $\Delta_2$ and $\Delta_2$ is an $n \times b$ design matrix for blocks, that is, the $(h,i)$th element of $\Delta_1$ (also of $\Delta_2$) is $1$ if the $i$th observation pertains to the $h$th line (also of block) and is zero otherwise. $\mu$ is a general mean, $g$ is a $p \times 1$ vector of line parameters, $b$ is a $b \times 1$ vector of block parameters and $e$ is an $n \times 1$ vector of residuals. It is assumed that vector $\tilde{f}$ is fixed and $e$ is normally distributed with mean $0$ and Var $(e) = \sigma^2$ with Cov $(\tilde{f}, e) = 0$, where $\tilde{f}$ is the identity matrix of conformable order.

For the analysis of proposed design $d$, the method of least squares leads to the following reduced normal equations for estimating the linear functions of the gca effects of lines under model (3.1).

$$C_d = G_d - N_a K_a N_d' = (c_{ij}) (i = 1, 2, \ldots, p)$$

where $G_d = \Delta_1, \Delta_1^t = (g_{ai}), g_{ai} = s_{ai}$ and for $i \neq i'$, $g_{ai}$ is the number of crosses in $d$ in which the lines $i$ and $i'$ appear together. $N_a = \Delta_1, \Delta_1 = (n_{ai}), n_{ai}$ is the number of times the line $i$ occurs in block $j$ of $d$. $K_a$ is the diagonal matrix of block sizes.

A design $d$ will be called connected if and only if rank $(C_d) = p - 1$, or equivalently, if and only if all elementary comparison among gca effects are estimable using $d$. We denote by $dD(p, b, k)$, the class of all such connected block design (d) with $p$ lines, $b$ blocks each of size $k$. To prove optimality of design $d$, we need the following well known lemma [10].

Lemma 3.1 For given positive integers $s$ and $t$, the minimum of $\sum n_i^2$ subject to $\sum n_i = s$, where $n_i$ are non-negative integers, is obtained when $t = s - t s\lceil t/s \rceil$ and $s = t + s\lceil t/s \rceil$, where $\lceil z \rceil$ denotes the largest integer not exceeding $z$. The corresponding minimum of

$$\sum n_i^2 = (2 s\lceil t/s \rceil + 1) - s\lceil t/s \rceil(\lceil t/s \rceil + 1)$$

Theorem 3.1: For any design $d \in D(p,b,k)$

$$\text{tr}(C_d) \leq mk \{k (2x + 1) - px (x + 1)\}$$

where $x = \lfloor mk/p \rfloor$.

Hence

$$\text{tr}(C_d) \leq mk - k^b(m k (k-1-2x) + px (x+1))$$

By Lemma 3.1, the above equality is attained if and only if $n_{ai} = x \text{or } x+1$, for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, b$

Theorem 3.1: For any design $d \in D(p,b,k)$

$$\text{tr}(C_d) = mk \{k (2x + 1) + px (x + 1)\}$$

where $x = \lfloor mk/p \rfloor$ and for a square matrix $A$, $\text{tr}(A)$ stands for the sum of the diagonal elements of $A$. Kiefer [11] showed that a design is universally optimal in a relevant class of competing design if (i) the information matrix (the $C_a$ matrix) of the design is completely symmetric in the sense that $C_a$ has all its diagonal elements equal and all of its off-diagonal elements equal, and (ii) the matrix $C_a$ has maximum trace in the class of competing designs, that is, such a design minimize the average variance of the best linear unbiased estimators of all elementary contrasts among the parameters of interest i.e. the general combining ability in our context. On the basis of the theorem 3.1 and Kiefer criterion of the universal optimality, we can state Theorem 3.2.
Theorem 3.2: For any design $d^* \in D(p, b, k)$ be a block design for m-allel crosses satisfying

(i) $\text{tr}(C_{d^*}) = k^{-1}b(m(k-1-2x) + p(x+1))$

(ii) $C_{d^*} = (p-1)^{-1}k^{-1}b(m(k-1-2x) + p(x+1))(lp-lp1p/p)$ is completely symmetric.

where $lp$ is an identity matrix of order $p$ and $1p1\prime p$ is a $p \times p$ matrix of all ones. Furthermore, using $d^*$ all elementary contrasts among gca effects are estimated with variance

$$[2b^{-1}(p-1)k/(m(k-1-2x) + p(x+1))]\sigma^2.$$

Then $d^*$ is universally optimal in $D(p, b, k)$, and in particular minimizes the average variance of the best linear unbiased estimator of all elementary contrasts among the general combining ability effects.

**Universal optimality of designs for 2-way- heterogeneity**

Consider an m-allel cross involving $p$ lines in a row-column design with $k$ rows and $b$ columns. The model can be written as given below

$$Y = \mu I_n + \Delta_1'\tau + \Delta_2'\beta + \Delta_3'\gamma + e \quad (4.1)$$

where $Y$ is an $n \times 1$ vector of observed responses, $\mu$ is the general mean, $\tau$, $\beta$, and $\gamma$ are the column vectors of pgp parameters, krow effects and bcolumn effects, respectively. $\Delta_1(n \times p), \Delta_2(n \times k), \Delta_3(n \times b)$ are the corresponding design matrices, respectively and $e$ denotes the vector of independent random errors having mean 0 and covariance matrix $\sigma^2I_n$.

Let $N_{si} = \Delta_1i$, be a $p \times k$ incidence matrix of lines versus rows and $N_{sj} = \Delta_2j$, be a $p \times k$ incidence matrix of treatments versus columns and $\Delta_3i = 1_{11}$. Let $r_{il}$ denote the number of times the $i$th cross appears in the design $d$, $j = 1, 2, . . . ,v$ and similarly $s_{lj}$ denote the number of times the $l$th line occurs in the design $d$, $i = 0, 1, . . . ,p-1$. Under (4.1), it can be shown that the reduced normal equations for estimating the treatment effects, after eliminating the effect of rows and columns, are

$$C_{d}^{\hat{\beta}} = Q \quad (4.2)$$

where $C_{d} = G_{d}(1/b)N_{s}N_{s}1'1 - 1/k N_{s}N_{s} + s^2 d_{s} s_{d}1'1$, $Q = T - 1/b N_{s}R - 1/k N_{s}C + (G/bk) s_{d}$, $G_{d} = \Delta_{3}'\Delta_{3} = (g_{dr}), N_{s}1 = (n_{s}), n$. is the number of times the line $i$ occurs in row $j$ of $d$, $N_{s} = (n_{s})$, $n_{s}$ is the number of times cross isocross in column $t$. $Q$ is a $p \times 1$ vector of adjusted treatments (crosses) total, $T$ is a $p \times 1$ vector of treatments (lines) totals, $R$ is a $k \times 1$ vector of rows totals, $C$ is a $b \times 1$ vector of columns totals, and $G$ is a grand total of all observations.

We use the following theorems of Prasad et al. [4] to prove optimality of the proposed designs.

Theorem 4.1: For any design $d \in D(p, b, k)$, trace $(C_{d})$ will be maximum when

(i) $n_{si} = n_{sj}$ i.e., the design is orthogonal with respect to the lines vs rows as blocks classification or a row – regular setting with respect to lines.

(ii) $n_{si} = xorp(x+1)$ where $x = \text{int}(mk/p)$ i.e., the lines appear either $x$ or $x+1$ time in columns as blocks classification, where $m = 2$ or 4.

Theorem 4.2: Let $d^* \in D(p, b, k)$ be a row – column design satisfying

a. $\text{tr}(C_{d^*}) = k^{-1}b(m(k-1-2x) + p(x+1))$

b. $C_{d^*}$ is completely symmetric.

Then $d^*$ is universally optimal in $D_{i}(p, b, k)$.

Now we will show a connection between optimal block and row-column design for optimal partial double cross design with nested balanced incomplete block designs of Preece (1967). Consider now a NBIB design $d$ obtained by developing mod (v) initial blocks, each sub-divided into $t$ sub-blocks. The parameters of such an NBIB design are $v = p$, $b = vt$, $b_1$, $b_2$, $b_1t_k$, $r_2t^*$, $k = 2$. If we identify the treatments of $d$ as lines of a diallel experiment and perform double crosses among the lines appearing in the same sub block of data arrange these sub-blocks into one bigger block and develop mod (v), we get a design $d^*$ for a double cross experiment involving $p$ lines with $b$ crosses arranged in $b = b_2/2$ blocks. Each double cross is replicated once. Such a design $d^*$ belongs to $D_{i}(p, b, k)$. For such a design $n_{xy} = 0$ or $1$ for $i = 1, 2, . . . ,p$, $j = 1, 2, . . . ,k$, and $G_{d} = (p-1)^{-1}[1(p-5)](p-5)1p1'1(4.3)$

Clearly $C_{d^*}$ given by (4.3) is completely symmetric and $\text{tr}(C_{d^*}) = p(p-5)$ which equals the equality given in theorem 4.2. Thus the design $d^*$ is optimal in $D_{i}(p, b, k)$ and using $d^*$ each elementary contrast among gca effects is estimated with a variance

$$2\sigma^2/p$$ $p(5)(4.4)$

If $p$ is even then the NBIB design has the following parameters.

$v = p$, $b_1 = (v-1)1', b_2$, $r_1$, $k = 2r_1$, $k_2 = 2r_1$, $r_2$, and $r$ is the number of replication of double cross. If we perform the same procedure given above, we get a design $d^*$ for a partial double cross experiment involving $p$ lines with $b_2$ crosses arranged in $b = b_2/2$ blocks. Such a design $d^* \in D(p, b, k)$ and using $d^*$ each elementary contrast among gca effects is estimated with a variance

$$2\sigma^2/p$$ $p(5)(4.4)$

Hence, we state the following theorems.
Theorem 4.3: The existence of a nested balanced incomplete block design d with parameters
\[ v, b, r, k, \lambda \] implies the existence of optimal block and row-column designs for partial double cross.

Theorem 4.4: The existence of a nested balanced incomplete block design d with parameters
\[ v, b, r, k, \lambda \] implies the existence of optimal block design for partial double cross.

Method of Construction

Series 1: Let \( p = 4m+1, m \geq 1 \) be a prime or a prime power and \( x \) be a primitive element of the Galois field of order \( p \). Consider the following \( m \) initial blocks.

\[
\left\{ \left( x^i, x^{i+2m} \right), \left( x^{i+m}, x^{i+3m} \right) \right\}, i = 0, 1, 2, \ldots, m - 1
\]

As shown by Dey et al. [9], these initial blocks, when developed in the sense of Bose [7], give rise to a nested balanced incomplete block design with parameters \( v = p = 4m + 1, b = 4m + 1, k = \frac{k}{2} \).

Note 1: For construction of double cross design it is necessary that the block size of a single block must be an even number i.e. \( m \) must be a multiple of 2.

The procedure to obtain the above designs has been explained by the following illustrative example.

Example 1: If we let \( m = 2 \), we get the following two blocks.

\[
\begin{pmatrix}
(1,2) & (1,2) \\
(2,1,1,1) & (2,1,1,2)
\end{pmatrix}
\]

We can now write both blocks in a single block as given below.

\[
\begin{pmatrix}
(1,2) & (x, x) \\
(2x + 1, x + 2) & (2x + 2, x + 1)
\end{pmatrix}
\]

where \( x \) is a primitive element of \( GF(3^2) \). Now cross the elements of the individual block and put these crosses in a single block. Adding successively the non-zero elements of \( GF(3^2) \) to the contents of the single block, we obtain block and row-column design for partial double cross experiment with parameters \( p = 9, b = 9, k = 2 \). The design is exhibited below, where the lines have been relabelled 1-9, using the correspondence 0→1, 1→2, 2→3, x→4, x+1→5, x+2→6, 2x→7, 2x+1→8, 2x+2→9:

\[
\begin{pmatrix}
(1,2) & (3,10) \\
(2,11) & (4,9)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(4,9) & (6,7) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(1,2) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(2,11) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(4,9) & (6,7) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(1,2) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(2,11) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(4,9) & (6,7) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(1,2) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
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\[
\begin{pmatrix}
(2,11) & (4,9) \\
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\end{pmatrix}
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\[
\begin{pmatrix}
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(3,10) & (6,7)
\end{pmatrix}
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\[
\begin{pmatrix}
(1,2) & (4,9) \\
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\end{pmatrix}
\]

\[
\begin{pmatrix}
(2,11) & (4,9) \\
(3,10) & (6,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(4,9) & (6,7) \\
(3,10) & (6,7)
\end{pmatrix}
\]

Series 2: Let \( p = 6m+1, m \geq 1 \) be a prime or a prime power and \( x \) be a primitive element of the Galois field of order \( p \). Consider the initial blocks

\[
\left\{ \left( x^i, x^{i+3m} \right), \left( x^{i+m}, x^{i+4m} \right), \left( x^{i+2m}, x^{i+5m} \right) \right\}, i = 0, 1, 2, \ldots, m - 1
\]

Dey et al. (1986)[4] showed when these initial blocks developed over \( mod(p) \), give a solution of a nested incomplete block design with parameters \( v = p = 6m+1, b = m(6m+1), k = 6, k = 2, \lambda = 1 \).

We can then arrange the above initial blocks into a single block as given below.

\[
\begin{pmatrix}
(1,2) & (2,11) \\
(4,9) & (8,5)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(3,10) & (6,7)
\end{pmatrix}
\]

We can arrange these two blocks in a single block as given below.

\[
\begin{pmatrix}
(1,2) & (2,11) \\
(4,9) & (8,5)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(3,10) & (6,7)
\end{pmatrix}
\]

Now performing crosses in pairs and developing these crosses over \( mod(p) \), we obtain block and row-column design with parameters \( p = 13, b = 13 \) and \( k = 3 \).

Series 3: Let \( p = 2m + 1, m \geq 2 \) be a prime or a prime power and consider the following m blocks

\[
(0, 2m), (1, 2m-1), (2, 2m-2), \ldots, (m-1, m+1) \ mod(2m+1)
\]

Example 3: If we let \( m = 4 \), then the single block will be as given below.

\[
\begin{pmatrix}
(0,8) \\
(1,7)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(2,6) \\
(3,5)
\end{pmatrix}
\]
Now applying the procedure given in example 1, we can obtain an optimal block design for double cross experiment with parameters \( p = 9, k = 2, b = 9 \).

Note 2: The \( m \) blocks given in series 3 form a nested balanced incomplete block design with parameters \( v = p = 2m + 1, b = m, k = m(2m+1), \lambda_2 = 2, \lambda_3 = 1 \) given by Dey et al.[9].

Example 4: Consider MRP 33 (2001) design with initial blocks given by

\[
\begin{bmatrix}
\end{bmatrix} \mod 15.
\]

Arranging these initial blocks in a single block and performing crosses between sub-blocks and developing \( \mod(15) \), treatment \( \infty \) is invariant under cyclic development of the initial blocks and we get optimal partial double crossblock design with parameters \( p = 16, b = 15, k = 4 \).

\[
\begin{bmatrix}
(\infty \times 0)(5 & 10)(1 & 2)(4 & 8)(6 & 9)(7 & 13)(11 & 3)(12 & 14)
\end{bmatrix} \mod (15)
\]

Conclusion

We have presented a method of construction of three series of optimal block and row-column designs for partial double cross by using nested balanced incomplete block designs. Our proposed optimal block and row-column designs for partial double cross are new and not available in statistical literature.

Acknowledgement

None

Conflict of Interest

No conflict of interest.

References


Table 1: Universally optimal block design for double cross with \( p \leq 16, s \leq 30 \) obtained by the above method from NBIB designs of Morgan, Preece & Rees [5], are listed in the following Table 1. These are the designs other than the designs catalogued by Das, Dey & Dean [12] and Parsad, et al.[4]. These are the new designs and successfully can be used in agricultural experimentation.

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